Chapter 1:

Ordinary Differential Equations

(ODEs)

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1.1. Basic Concepts. Modeling

mathematical modeling

often an equation containing derivatives of an unknown function

differential equation

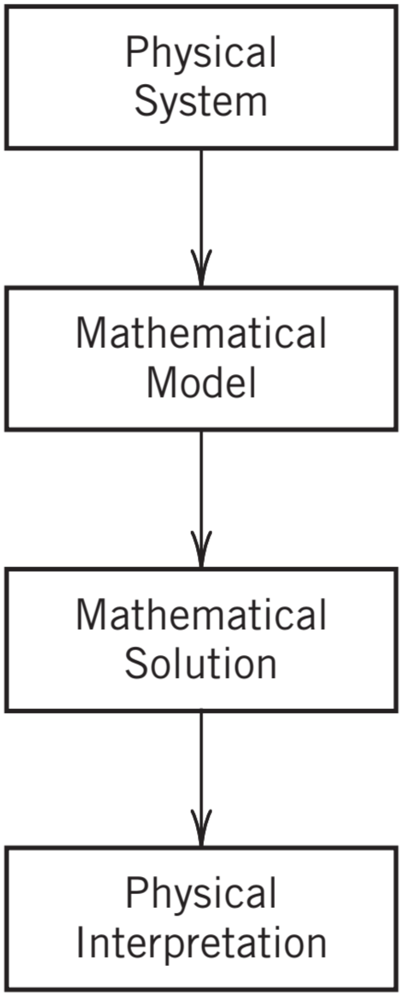
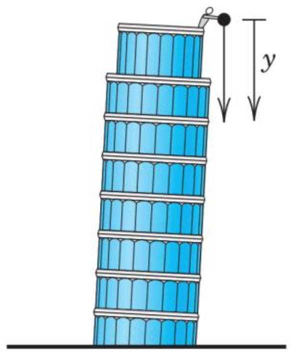


Fig. 1. Modeling, solving, interpreting

Some applications of differential equations



Falling stone

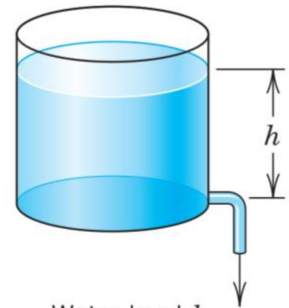
$y'' = g = const.$

(Sec. 1.1)

Parachutist

$mv' = mg - bv^2$

(Sec. 1.2)

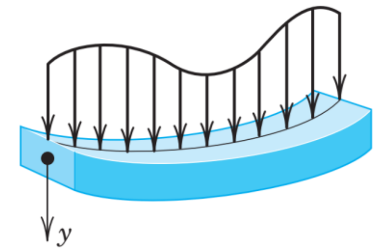


Water level h

Outflowing water

$h' = -k\sqrt{h}$

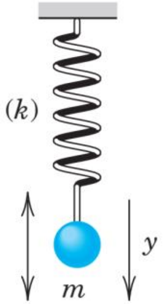
(Sec. 1.3)



Deformation of a beam

$EIy^{iv} = f(x)$

(Sec. 3.3)

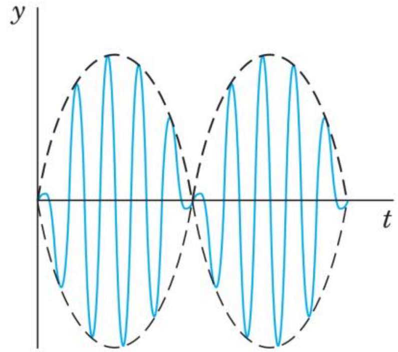


Displacement y

Vibrating mass on a spring

$my'' + ky = 0$

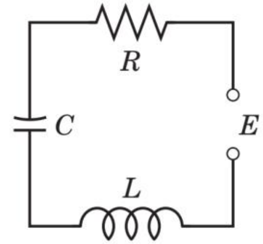
(Secs. 2.4, 2.8)



Beats of a vibrating system

$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 \approx \omega$

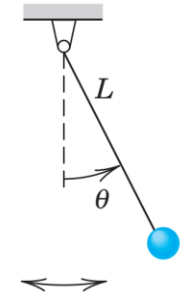
(Sec. 2.8)



Current I in an RLC circuit

$LI'' + RI' + \frac{1}{C} I = E'$

(Sec. 2.9)



Pendulum

$L\theta'' + g \sin \theta = 0$

(Sec. 4.5)

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(1) $$y' = \cos x$$

(2) $$y'' + 9y = e^{-2x}$$

(3) $$y'y''' - \tfrac{3}{2}y'^2 = 0$$

are ordinary differential equations (ODEs). Here, as in calculus, y' denotes dy/dx, $y'' = d^2y/dx^2$, etc. The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDEs have important engineering applications, but they are more complicated than ODEs; they will be considered in Chap. 12.

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

(4) $$F(x, y, y') = 0$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

A function $$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

EXAMPLE 1 Verification of Solution

Verify that $y = c/x$ (c an arbitrary constant) is a solution of the ODE $xy' = -y$ for all $x \neq 0$. Indeed, differentiate $y = c/x$ to get $y' = -c/x^2$. Multiply this by x, obtaining $xy' = -c/x$; thus, $xy' = -y$, the given ODE. ■

EXAMPLE 2 Solution by Calculus. Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions***. Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 3 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.

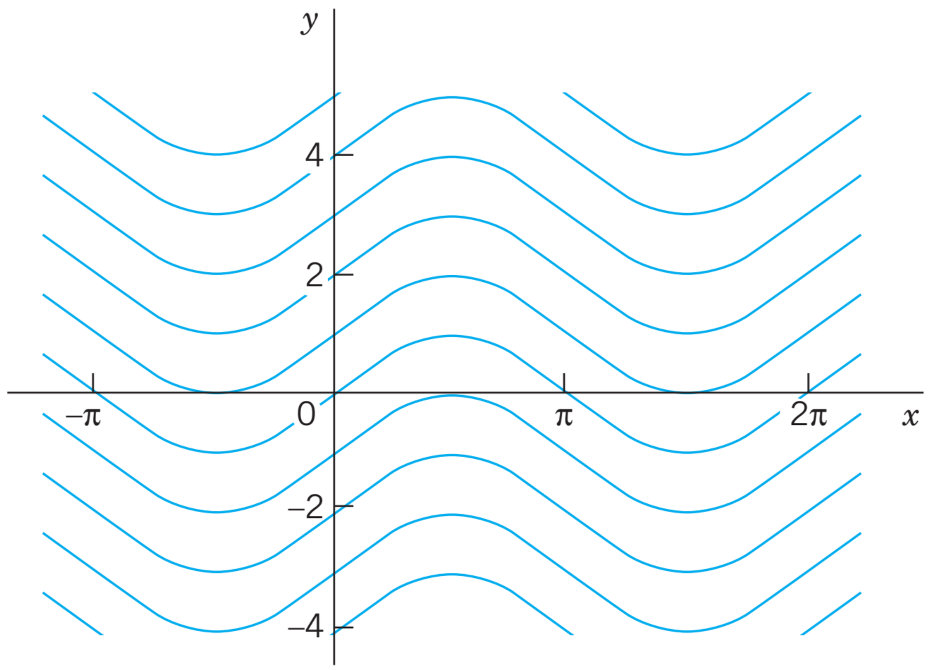


Fig. 3. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

EXAMPLE 3 (A) Exponential Growth. (B) Exponential Decay

From calculus we know that $y = ce^{0.2t}$ has the derivative

$$y' = \frac{dy}{dt} = 0.2e^{0.2t} = 0.2y.$$

Hence y is a solution of $y' = 0.2y$ (Fig. 4A). This ODE is of the form $y' = ky$. With positive-constant k it can model exponential growth, for instance, of colonies of bacteria or populations of animals. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as **Malthus's law**.[1] We shall say more about this topic in Sec. 1.5.

(B) Similarly, $y' = -0.2$ (with a minus on the right) has the solution $y = ce^{-0.2t}$, (Fig. 4B) modeling **exponential decay**, as, for instance, of a radioactive substance (see Example 5). ■

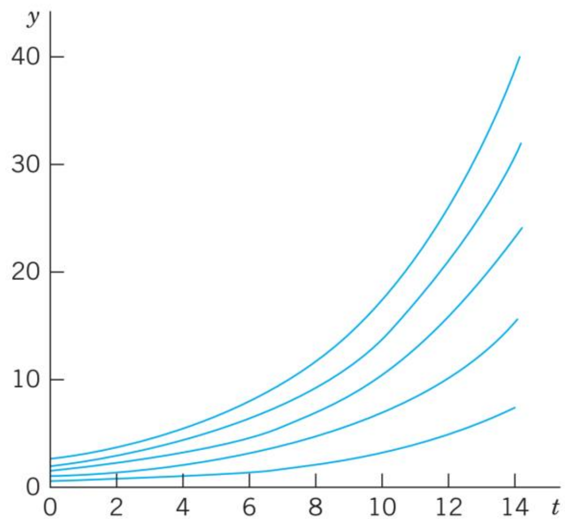


Fig. 4A. Solutions of $y' = 0.2y$ in Example 3 **(exponential growth)**

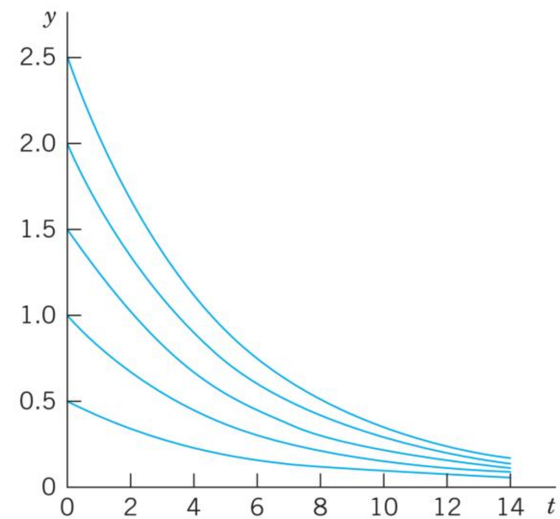


Fig. 4B. Solutions of $y' = -0.2y$ in Example 3 **(exponential decay)**

We see that each ODE in these examples has a solution that contains an arbitrary constant c. Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c. If we choose a specific c (e.g., $c = 6.45$ or 0 or -2.01) we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE, together with an initial condition, is called an **initial value problem**. Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{5}$$

EXAMPLE 4 Initial Value Problem

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7.$$

Solution. The general solution is $y(x) = ce^{3x}$; see Example 3. From this solution and the initial condition we obtain $y(0) = ce^{0} = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

1.3. Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

$$g(y)\, y' = f(x) \tag{1}$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

$$\int g(y)\, y' dx = \int f(x)\, dx + c. \tag{2}$$

On the left we can switch to y as the variable of integration. By calculus, $y' dx = dy$, so that

$$\int g(y)\, dy = \int f(x)\, dx + c. \tag{3}$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: x appears only on the right and y only on the left.

EXAMPLE 1 Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1 + y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

EXAMPLE 2 Separable ODE

The ODE $y' = (x + 1)e^{-x}y^2$ is separable; we obtain $y^{-2}\,dy = (x + 1)e^{-x}\,dx$.

By integration, $-y^{-1} = -(x + 2)e^{-x} + c, \qquad y = \dfrac{1}{(x + 2)e^{-x} - c}$.

EXAMPLE 3 Initial Value Problem (IVP). Bell-Shaped Curve

Solve $y' = -2xy,\ y(0) = 1.8$.

Solution. By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + \tilde{c}, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Hence the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve (Fig. 10). ■

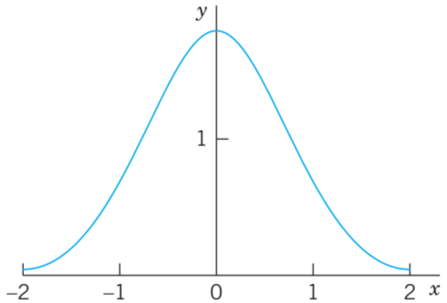


Fig. 10. Solution in Example 3 (bell-shaped curve)

Extended Method: Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE,* a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

The form of such an ODE suggests that we set $y/x = u$; thus,

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u. \tag{9}$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}. \tag{10}$$

EXAMPLE 8 Reduction to Separable Form

Solve

$$2xyy' = y^2 - x^2.$$

Solution. To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' from (9) and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2 to obtain (Fig. 14)

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis. ■

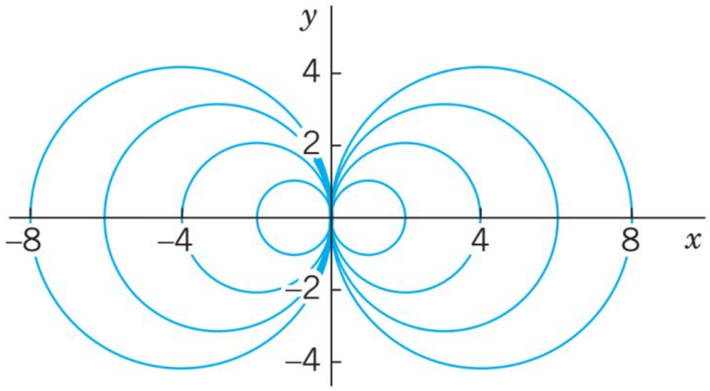


Fig. 14. General solution (family of circles) in Example 8

1.4. Exact ODEs. Integrating Factors

We recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c = \text{const}$, then $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2},$$

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'dx$ as in Sec. 1.3)

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{2}$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

(3) $$u(x, y) = c.$$

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

(4) (a) $\dfrac{\partial u}{\partial x} = M,$ (b) $\dfrac{\partial u}{\partial y} = N.$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

$\Rightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation.

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

(6)
$$u = \int M\,dx + k(y);$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k. (See Example 1, below.)

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use (4b). Then, instead of (6), we first have by integration with respect to y

(6*)
$$u = \int N\,dy + l(x).$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

EXAMPLE 1 An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \cos(x + y)\,dx + (\cos(x + y) + 3y^2 + 2y)\,dy = 0. \tag{9}$$

This completes the check. ■

EXAMPLE 2 An Initial Value Problem

Solve the initial value problem

$$(\cos y \sinh x + 1)\, dx - \sin y \cosh x\, dy = 0, \qquad y(1) = 2. \tag{10}$$

Solution. You may verify that the given ODE is exact. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x\, dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = M = \cos y \sinh x + 1$. Hence $dl/dx = 1$. By integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$. Hence the answer is $\cos y \cosh x + x = 0.358$. Figure 17 shows the particular solutions for $c = 0, 0.358$ (thicker curve), 1, 2, 3. Check that the answer satisfies the ODE. (Proceed as in Example 1.) Also check that the initial condition is satisfied. ■

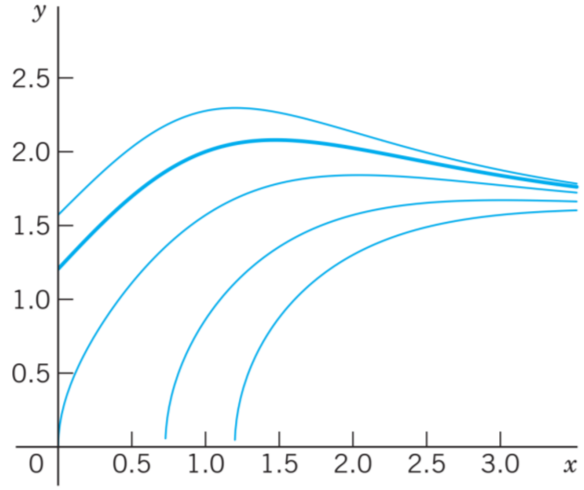


Fig. 17. Particular solutions in Example 2

EXAMPLE 3 WARNING! Breakdown in the Case of Nonexactness

The equation $-y\,dx + x\,dy = 0$ is not exact because $M = -y$ and $N = x$, so that in (5), $\partial M/\partial y = -1$ but $\partial N/\partial x = 1$. Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + k(y) = -xy + k(y), \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed. ■

Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\,dx + x\,dy = 0$. It is not exact. However, if we multiply it by $1/x^2$, we get an exact equation [check exactness by (5)!],

$$\text{(11)} \qquad \frac{-y\,dx + x\,dy}{x^2} = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right) = 0.$$

Integration of (11) then gives the general solution $y/x = c = \text{const}$.

This example gives the idea. All we did was to multiply a given nonexact equation, say,

(12) $$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

by a function F that, in general, will be a function of both x and y. The result was an equation

(13) $$FP\,dx + FQ\,dy = 0$$

that is exact, so we can solve it as just discussed. Such a function $F(x, y)$ is then called an **integrating factor** of (12).

EXAMPLE 4 Integrating Factor

The integrating factor in (11) is $F = 1/x^2$. Hence in this case the exact equation (13) is

$$FP\,dx + FQ\,dy = \frac{-y\,dx + x\,dy}{x^2} = d\left(\frac{y}{x}\right) = 0. \qquad \text{Solution} \qquad \frac{y}{x} = c.$$

These are straight lines $y = cx$ through the origin. (Note that $x = 0$ is also a solution of $-y\,dx + x\,dy = 0$.)

It is remarkable that we can readily find other integrating factors for the equation $-y\,dx + x\,dy = 0$, namely, $1/y^2$, $1/(xy)$, and $1/(x^2 + y^2)$, because

(14) $$\frac{-y\,dx + x\,dy}{y^2} = d\left(\frac{x}{y}\right), \quad \frac{-y\,dx + x\,dy}{xy} = -d\left(\ln\frac{x}{y}\right), \quad \frac{-y\,dx + x\,dy}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right).$$ ■

How to Find Integrating Factors

For $M\,dx + N\,dy = 0$ the exactness condition (5) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FP\,dx + FQ\,dy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \tag{15}$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_yP + FP_y = F_xQ + FQ_x.$$

In the general case, this would be complicated and useless. So we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one*—the result may be useful (and may also help you later on). Hence we look for an integrating factor depending only on ***one*** variable: fortunately, in many practical cases, there are such factors, as we shall see. Thus, let $F = F(x)$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (15) becomes

$$FP_y = F'Q + FQ_x.$$

Dividing by FQ and reshuffling terms, we have

$$(16) \qquad \boxed{\frac{1}{F}\frac{dF}{dx} = R,} \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

THEOREM 1

Integrating Factor $F(x)$

If (12) is such that the right side R of (16) depends only on x, then (12) has an integrating factor $F = F(x)$, which is obtained by integrating (16) and taking exponents on both sides.

$$\textbf{(17)} \qquad F(x) = \exp \int R(x)\, dx.$$

Similarly, if $F^* = F^*(y)$, then instead of (16) we get

$$(18) \qquad \boxed{\frac{1}{F^*}\frac{dF^*}{dy} = R^*,} \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

and we have the companion

THEOREM 2

Integrating Factor $F^*(y)$

If (12) *is such that the right side R^* of* (18) *depends only on y, then* (12) *has an integrating factor $F^* = F^*(y)$, which is obtained from* (18) *in the form*

(19) $$F^*(y) = \exp \int R^*(y)\, dy.$$

EXAMPLE 5 Application of Theorems 1 and 2. Initial Value Problem

Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

(20) $$(e^{x+y} + ye^y)\, dx + (xe^y - 1)\, dy = 0, \quad y(0) = -1$$

Solution. *Step 1. Nonexactness*. The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^y) = e^{x+y} + e^y + ye^y \qquad \text{but} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y - 1) = e^y.$$

Step 2. Integrating factor. General solution. Theorem 1 fails because R [the right side of (16)] depends on both x and y.

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (18) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1.$$

Hence (19) gives the integrating factor $F^*(y) = e^{-y}$. From this result and (20) you get the exact equation

$$(e^x + y)\,dx + (x - e^{-y})\,dy = 0.$$

Test for exactness; you will get 1 on both sides of the exactness condition. By integration, using (4a),

$$u = \int (e^x + y)\,dx = e^x + xy + k(y).$$

Differentiate this with respect to y and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$\boxed{u(x, y) = e^x + xy + e^{-y} = c.}$$

1.5. Linear ODEs. Bernoulli Equation

Linear ODEs or ODEs that can be transformed to linear form are models of various phenomena, for instance, in physics, biology, population dynamics, and ecology, as we shall see. A first-order ODE is said to be **linear** if it can be brought into the form

(1) $$y' + p(x)y = r(x),$$

by algebra, and **nonlinear** if it cannot be brought into this form.

The defining feature of the linear ODE (1) is that it is linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be ***any*** given functions of x. If in an application the independent variable is time, we write t instead of x.

If the first term is $f(x)y'$ (instead of y'), divide the equation by $f(x)$ to get the **standard form** (1), with y' as the first term, which is practical.

For instance, $y' \cos x + y \sin x = x$ is a linear ODE, and its standard form is $y' + y \tan x = x \sec x$.

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \qquad \textbf{(2)}$$

and is called **homogeneous**. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)\,dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \qquad \textbf{(3)}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous**. It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the previous section or we can proceed directly, as follows. We multiply (1) by $F(x)$, obtaining

$$(1^*) \qquad Fy' + pFy = rF.$$

$$y' + p(x)y = r(x),$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h'e^h y = e^h y' + (e^h)'y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r\,dx + c.$$

Dividing by e^h, we obtain the desired solution formula

(4) $$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx.$$

The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

(4*) $$y(x) = e^{-h}\int e^h r\,dx + ce^{-h},$$

we see the following:

(5) Total Output = Response to the Input r + Response to the Initial Data.

EXAMPLE 1 First-Order ODE, General Solution, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$h = \int p\, dx = \int \tan x\, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\, dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$. ■

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**[7]

(9) $$y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}).$$

If $a = 0$ or $a = 1$, Equation (9) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (9), obtaining

$$u' = (1-a)y^{-a}y' = (1-a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1-a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

(10) $$u' + (1-a)pu = (1-a)g.$$

EXAMPLE 4 Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**[8]):

$$y' = Ay - By^2 \tag{11}$$

Solution. Write (11) in the form (9), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$, so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (11),

$$u' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}.$$

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

$$u' + Au = B.$$

The general solution is [by (4)]

$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (11),

$$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A} \tag{12}$$

(Fig. 21)

1.5. Existence and Uniqueness of Solutions for Initial Value Problems

The initial value problem

$$|y'| + |y| = 0, \qquad y(0) = 1$$

has no solution because $y = 0$ (that is, $y(x) = 0$ for all x) is the only solution of the ODE.

The initial value problem

$$y' = 2x, \qquad y(0) = 1$$

has precisely one solution, namely, $y = x^2 + 1$.

The initial value problem

$$xy' = y - 1, \qquad y(0) = 1$$

has infinitely many solutions, namely, $y = 1 + cx$, where c is an arbitrary constant because $y(0) = 1$ for all c.

From these examples we see that an **initial value problem**

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

may have no solution, precisely one solution, or more than one solution. This fact leads to the following two fundamental questions.

Problem of Existence

Under what conditions does an initial value problem of the form (1) *have at least one solution (hence one or several solutions)?*

Problem of Uniqueness

Under what conditions does that problem have at most one solution (hence excluding the case that is has more than one solution)?

Theorems that state such conditions are called **existence theorems** and **uniqueness theorems**, respectively.

THEOREM 1

Existence Theorem

Let the right side $f(x, y)$ *of the ODE in the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

be continuous at all points (x, y) *in some rectangle*

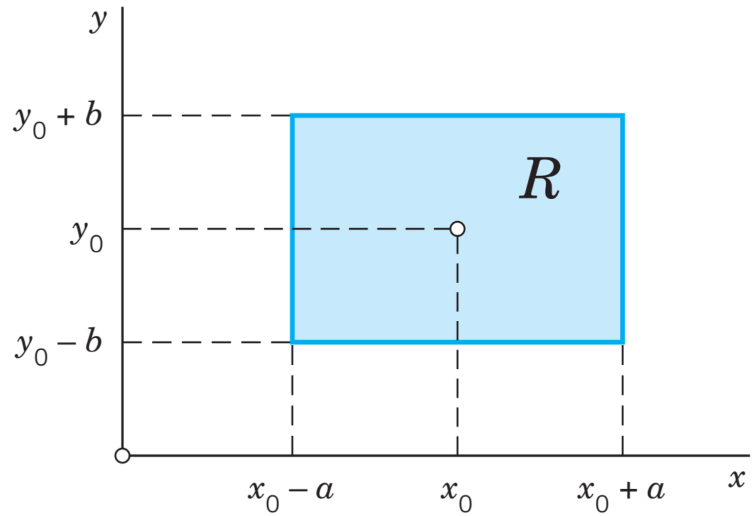
$$R: |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in R; that is, there is a number K such that*

$$|f(x, y)| \leqq K \qquad \textit{for all } (x, y) \textit{ in } R. \tag{2}$$

Then the initial value problem (1) *has at least one solution* $y(x)$. *This solution exists at least for all x in the subinterval* $|x - x_0| < \alpha$ *of the interval* $|x - x_0| < a$; *here,* α *is the smaller of the two numbers a and* b/K.

(*Example of Boundedness*. The function $f(x, y) = x^2 + y^2$ is bounded (with $K = 2$) in the square $|x| < 1, |y| < 1$. The function $f(x, y) = \tan(x + y)$ is not bounded for $|x + y| < \pi/2$. Explain!)



THEOREM 2

Uniqueness Theorem

Let f and its partial derivative $f_y = \partial f/\partial y$ *be continuous for all* (x, y) *in the rectangle* R *(Fig.* 26) *and bounded, say,*

$$(3) \qquad \text{(a)} \quad |f(x, y)| \leqq K, \qquad \text{(b)} \quad |f_y(x, y)| \leqq M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at most one solution* $y(x)$*. Thus, by Theorem* 1, *the problem has precisely one solution. This solution exists at least for all* x *in that subinterval* $|x - x_0| < \alpha$.

Understanding These Theorems

These two theorems take care of almost all practical cases. Theorem 1 says that if $f(x, y)$ is continuous in some region in the xy-plane containing the point (x_0, y_0), then the initial value problem (1) has at least one solution.

Theorem 2 says that if, moreover, the partial derivative $\partial f/\partial y$ of f with respect to y exists and is continuous in that region, then (1) can have at most one solution; hence, by Theorem 1, it has precisely one solution.

Since $y' = f(x, y)$, the condition (2) implies that $|y'| \leqq K$; that is, the slope of any solution curve $y(x)$ in R is at least $-K$ and at most K. Hence a solution curve that passes through the point (x_0, y_0) must lie in the colored region in Fig. 27 bounded by the lines l_1 and l_2 whose slopes are $-K$ and K, respectively. Depending on the form of R, two different cases may arise. In the first case, shown in Fig. 27a, we have $b/K \geqq a$ and therefore $\alpha = a$ in the existence theorem, which then asserts that the solution exists for all x between $x_0 - a$ and $x_0 + a$. In the second case, shown in Fig. 27b, we have $b/K < a$. Therefore, $\alpha = b/K < a$, and all we can conclude from the theorems is that the solution exists for all x between $x_0 - b/K$ and $x_0 + b/K$.

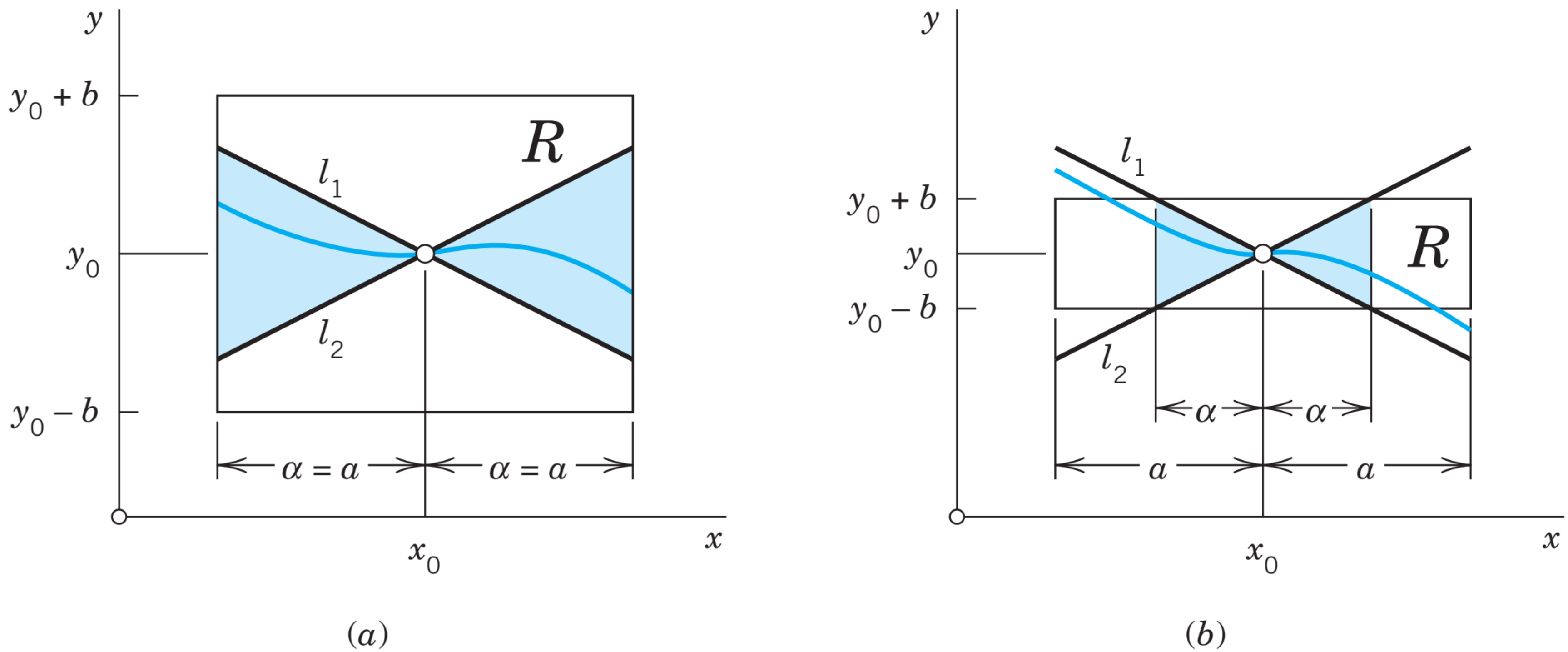


Fig. 27. The condition (2) of the existence theorem. (a) First case. (b) Second case

EXAMPLE 1 Choice of a Rectangle

Consider the initial value problem

$$y' = 1 + y^2, \qquad y(0) = 0$$

and take the rectangle R; $|x| < 5$, $|y| < 3$. Then $a = 5$, $b = 3$, and

$$|f(x, y)| = |1 + y^2| \leqq K = 10,$$

$$\left|\frac{\partial f}{\partial y}\right| = 2|y| \leqq M = 6,$$

$$\alpha = \frac{b}{K} = 0.3 < a.$$

Indeed, the solution of the problem is $y = \tan x$ (see Sec. 1.3, Example 1). This solution is discontinuous at $\pm\pi/2$, and there is no *continuous* solution valid in the entire interval $|x| < 5$ from which we started. ■