Some Formulas

Matrix representations
A ket is represented by a column vector $(c_{a'})$:

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle \;, \qquad c_{a'} = \langle a'|\alpha\rangle$$

Probability for $a' = |\langle a'|\alpha\rangle|^2$
Completeness relation:

$$\sum_{a'} |a'\rangle\langle a'| = 1$$

$$\langle\alpha|\alpha\rangle = \sum_{a'} |\langle a'|\alpha\rangle|^2$$

An operator is represented by a matrix:

$$A_{k,l} = \langle a_k|A|a_l\rangle$$

The commutator $[,]$ and the anticommutator $\{,\}$ are defined by:

$$[A,B] = AB - BA\,, \qquad \{A,B\} = AB + BA$$

Spin-1/2
In the basis of S_z , spin up and spin down along the z-axis, we have:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_\pm = S_x \pm i S_y \quad , \quad [S_i, S_j] = i\epsilon_{ijk}\hbar S_k \quad , \quad \{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}$$

Expectation values

$$\langle A\rangle_\alpha \doteq \langle\alpha|A|\alpha\rangle = \sum_{a'} a' |\langle a'|\alpha\rangle|^2$$

Heisenberg uncertainty relation (in a more general formulation):

$$\Delta_\alpha A \doteq A - \langle A\rangle_\alpha \;, \qquad \langle(\Delta_\alpha A)^2\rangle = \langle A^2\rangle_\alpha - \langle A\rangle_\alpha^2$$

$$\langle(\Delta_\alpha A)^2\rangle\langle(\Delta_\alpha B)^2\rangle \;\geq\; \frac{1}{4}|\langle[A,B]\rangle_\alpha^2$$

Change of basis

$$|b^l\rangle = U|a^l\rangle\;, \qquad U = \sum_k |b^k\rangle\langle a^k|\;, \qquad U^\dagger U = UU^\dagger = 1$$

Representation of U in the old basis:

$$U_{kl} = \langle a^k|U|a^l\rangle = \langle a^k|b^l\rangle$$

$$\langle b^k|\alpha\rangle = \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$$

In matrix notation:

$$\Big(\text{new}\Big) = \Big(U^\dagger\Big)\Big(\text{old}\Big)$$

The trace of an operator X is defined as the sum of diagonal elements:

$$\text{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle$$

Diagonalization

$$\det\Big(B - \lambda 1\Big) = 0$$

Position Eigenkets

$$x|x'\rangle = x'|x'\rangle\,, \qquad \langle x''|x'\rangle = \delta(x'' - x')\,, \qquad |\alpha\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\alpha\rangle$$

Point particle in Euclidean space

The Hilbert space is defined by the set of 3 commuting position observables:

$$[x_i, x_j] = 0\,, \qquad [p_i, p_j] = 0\,, \qquad [x_i, p_j] = i\hbar\delta_{ij}$$

From this follows:

$$[x_i, F(\boldsymbol{p})] = i\hbar\frac{\partial F}{\partial p_i}\,, \qquad [p_i, G(\boldsymbol{x})] = -i\hbar\frac{\partial G}{\partial x_i}$$

The wave function is simply the position representation:

$$\psi_\alpha(\boldsymbol{x}') = \langle \boldsymbol{x}'|\alpha\rangle$$

$$\langle\beta|\alpha\rangle = \int d\boldsymbol{x}'\langle\beta|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle = \int d\boldsymbol{x}'\psi_\beta^*(\boldsymbol{x}')\psi_\alpha(\boldsymbol{x}')$$

The wave function of a momentum eigenstate in one dimension:

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

In 3 dimensions, this becomes:

$$\langle\boldsymbol{x}'|\boldsymbol{p}'\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\exp\left(\frac{i\boldsymbol{p}'.\boldsymbol{x}'}{\hbar}\right)$$