Chapter 9:

Vector Differential Calculus.

Grad, Div., Curl

E. Kreyszig

9.1. Vectors in 2-Space and 3-Space

In engineering, physics, mathematics, and other areas we encounter two kinds of quantities. They are scalars and vectors.

A **scalar** is a quantity that is determined by its magnitude. It takes on a numerical value, i.e., a number. Examples of scalars are time, temperature, length, distance, speed, density, energy, and voltage.

In contrast, a **vector** is a quantity that has both magnitude and direction. We can say that a vector is an ***arrow*** or a ***directed line segment***. For example, a velocity vector has length or magnitude, which is speed, and direction, which indicates the direction of motion. Typical examples of vectors are displacement, velocity, and force, see Fig. 164 as an illustration.

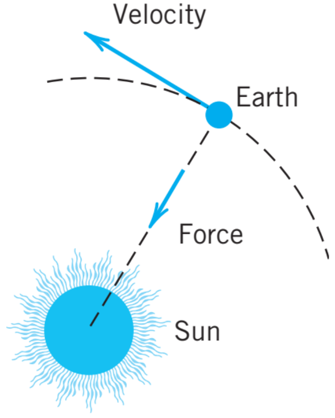


Fig. 164. Force and velocity

A vector (arrow) has a tail, called its **initial point**, and a tip, called its **terminal point**. This is motivated in the **translation** (displacement without rotation) of the triangle in Fig. 165, where the initial point P of the vector $\mathbf{a}$ is the original position of a point, and the terminal point Q is the terminal position of that point, its position *after* the translation. The length of the arrow equals the distance between P and Q. This is called the **length** (or *magnitude*) of the vector $\mathbf{a}$ and is denoted by $|\mathbf{a}|$. Another name for *length* is **norm** (or *Euclidean norm*).

A vector of length 1 is called a **unit vector**.

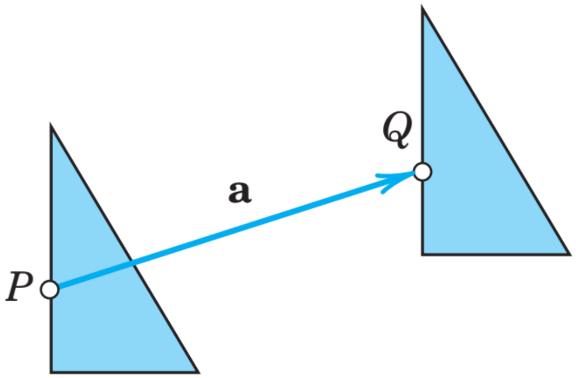


Fig. 165. Translation

DEFINITION

Equality of Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, written $\mathbf{a} = \mathbf{b}$, if they have the same length and the same direction [as explained in Fig. 166; in particular, note (B)]. Hence a vector can be arbitrarily translated; that is, its initial point can be chosen arbitrarily.

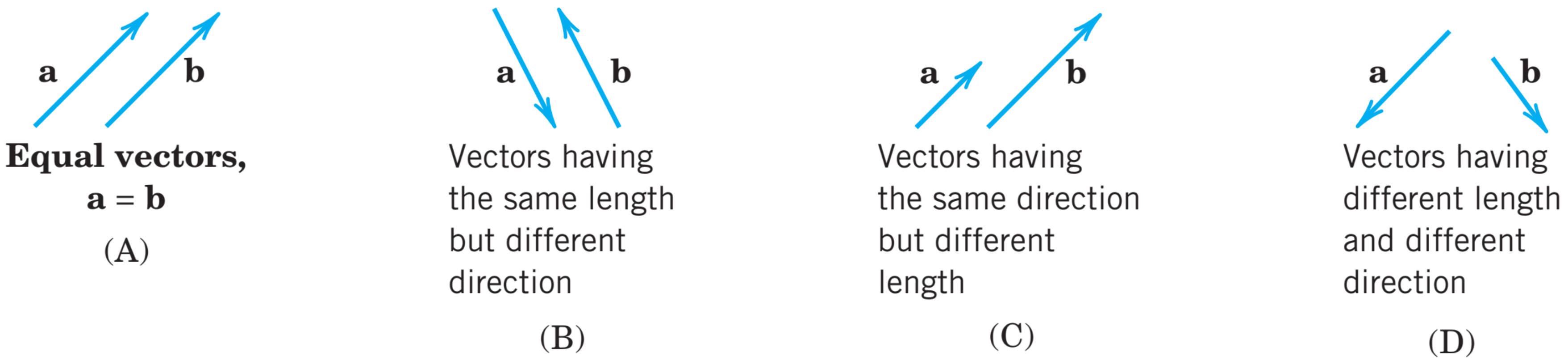


Fig. 166. (A) Equal vectors. (B)–(D) Different vectors

Components of a Vector

We choose an *xyz* **Cartesian coordinate system**[1] in space (Fig. 167), that is, a usual rectangular coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes. Let **a** be a given vector with initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2). Then the three coordinate differences

(1) $$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1$$

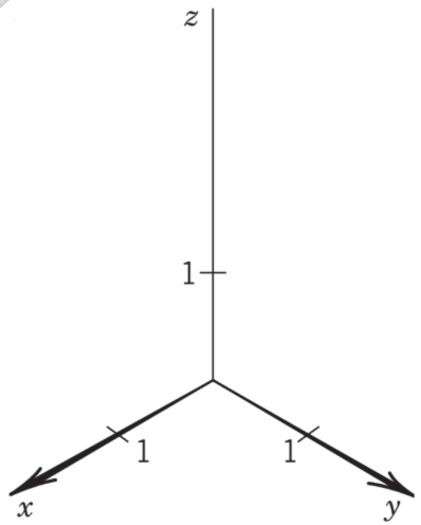


Fig. 167. Cartesian coordinate system

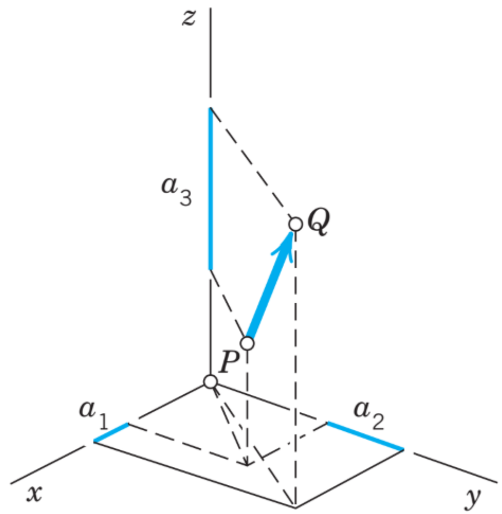


Fig. 168. Components of a vector

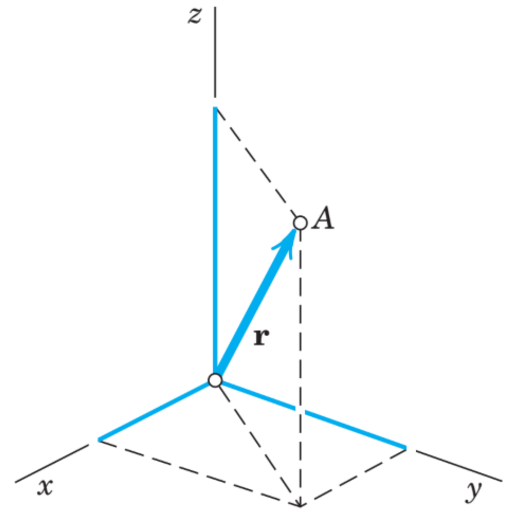


Fig. 169. Position vector **r** of a point A: (x, y, z)

are called the **components** of the vector **a** with respect to that coordinate system, and we write simply $\mathbf{a} = [a_1, a_2, a_3]$. See Fig. 168.

The **length** $|\mathbf{a}|$ of **a** can now readily be expressed in terms of components because from (1) and the Pythagorean theorem we have

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{2}$$

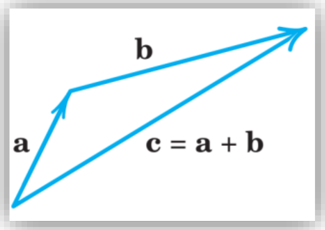
A Cartesian coordinate system being given, the **position vector r** of a point A: (x, y, z) is the vector with the origin $(0, 0, 0)$ as the initial point and A as the terminal point (see Fig. 169). Thus in components, $\mathbf{r} = [x, y, z]$. This can be seen directly from (1) with $x_1 = y_1 = z_1 = 0$.

Vector Addition, Scalar Multiplication

DEFINITION

Addition of Vectors

The **sum a** + **b** of two vectors **a** $= [a_1, a_2, a_3]$ and **b** $= [b_1, b_2, b_3]$ is obtained by adding the corresponding components,



(3) $$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3].$$

Geometrically, place the vectors as in Fig. 170 (the initial point of **b** at the terminal point of **a**); then **a** + **b** is the vector drawn from the initial point of **a** to the terminal point of **b**.

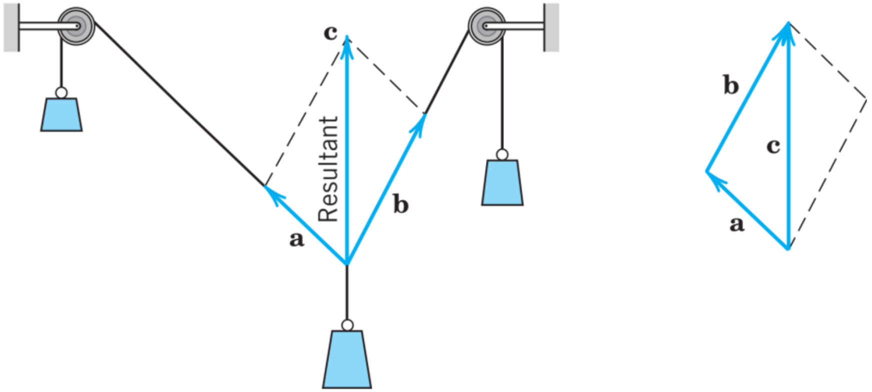


Fig. 171. Resultant of two forces (**parallelogram law**)

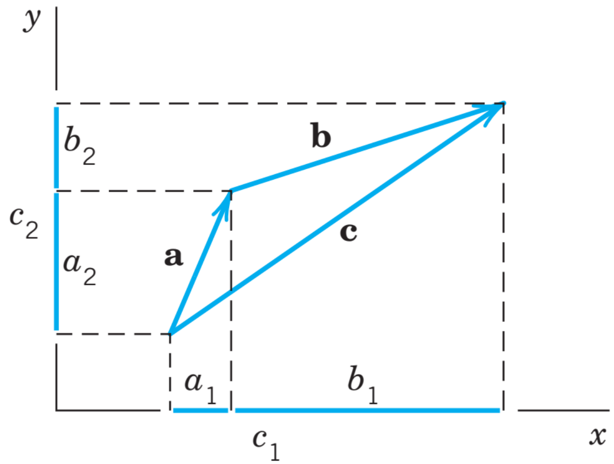


Fig. 172. Vector addition

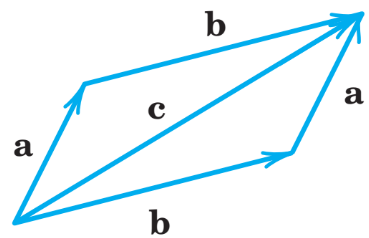


Fig. 173. Cummutativity of vector addition

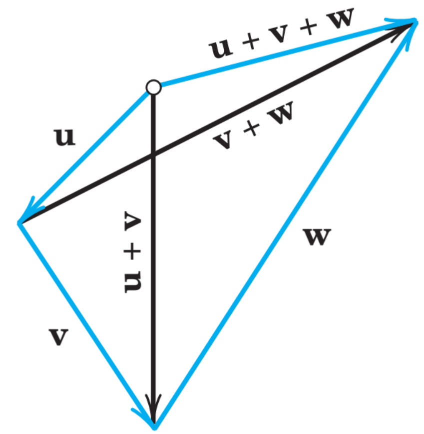


Fig. 174. Associativity of vector addition

Basic Properties of Vector Addition. Familiar laws for real numbers give immediately

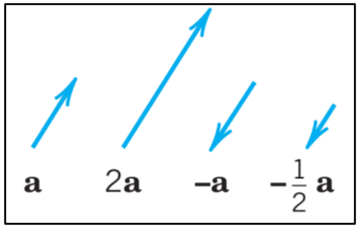
$$
\textbf{(4)}\quad
\begin{aligned}
&\text{(a)} && \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} && (\textit{Commutativity})\\
&\text{(b)} && (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) && (\textit{Associativity})\\
&\text{(c)} && \mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a} &&\\
&\text{(d)} && \mathbf{a} + (-\mathbf{a}) = \mathbf{0}. &&
\end{aligned}
$$

Properties (a) and (b) are verified geometrically in Figs. 173 and 174. Furthermore, $-\mathbf{a}$ denotes the vector having the length $|\mathbf{a}|$ and the direction opposite to that of $\mathbf{a}$.

DEFINITION

Scalar Multiplication (Multiplication by a Number)

The product $c\mathbf{a}$ of any vector $\mathbf{a} = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of $\mathbf{a}$ by c,



(5)
$$c\mathbf{a} = [ca_1, ca_2, ca_3].$$

Geometrically, if $\mathbf{a} \neq \mathbf{0}$, then $c\mathbf{a}$ with $c > 0$ has the direction of $\mathbf{a}$ and with $c < 0$ the direction opposite to $\mathbf{a}$. In any case, the length of $c\mathbf{a}$ is $|c\mathbf{a}| = |c||\mathbf{a}|$, and $c\mathbf{a} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $c = 0$ (or both). (See Fig. 175.)

Basic Properties of Scalar Multiplication. From the definitions we obtain directly

(6)

(a) $$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$$

(b) $$(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$$

(c) $$c(k\mathbf{a}) = (ck)\mathbf{a}$$ (written $ck\mathbf{a}$)

(d) $$1\mathbf{a} = \mathbf{a}.$$

EXAMPLE 2 Vector Addition. Multiplication by Scalars

With respect to a given coordinate system, let

$$\mathbf{a} = [4, 0, 1] \qquad \text{and} \qquad \mathbf{b} = [2, -5, \tfrac{1}{3}].$$

Then $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \tfrac{4}{3}]$, and

$$2(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{3}] = [4, 10, \tfrac{4}{3}] = 2\mathbf{a} - 2\mathbf{b}.$$

■

Unit Vectors i, j, k. Besides $\mathbf{a} = [a_1, a_2, a_3]$ another popular way of writing vectors is

(8) $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

In this representation, **i**, **j**, **k** are the unit vectors in the positive directions of the axes of a Cartesian coordinate system (Fig. 177). Hence, in components,

(9) $$\mathbf{i} = [1, 0, 0], \qquad \mathbf{j} = [0, 1, 0], \qquad \mathbf{k} = [0, 0, 1]$$

and the right side of (8) is a sum of three vectors parallel to the three axes.

All the vectors $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ (with real numbers as components) form the **real vector space** R^3 with the two *algebraic operations* of vector addition and scalar multiplication as just defined. R^3 has **dimension** 3. The triple of vectors **i**, **j**, **k** is called a **standard basis** of R^3. Given a Cartesian coordinate system, the representation (8) of a given vector is unique.

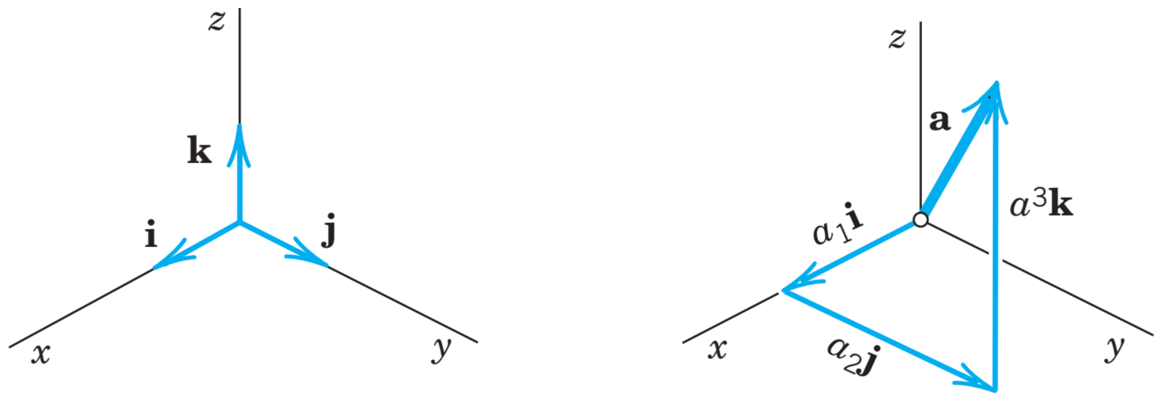


Fig. 177. The unit vectors **i**, **j**, **k** and the representation (8)

9.2. Inner Product (Dot Product)

DEFINITION

Inner Product (Dot Product) of Vectors

The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ (read "$\mathbf{a}$ dot $\mathbf{b}$") of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the product of their lengths times the cosine of their angle (see Fig. 178),

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma \qquad \text{if} \quad \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$$
(1)
$$\mathbf{a} \cdot \mathbf{b} = 0 \qquad \text{if} \quad \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}.$$

The angle γ, $0 \leqq \gamma \leqq \pi$, between $\mathbf{a}$ and $\mathbf{b}$ is measured when the initial points of the vectors coincide, as in Fig. 178. In components, $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and

(2)
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The second line in (1) is needed because γ is undefined when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. The derivation of (2) from (1) is shown below.

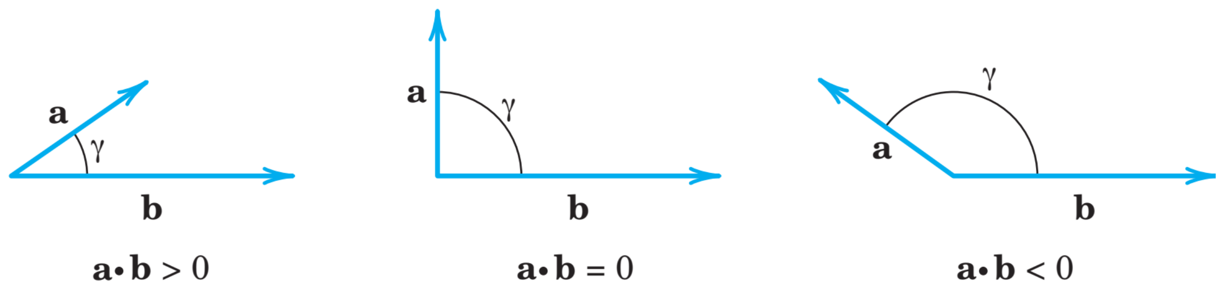


Fig. 178. Angle between vectors and value of inner product

A vector **a** is called **orthogonal** to a vector **b** if $\mathbf{a} \cdot \mathbf{b} = 0$. Then **b** is also orthogonal to **a**, and we call **a** and **b** **orthogonal vectors**. Clearly, this happens for nonzero vectors if and only if $\cos \gamma = 0$; thus $\gamma = \pi/2$ (90°). This proves the important

THEOREM 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

Length and Angle. Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. Hence

(3) $$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

(4) $$\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}\sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$

EXAMPLE 1 Inner Product. Angle Between Vectors

Find the inner product and the lengths of $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$ as well as the angle between these vectors.

Solution. $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$, $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{5}$, $|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} = \sqrt{14}$, and (4) gives the angle

$$\gamma = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \arccos(-0.11952) = 1.69061 = 96.865°.$$

From the definition we see that the inner product has the following properties. For any vectors **a**, **b**, **c** and scalars q_1, q_2,

(5)

(a) $$(q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_1\mathbf{b} \cdot \mathbf{c}$$ (*Linearity*)

(b) $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$ (*Symmetry*)

(c) $$\left.\begin{aligned} &\mathbf{a} \cdot \mathbf{a} \geqq 0 \\ &\mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{aligned}\right\}$$ (*Positive-definiteness*).

Hence *dot multiplication is commutative* as shown by (5b). Furthermore, it is *distributive with respect to vector addition.* This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

(5a*) $$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$$ (*Distributivity*).

Furthermore, from (1) and $|\cos \gamma| \leqq 1$ we see that

(6) $$|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}|$$ (*Cauchy–Schwarz inequality*).

Using this and (3), you may prove (see Prob. 16)

$$|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}| \qquad \textbf{(Triangle inequality)}. \tag{7}$$

Geometrically, (7) with $<$ says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \qquad (\textit{Parallelogram equality}). \tag{8}$$

EXAMPLE 2 Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force $\mathbf{p}$ acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement $\mathbf{d}$. Then the work done by $\mathbf{p}$ in the displacement is defined as

$$W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \bullet \mathbf{d}, \tag{9}$$

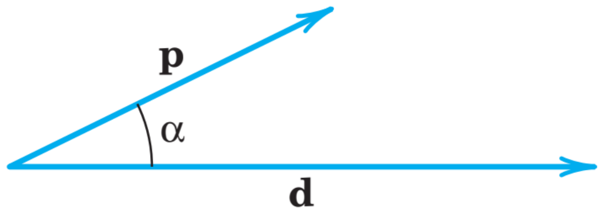


Fig. 179. Work done by a force

that is, magnitude $|\mathbf{p}|$ of the force times length $|\mathbf{d}|$ of the displacement times the cosine of the angle α between $\mathbf{p}$ and $\mathbf{d}$ (Fig. 179). If $\alpha < 90°$, as in Fig. 179, then $W > 0$. If **p and d are orthogonal**, then the work is zero (why?). If $\alpha > 90°$, then $W < 0$, which means that in the displacement one has to do work against the force. For example, think of swimming across a river at some angle α against the current.

EXAMPLE 3 Component of a Force in a Given Direction

What force in the rope in Fig. 180 will hold a car of 5000 lb in equilibrium if the ramp makes an angle of 25° with the horizontal?

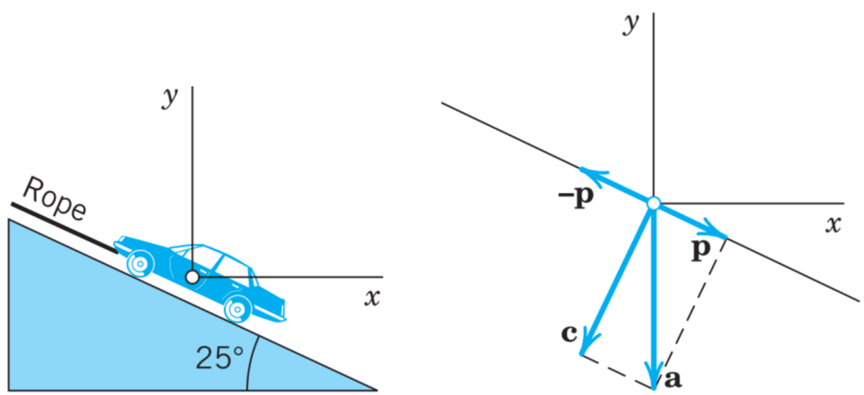


Fig. 180. Example 3

Solution. Introducing coordinates as shown, the weight is $\mathbf{a} = [0, -5000]$ because this force points downward, in the negative y-direction. We have to represent $\mathbf{a}$ as a sum (resultant) of two forces, $\mathbf{a} = \mathbf{c} + \mathbf{p}$, where $\mathbf{c}$ is the force the car exerts on the ramp, which is of no interest to us, and $\mathbf{p}$ is parallel to the rope. A vector in the direction of the rope is (see Fig. 180)

$$\mathbf{b} = [-1, \tan 25°] = [-1, 0.46631], \qquad \text{thus} \qquad |\mathbf{b}| = 1.10338,$$

The direction of the unit vector $\mathbf{u}$ is opposite to the direction of the rope so that

$$\mathbf{u} = -\frac{1}{|\mathbf{b}|}\mathbf{b} = [0.90631, -0.42262].$$

Since $|\mathbf{u}| = 1$ and $\cos \gamma > 0$, we see that we can write our result as

$$|\mathbf{p}| = (|\mathbf{a}| \cos \gamma)|\mathbf{u}| = \mathbf{a} \cdot \mathbf{u} = -\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{5000 \cdot 0.46631}{1.10338} = 2113 \text{ [1b]}.$$

We can also note that $\gamma = 90° - 25° = 65°$ is the angle between $\mathbf{a}$ and $\mathbf{p}$ so that

$$|\mathbf{p}| = |\mathbf{a}| \cos \gamma = 5000 \cos 65° = 2113 \text{ [1b]}.$$

Answer: About 2100 lb.

Example 3 is typical of applications that deal with the **component** or **projection** *of a vector* **a** *in the direction of a vector* **b** $(\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of **a** on a straight line l parallel to **b** as shown in Fig. 181, then

(10) $$p = |\mathbf{a}| \cos \gamma.$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of **b** and with the minus sign if $p\mathbf{b}$ has the direction opposite to **b**.

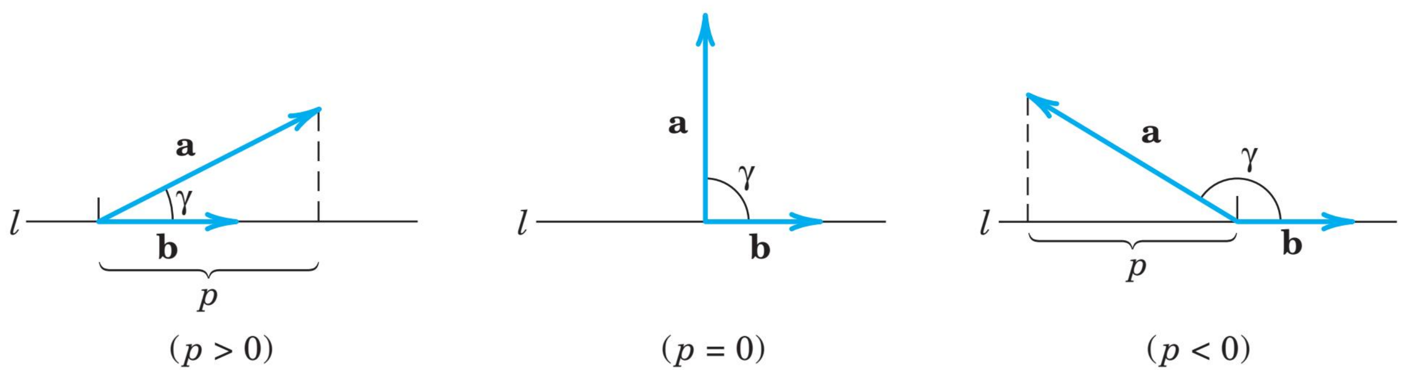


Fig. 181. Component of a vector **a** in the direction of a vector **b**

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

$$p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \qquad (\mathbf{b} \neq \mathbf{0}). \tag{11}$$

If **b** is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$p = \mathbf{a} \cdot \mathbf{b} \qquad (|\mathbf{b}| = 1). \tag{12}$$

Figure 182 shows the projection p of **a** in the direction of **b** (as in Fig. 181) and the projection $q = |\mathbf{b}| \cos \gamma$ of **b** in the direction of **a**.

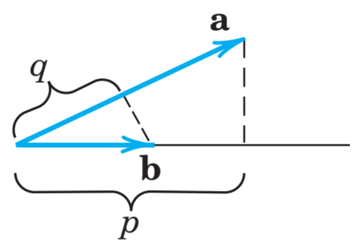


Fig. 182. Projections p of **a** on **b** and q of **b** on **a**

9.3. Vector Product (Cross Product)

DEFINITION

Vector Product (Cross Product, Outer Product) of Vectors

The **vector product** or **cross product** $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**") of two vectors **a** and **b** is the vector **v** denoted by

$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

I. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then we define $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

II. If both vectors are nonzero vectors, then vector **v** has the length

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma, \tag{1}$$

where γ is the angle between **a** and **b** as in Sec. 9.2.

Furthermore, by design, **a** and **b** form the sides of a parallelogram on a plane in space. The parallelogram is shaded in blue in Fig. 185. The area of this blue parallelogram is precisely given by Eq. (1), so that the length $|\mathbf{v}|$ of the vector **v** is equal to the area of that parallelogram.

III. If **a** and **b** lie in the same straight line, i.e., **a** and **b** have the same or opposite directions, then γ is 0° or 180° so that $\sin \gamma = 0$. In that case $|\mathbf{v}| = 0$ so that $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

IV. If cases I and III do not occur, then **v** is a nonzero vector. The direction of $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b** such that **a**, **b**, **v**—precisely in this order (!)—form a right-handed triple as shown in Figs. 185–187 and explained below.

Another term for vector product is outer product.

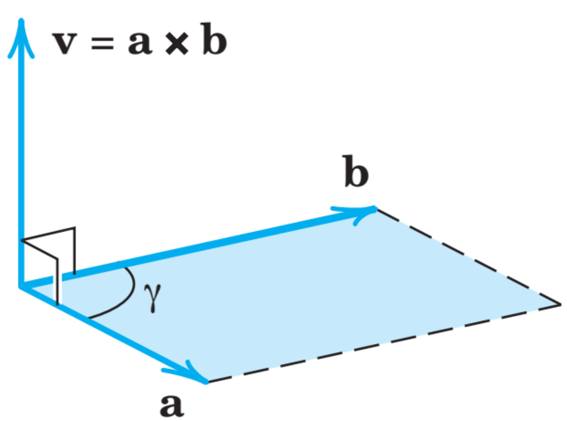


Fig. 185. Vector product

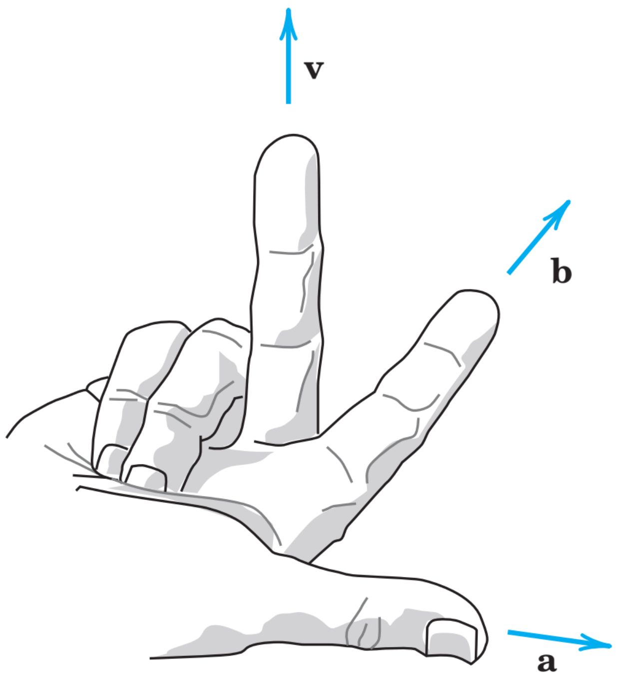
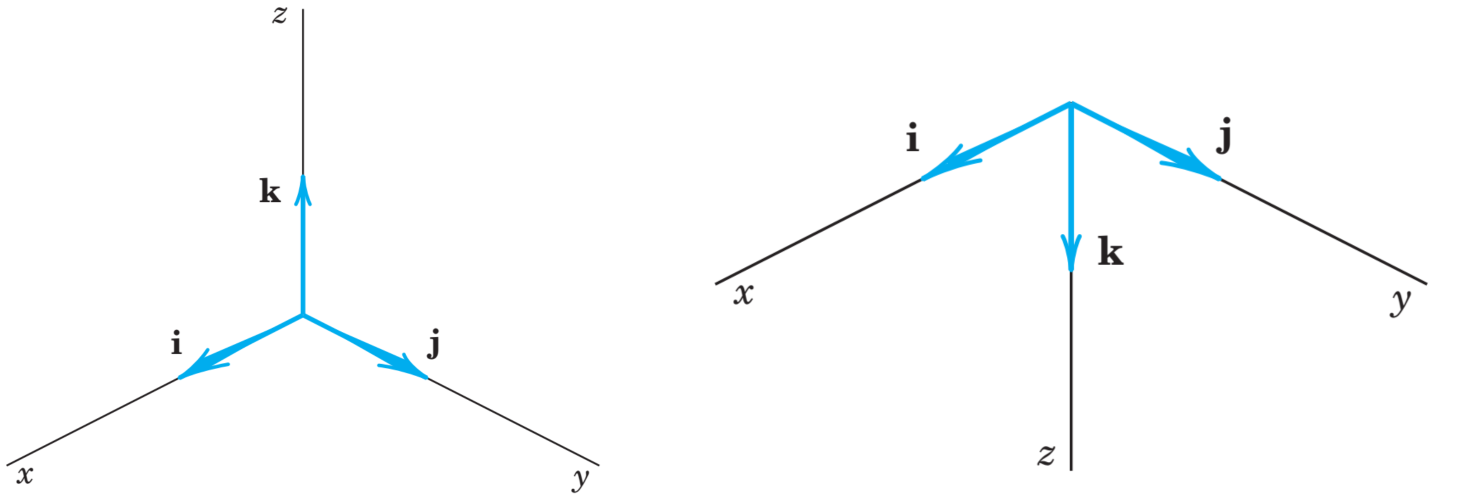


Fig. 186. Right-handed triple of vectors **a**, **b**, **v**

Just as we did with the dot product, we would also like to express the cross product in components. Let $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$. Then $\mathbf{v} = [v_1, v_2, v_3] = \mathbf{a} \times \mathbf{b}$ has the components

$$v_1 = a_2 b_3 - a_3 b_2, \qquad v_2 = a_3 b_1 - a_1 b_3, \qquad v_3 = a_1 b_2 - a_2 b_1. \tag{2}$$

Right-Handed Cartesian Coordinate System. The system is called **right-handed** if the corresponding unit vectors **i**, **j**, **k** in the positive directions of the axes (see Sec. 9.1) form a right-handed triple as in Fig. 188a. The system is called **left-handed** if the sense of **k** is reversed, as in Fig. 188b. In applications, we prefer right-handed systems.



(a) Right-handed (b) Left-handed

Fig. 188. The two types of Cartesian coordinate systems

How to Memorize (2). If you know second- and third-order determinants, you see that (2) can be written

$$(2^*) \qquad v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad v_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = +\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

and $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is the expansion of the following symbolic determinant by its first row. (We call the determinant "symbolic" because the first row consists of vectors rather than of numbers.)

$$(2^{**}) \qquad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

For a left-handed system the determinant has a minus sign in front.

EXAMPLE 1 Vector Product

For the vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ of $\mathbf{a} = [1, 1, 0]$ and $\mathbf{b} = [3, 0, 0]$ in right-handed coordinates we obtain from (2)

$$v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 1 \cdot 0 - 1 \cdot 3 = -3.$$

We confirm this by (2**):

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \mathbf{k} = -3\mathbf{k} = [0, 0, -3].$$

EXAMPLE 2 Vector Products of the Standard Basis Vectors

$$\text{(3)} \qquad \begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

THEOREM 1

General Properties of Vector Products

(a) *For every scalar l,*

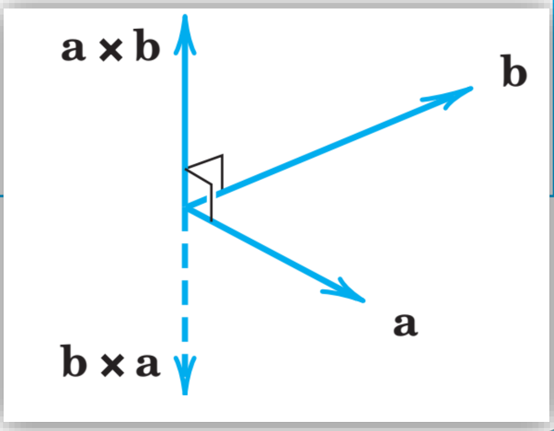
(4) $$(l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

(5) $$(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$
$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

(c) *Cross multiplication is* ***not commutative*** *but* ***anticommutative;*** *that is,*

(6) $$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$$



(d) *Cross multiplication is* ***not associative****; that is, in general,*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \tag{7}$$

so that the parentheses cannot be omitted.

EXAMPLE 3 Moment of a Force

In mechanics the moment m of a force $\mathbf{p}$ about a point Q is defined as the product $m = |\mathbf{p}|d$, where d is the (perpendicular) distance between Q and the line of action L of $\mathbf{p}$ (Fig. 190). If $\mathbf{r}$ is the vector from Q to any point A on L, then $d = |\mathbf{r}| \sin \gamma$, as shown in Fig. 190, and

$$m = |\mathbf{r}||\mathbf{p}| \sin \gamma.$$

Since γ is the angle between $\mathbf{r}$ and $\mathbf{p}$, we see from (1) that $m = |\mathbf{r} \times \mathbf{p}|$. The vector

$$\mathbf{m} = \mathbf{r} \times \mathbf{p}$$

is called the **moment vector** or **vector moment** of $\mathbf{p}$ about Q. Its magnitude is m.

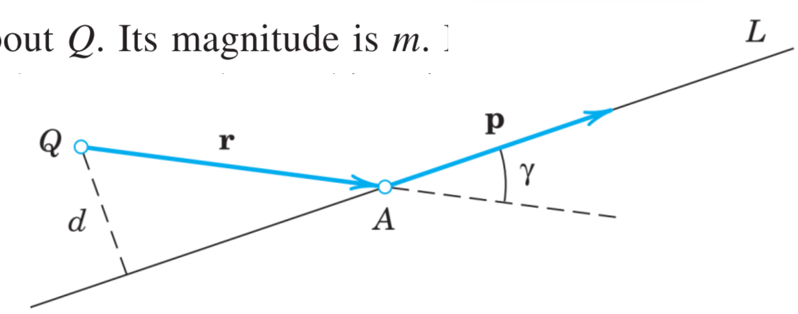


Fig. 190. Moment of a force $\mathbf{p}$

EXAMPLE 4 Moment of a Force

Find the moment of the force **p** about the center Q of a wheel, as given in Fig. 191.

Solution. Introducing coordinates as shown in Fig. 191, we have

$$\mathbf{p} = [1000 \cos 30°, \quad 1000 \sin 30°, \quad 0] = [866, \quad 500, \quad 0], \qquad \mathbf{r} = [0, \quad 1.5, \quad 0].$$

(Note that the center of the wheel is at $y = -1.5$ on the y-axis.) Hence (8) and (2**) give

$$\mathbf{m} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 0 \\ 866 & 500 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \begin{vmatrix} 0 & 1.5 \\ 866 & 500 \end{vmatrix} \mathbf{k} = [0, 0, -1299].$$

This moment vector **m** is normal, i.e., perpendicular to the plane of the wheel. Hence it has the direction of the axis of rotation about the center Q of the wheel that the force **p** has the tendency to produce. The moment **m** points in the negative z-direction, This is, the direction in which a right-handed screw would advance if turned in that way. ■

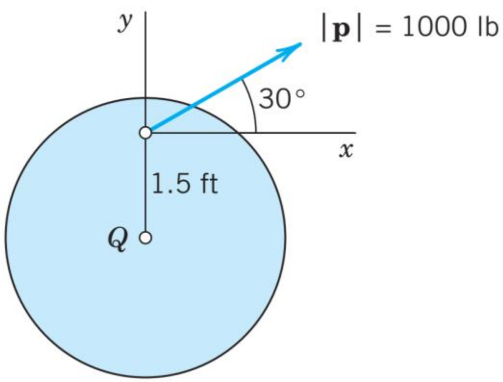


Fig. 191. Moment of a force **p**

Scalar Triple Product

Certain products of vectors, having three or more factors, occur in applications. The most important of these products is the scalar triple product or mixed product of three vectors **a**, **b**, **c**.

(10*) $$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}).$$

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \bullet \mathbf{v} = a_1v_1 + a_2v_2 + a_3v_3$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

(10) $$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 2

Properties and Applications of Scalar Triple Products

(a) *In* (10) *the dot and cross can be interchanged:*

$$(\mathbf{a}\quad\mathbf{b}\quad\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}. \tag{11}$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a}\quad\mathbf{b}\quad\mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

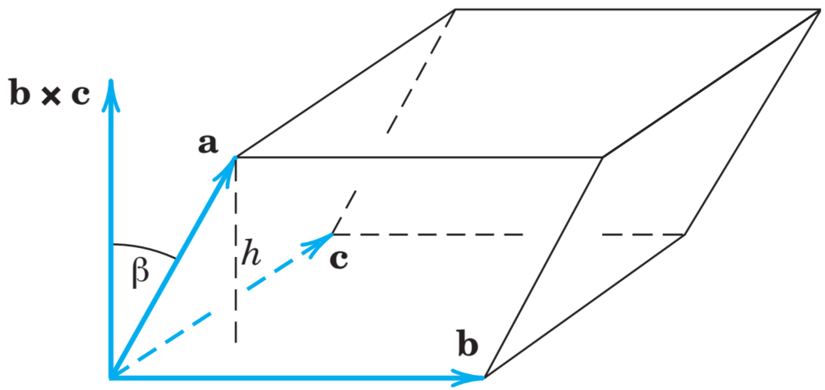


Fig. 193. Geometric interpretation of a scalar triple product

9.4. Vector and Scalar Functions and Their Fields. Vector Calculus: Derivatives

Our discussion of vector calculus begins with identifying the two types of functions on which it operates. Let P be any point in a domain of definition. Typical domains in applications are three-dimensional, or a surface or a curve in space. Then we define a **vector function** $\mathbf{v}$, whose values are vectors, that is,

$$\mathbf{v} = \mathbf{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

that depends on points P in space. We say that a vector function defines a **vector field** in a domain of definition. Typical domains were just mentioned. Examples of vector fields are the field of tangent vectors of a curve (shown in Fig. 195), normal vectors of a surface (Fig. 196), and velocity field of a rotating body (Fig. 197). Note that vector functions may also depend on time t or on some other parameters.

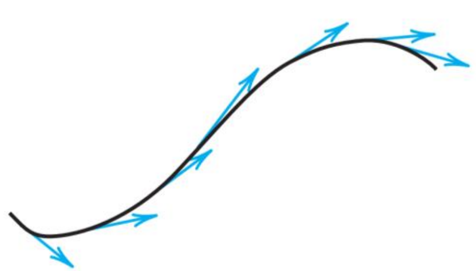


Fig. 195. Field of tangent vectors of a curve

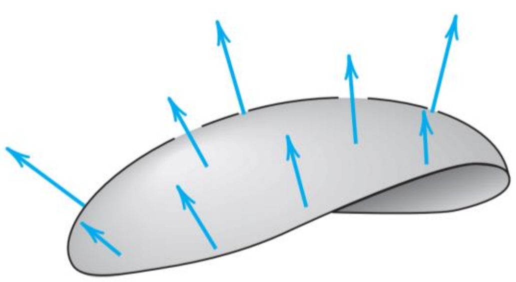


Fig. 196. Field of normal vectors of a surface

Similarly, we define a **scalar function** f, whose values are scalars, that is,

$$f = f(P)$$

that depends on P. We say that a scalar function defines a scalar field in that three-dimensional domain or surface or curve in space. Two representative examples of scalar fields are the temperature field of a body and the pressure field of the air in Earth's atmosphere. Note that scalar functions may also depend on some parameter such as time t.

Notation. If we introduce Cartesian coordinates x, y, z, then, instead of writing $\mathbf{v}(P)$ for the vector function, we can write

$$\mathbf{v}(x, y, z) = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)].$$

We have to keep in mind that the components depend on our choice of coordinate system, whereas a vector field that has a physical or geometric meaning should have magnitude and direction depending only on P, not on the choice of coordinate system.

Similarly, for a scalar function, we write

$$f(P) = f(x, y, z).$$

EXAMPLE 2 Vector Field (Velocity Field)

At any instant the velocity vectors $\mathbf{v}(P)$ of a rotating body B constitute a vector field, called the **velocity field** of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then (see Example 5 in Sec. 9.3)

$$\mathbf{v}(x, y, z) = \mathbf{w} \times \mathbf{r} = \mathbf{w} \times [x, y, z] = \mathbf{w} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{1}$$

where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinates are such that the z-axis is the axis of rotation and $\mathbf{w}$ points in the positive z-direction, then $\mathbf{w} = \omega\mathbf{k}$ and

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega[-y, x, 0] = \omega(-y\mathbf{i} + x\mathbf{j}).$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 197. ■

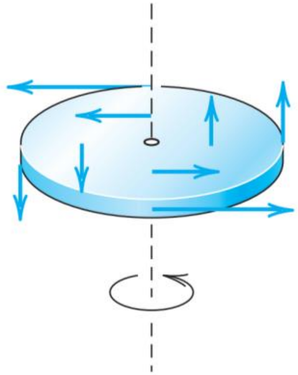


Fig. 197. Velocity field of a rotating body

Vector Calculus

Convergence. An infinite sequence of vectors $\mathbf{a}_{(n)}$, $n = 1, 2, \cdots,$ is said to **converge** if there is a vector $\mathbf{a}$ such that

$$\lim_{n\to\infty} |\mathbf{a}_{(n)} - \mathbf{a}| = 0. \tag{4}$$

$\mathbf{a}$ is called the **limit vector** of that sequence, and we write

$$\lim_{n\to\infty} \mathbf{a}_{(n)} = \mathbf{a}. \tag{5}$$

Similarly, a vector function $\mathbf{v}(t)$ of a real variable t is said to have the **limit** $\boldsymbol{l}$ as t approaches t_0, if $\mathbf{v}(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t\to t_0} |\mathbf{v}(t) - \boldsymbol{l}| = 0. \tag{6}$$

Then we write

$$\lim_{t\to t_0} \mathbf{v}(t) = \boldsymbol{l}. \tag{7}$$

Here, a *neighborhood* of t_0 is an interval (segment) on the t-axis containing t_0 as an interior point (not as an endpoint).

Continuity. A vector function $\mathbf{v}(t)$ is said to be **continuous** at $t = t_0$ if it is defined in some neighborhood of t_0 (including at t_0 itself!) and

$$\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0). \tag{8}$$

If we introduce a Cartesian coordinate system, we may write

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

Then $\mathbf{v}(t)$ is continuous at t_0 if and only if its three components are continuous at t_0.

DEFINITION

Derivative of a Vector Function

A vector function $\mathbf{v}(t)$ is said to be **differentiable** at a point t if the following limit exists:

$$\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}. \tag{9}$$

This vector $\mathbf{v}'(t)$ is called the **derivative** of $\mathbf{v}(t)$. See Fig. 199.

In components with respect to a given Cartesian coordinate system,

$$\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)]. \tag{10}$$

Hence the derivative $\mathbf{v}'(t)$ ***is obtained by differentiating each component separately.*** For instance, if $\mathbf{v} = [t, t^2, 0]$, then $\mathbf{v}' = [1, 2t, 0]$.

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

and in particular

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \tag{11}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \tag{12}$$

$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}'). \tag{13}$$

EXAMPLE 4 Derivative of a Vector Function of Constant Length

Let $\mathbf{v}(t)$ be a vector function whose length is constant, say, $|\mathbf{v}(t)| = c$. Then $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = c^2$, and $(\mathbf{v} \cdot \mathbf{v})' = 2\mathbf{v} \cdot \mathbf{v}' = 0$, by differentiation [see (11)]. This yields the following result. *The derivative of a vector function* $\mathbf{v}(t)$ *of constant length is either the zero vector or is perpendicular to* $\mathbf{v}(t)$. ■

?

Partial Derivatives of a Vector Function

Our present discussion shows that partial differentiation of vector functions of two or more variables can be introduced as follows. Suppose that the components of a vector function

$$\mathbf{v} = [v_1, \quad v_2, \quad v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of n variables $t_1, \cdots, t_n$. Then the **partial derivative** of $\mathbf{v}$ with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

Similarly, second partial derivatives are

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k},$$

EXAMPLE 5 Partial Derivatives

Let $\mathbf{r}(t_1, t_2) = a\cos t_1\,\mathbf{i} + a\sin t_1\,\mathbf{j} + t_2\,\mathbf{k}$.

$$\frac{\partial \mathbf{r}}{\partial t_1} = -a\sin t_1\,\mathbf{i} + a\cos t_1\,\mathbf{j} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}.$$

$$\frac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}.$$

9.5. Curves. Arc Length. Curvature. Torsion

Bodies that move in space form paths that may be represented by curves C. This and other applications show the need for **parametric representations** of C with **parameter** t, which may denote time or something else (see Fig. 200). A typical parametric representation is given by

$$\mathbf{r}(t) = [x(t), \quad y(t), \quad z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{1}$$

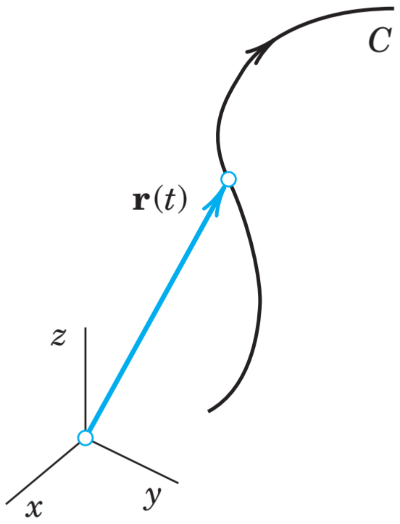


Fig. 200. Parametric representation of a curve

EXAMPLE 1 Circle. Parametric Representation. Positive Sense

The circle $x^2 + y^2 = 4$, $z = 0$ in the xy-plane with center 0 and radius 2 can be represented parametrically by

$$\mathbf{r}(t) = [2\cos t, 2\sin t, 0] \qquad \text{or simply by} \qquad \mathbf{r}(t) = [2\cos t, 2\sin t] \qquad \text{(Fig. 201)}$$

where $0 \leqq t \leqq 2\pi$. Indeed, $x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$, For $t = 0$ we have $\mathbf{r}(0) = [2, 0]$, for $t = \frac{1}{2}\pi$ we get $\mathbf{r}(\frac{1}{2}\pi) = [0, 2]$, and so on. The positive sense induced by this representation is the counterclockwise sense.

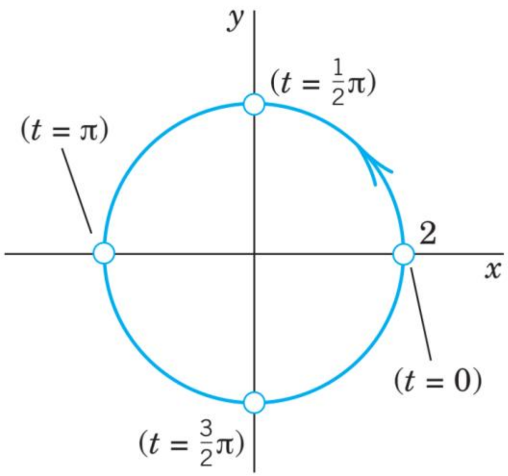


Fig. 201. Circle in Example 1

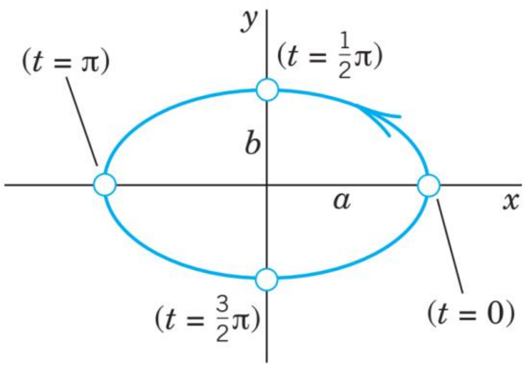


Fig. 202. Ellipse in Example 2

EXAMPLE 2 Ellipse

The vector function

$$\mathbf{r}(t) = [a \cos t, \quad b \sin t, \quad 0] = a \cos t\, \mathbf{i} + b \sin t\, \mathbf{j} \qquad \text{(Fig. 202)} \tag{3}$$

represents an ellipse in the *xy*-plane with center at the origin and principal axes in the direction of the *x*- and *y*-axes. In fact, since $\cos^2 t + \sin^2 t = 1$, we obtain from (3)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = 0.$$

If $b = a$, then (3) represents a *circle* of radius a. ■

EXAMPLE 4 Circular Helix

The twisted curve C represented by the vector function

$$\mathbf{r}(t) = [a \cos t, \quad a \sin t, \quad ct] = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + ct\, \mathbf{k} \qquad (c \neq 0) \tag{5}$$

is called a *circular helix*. It lies on the cylinder $x^2 + y^2 = a^2$. If $c > 0$, the helix is shaped like a right-handed screw (Fig. 204). If $c < 0$, it looks like a left-handed screw (Fig. 205). If $c = 0$, then (5) is a circle. ■

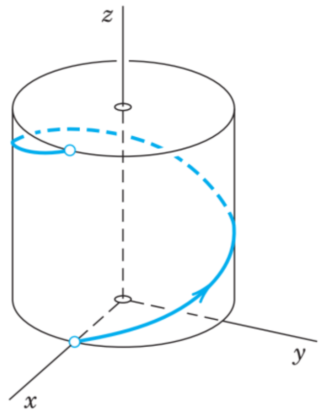


Fig. 204. Right-handed circular helix

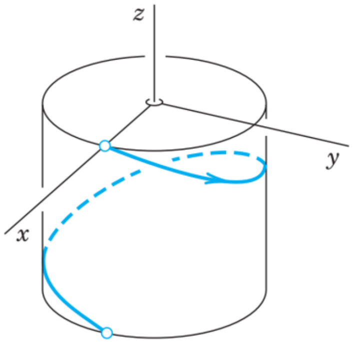


Fig. 205. Left-handed circular helix

A **simple curve** is a curve without **multiple points**, that is, without points at which the curve intersects or touches itself. Circle and helix are simple curves. Figure 206 shows curves that are not simple. An example is [sin $2t$, cos t, 0]. Can you sketch it?

An **arc** of a curve is the portion between any two points of the curve. For simplicity, we say "curve" for curves as well as for arcs.

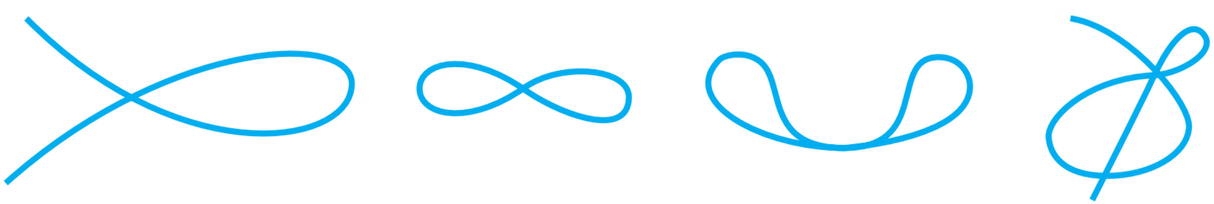


Fig. 206. Curves with multiple points

Tangent to a Curve

Tangents are straight lines touching a curve.

Let us formalize this concept. If C is given by $\mathbf{r}(t)$, and P and Q correspond to t and $t + \Delta t$, then a vector in the direction of L is

(6) $$\frac{1}{\Delta t}[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)].$$

In the limit this vector becomes the derivative

(7) $$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)],$$

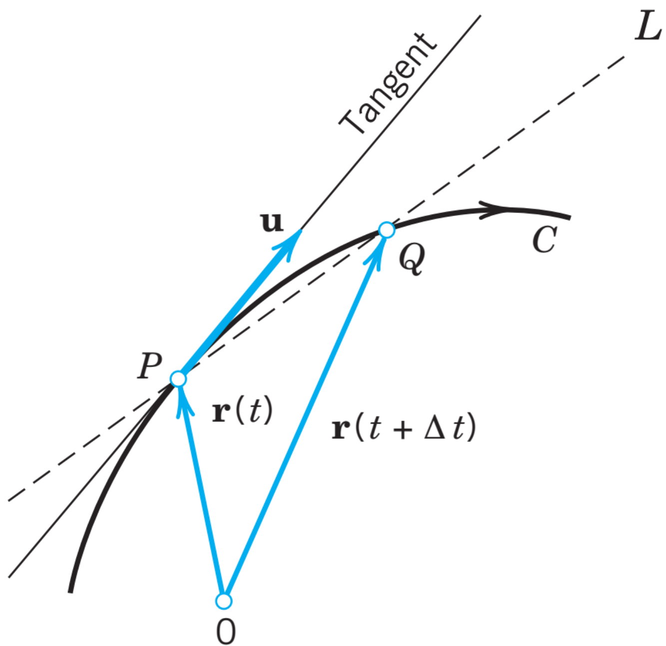


Fig. 207. Tangent to a curve

provided $\mathbf{r}(t)$ is differentiable, as we shall assume from now on. If $\mathbf{r}'(t) \neq \mathbf{0}$, we call $\mathbf{r}'(t)$ a **tangent vector** of C at P because it has the direction of the tangent. The corresponding unit vector is the **unit tangent vector** (see Fig. 207)

$$\mathbf{u} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'. \tag{8}$$

EXAMPLE 5 Tangent to an Ellipse

Find the tangent to the ellipse $\frac{1}{4}x^2 + y^2 = 1$ at P: $(\sqrt{2}, 1/\sqrt{2})$.

Solution. Equation (3) with semi-axes $a = 2$ and $b = 1$ gives $\mathbf{r}(t) = [2\cos t, \sin t]$. The derivative is $\mathbf{r}'(t) = [-2\sin t, \cos t]$. Now P corresponds to $t = \pi/4$ because

$$\mathbf{r}(\pi/4) = [2\cos(\pi/4), \sin(\pi/4)] = [\sqrt{2}, \quad 1/\sqrt{2}].$$

Hence $\mathbf{r}'(\pi/4) = [-\sqrt{2}, \quad 1/\sqrt{2}]$. From (9) we thus get the *answer*

$$\mathbf{q}(w) = [\sqrt{2}, \quad 1/\sqrt{2}] + w[-\sqrt{2}, \quad 1/\sqrt{2}] = [\sqrt{2}(1 - w), \quad (1/\sqrt{2})(1 + w)].$$

To check the result, sketch or graph the ellipse and the tangent. ■

Length of a Curve

We are now ready to define the length l of a curve. l will be the limit of the lengths of broken lines of n chords (see Fig. 209, where $n = 5$) with larger and larger n. For this, let $\mathbf{r}(t)$, $a \leqq t \leqq b$, represent C. For each $n = 1, 2, \cdots$, we subdivide ("partition") the interval $a \leqq t \leqq b$ by points

$$t_0 (= a), \quad t_1, \cdots, t_{n-1}, \quad t_n (= b), \qquad \text{where} \qquad t_0 < t_1 < \cdots < t_n.$$

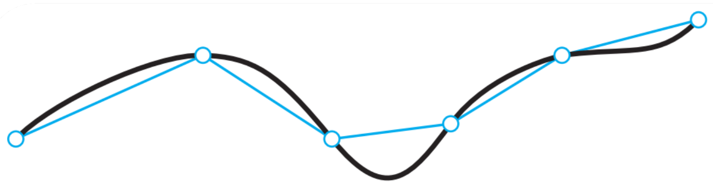


Fig. 209. Length of a curve

(10) $$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, dt \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{dt}\right)$$

Curves in Mechanics. Velocity. Acceleration

Curves play a basic role in mechanics, where they may serve as paths of moving bodies. Then such a curve C should be represented by a parametric representation $\mathbf{r}(t)$ with **time** t as parameter. The tangent vector (7) of C is then called the **velocity vector v** because, being tangent, it points in the instantaneous direction of motion and its length gives the **speed** $|\mathbf{v}| = |\mathbf{r}'| = \sqrt{\mathbf{r}' \bullet \mathbf{r}'} = ds/dt$; see (12). The second derivative of $\mathbf{r}(t)$ is called the **acceleration vector** and is denoted by **a**. Its length $|\mathbf{a}|$ is called the **acceleration** of the motion. Thus

$$\text{(16)} \qquad \mathbf{v}(t) = \mathbf{r}'(t), \qquad \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Tangential and Normal Acceleration. Whereas the velocity vector is always tangent to the path of motion, the acceleration vector will generally have another direction. We can split the acceleration vector into two directional components, that is,

$$\text{(17)} \qquad \mathbf{a} = \mathbf{a}_{\tan} + \mathbf{a}_{\text{norm}},$$

where the **tangential acceleration vector** $\mathbf{a}_{\tan}$ is tangent to the path (or, sometimes, **0**) and the **normal acceleration vector** $\mathbf{a}_{\text{norm}}$ is normal (perpendicular) to the path (or, sometimes, **0**).

Expressions for the vectors in (17) are obtained from (16) by the chain rule. We first have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt}$$

where $\mathbf{u}(s)$ is the unit tangent vector (14). Another differentiation gives

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2s}{dt^2}. \tag{18}$$

Since the tangent vector $\mathbf{u}(s)$ has constant length (length one), its derivative $d\mathbf{u}/ds$ is perpendicular to $\mathbf{u}(s)$, from the result in Example 4 in Sec. 9.4. Hence the first term on the right of (18) is the normal acceleration vector, and the second term on the right is the tangential acceleration vector, so that (18) is of the form (17).

Now the length $|\mathbf{a}_{\tan}|$ is the absolute value of the projection of $\mathbf{a}$ in the direction of $\mathbf{v}$, given by (11) in Sec. 9.2 with $\mathbf{b} = \mathbf{v}$; that is, $|\mathbf{a}_{\tan}| = |\mathbf{a} \cdot \mathbf{v}|/|\mathbf{v}|$. Hence $\mathbf{a}_{\tan}$ is this expression times the unit vector $(1/|\mathbf{v}|)\mathbf{v}$ in the direction of $\mathbf{v}$, that is,

(18*) $$\mathbf{a}_{\tan} = \frac{\mathbf{a} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$ Also, $$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\tan}.$$

EXAMPLE 7 Centripetal Acceleration. Centrifugal Force

The vector function

$$\mathbf{r}(t) = [R\cos\omega t,\quad R\sin\omega t] = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \qquad \text{(Fig. 210)}$$

(with fixed **i** and **j**) represents a circle C of radius R with center at the origin of the xy-plane and describes the motion of a small body B counterclockwise around the circle. Differentiation gives the velocity vector

$$\mathbf{v} = \mathbf{r}' = [-R\omega\sin\omega t,\quad R\omega\cos\omega t] = -R\omega\sin\omega t\,\mathbf{i} + R\omega\cos\omega t\,\mathbf{j} \qquad \text{(Fig. 210)}$$

v is tangent to C. Its magnitude, the speed, is

$$|\mathbf{v}| = |\mathbf{r}'| = \sqrt{\mathbf{r}'\cdot\mathbf{r}'} = R\omega.$$

Hence it is constant. The speed divided by the distance R from the center is called the **angular speed**. It equals ω, so that it is constant, too. Differentiating the velocity vector, we obtain the acceleration vector

$$\mathbf{a} = \mathbf{v}' = [-R\omega^2\cos\omega t,\quad -R\omega^2\sin\omega t] = -R\omega^2\cos\omega t\,\mathbf{i} - R\omega^2\sin\omega t\,\mathbf{j}. \tag{19}$$

This shows that $\mathbf{a} = -\omega^2\mathbf{r}$ (Fig. 210), so that there is an acceleration toward the center, called the **centripetal acceleration** of the motion. It occurs because the velocity vector is changing direction at a constant rate. Its magnitude is constant, $|\mathbf{a}| = \omega^2|\mathbf{r}| = \omega^2 R$. Multiplying **a** by the mass m of B, we get the **centripetal force** $m\mathbf{a}$. The opposite vector $-m\mathbf{a}$ is called the **centrifugal force**. At each instant these two forces are in equilibrium.

9.6. Calculus Review: Functions of Several Variables

Chain Rules

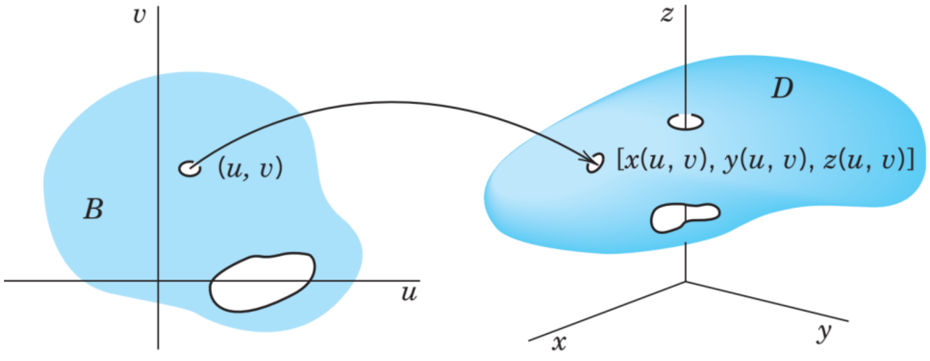


Fig. 213. Notations in Theorem 1

Chain Rule

Let $w = f(x, y, z)$ *be continuous and have continuous first partial derivatives in a domain* D *in* xyz*-space. Let* $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ *be functions that are continuous and have first partial derivatives in a domain* B *in the* uv*-plane, where* B *is such that for every point* (u, v) *in* B, *the corresponding point* $[x(u, v), y(u, v), z(u, v)]$ *lies in* D. See Fig. 213. *Then the function*

$$w = f(x(u, v), y(u, v), z(u, v))$$

is defined in B, has first partial derivatives with respect to u and v in B, and

(1)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

In calculus, x, y, z are often called the **intermediate variables**, in contrast with the **independent variables** u, v and the **dependent variable** w.

Special Cases of Practical Interest

If $w = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$ as before, then (1) becomes

(2)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}.$$

If $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$, then (1) gives

(3)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

If $w = f(x, y)$ and $x = x(t)$, $y = y(t)$, then (3) reduces to

(4)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Finally, the simplest case $w = f(x)$, $x = x(t)$ gives

(5)

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

EXAMPLE 1 Chain Rule

If $w = x^2 - y^2$ and we define polar coordinates r, θ by $x = r\cos\theta$, $y = r\sin\theta$, then (2) gives

$$\frac{\partial w}{\partial r} = 2x\cos\theta - 2y\sin\theta = 2r\cos^2\theta - 2r\sin^2\theta = 2r\cos 2\theta$$

$$\frac{\partial w}{\partial \theta} = 2x(-r\sin\theta) - 2y(r\cos\theta) = -2r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta = -2r^2\sin 2\theta.$$

Partial Derivatives on a Surface

Let $w = f(x, y, z)$ and let $z = g(x, y)$ represent a surface S in space. Then on S the function becomes

$$\tilde{w}(x, y) = f(x, y, g(x, y)).$$

Hence, by (1), the partial derivatives are

$$\text{(6)} \qquad \frac{\partial \tilde{w}}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}, \qquad \frac{\partial \tilde{w}}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \qquad [z = g(x, y)].$$

EXAMPLE 2 Partial Derivatives on Surface

Let $w = f = x^3 + y^3 + z^3$ and let $z = g = x^2 + y^2$. Then (6) gives

$$\frac{\partial \tilde{w}}{\partial x} = 3x^2 + 3z^2 \cdot 2x = 3x^2 + 3(x^2 + y^2)^2 \cdot 2x,$$

$$\frac{\partial \tilde{w}}{\partial y} = 3y^2 + 3z^2 \cdot 2y = 3y^2 + 3(x^2 + y^2)^2 \cdot 2y.$$

We confirm this by substitution, using $w(x, y) = x^3 + y^3 + (x^2 + y^2)^3$, that is,

$$\frac{\partial \tilde{w}}{\partial x} = 3x^2 + 3(x^2 + y^2)^2 \cdot 2x, \qquad \frac{\partial \tilde{w}}{\partial y} = 3y^2 + 3(x^2 + y^2)^2 \cdot 2y.$$

Mean Value Theorems

THEOREM 2

Mean Value Theorem

Let $f(x, y, z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let P_0: (x_0, y_0, z_0) and P: $(x_0 + h, y_0 + k, z_0 + l)$ be points in D such that the straight line segment P_0P joining these points lies entirely in D. Then

(7)
$$f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z},$$

the partial derivatives being evaluated at a suitable point of that segment.

For a function (x, y) of two variables
$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y},$$

For a function (x) of a single variable
$$f(x_0 + h) - f(x_0) = h\frac{\partial f}{\partial x},$$

9.7. Gradient of a Scalar Field. Directional Derivative

DEFINITION 1

Gradient

The setting is that we are given a scalar function $f(x, y, z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates x, y, z. We denote the **gradient** of that function by grad f or ∇f (read **nabla** f). Then the qradient of $f(x, y, z)$ is defined as the vector function

(1)
$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Remarks. For a definition of the gradient in curvilinear coordinates, see App. 3.4. As a quick example, if $f(x, y, z) = 2y^3 + 4xz + 3x$, then grad $f = [4z + 3, 6y^2, 4x]$.

Furthermore, we will show later in this section that (1) actually does define a vector.

The notation ∇f is suggested by the *differential operator* ∇ (read *nabla*) defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{1*}$$

Directional Derivative

From calculus we know that the partial derivatives in (1) give the rates of change of $f(x, y, z)$ in the directions of the three coordinate axes. It seems natural to extend this and ask for the rate of change of f *in an arbitrary direction* in space. This leads to the following concept.

DEFINITION 2

Directional Derivative

The directional derivative $D_{\mathbf{b}}f$ or df/ds of a function $f(x, y, z)$ at a point P in the direction of a vector $\mathbf{b}$ is defined by (see Fig. 215)

$$D_{\mathbf{b}}f = \frac{df}{ds} = \lim_{s\to 0}\frac{f(Q) - f(P)}{s}. \tag{2}$$

Here Q is a variable point on the straight line L in the direction of $\mathbf{b}$, and $|s|$ is the distance between P and Q. Also, $s > 0$ if Q lies in the direction of $\mathbf{b}$ (as in Fig. 215), $s < 0$ if Q lies in the direction of $-\mathbf{b}$, and $s = 0$ if $Q = P$.

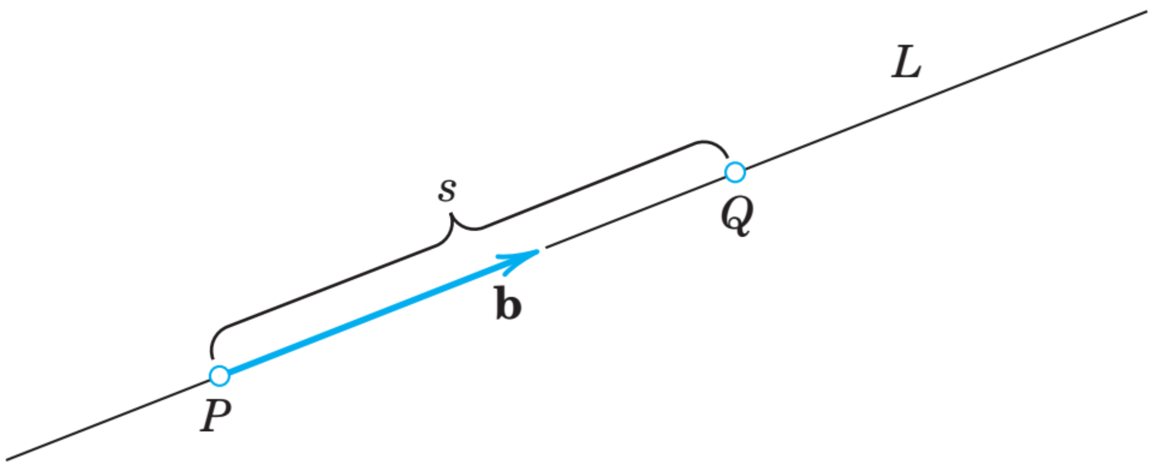


Fig. 215. Directional derivative

The next idea is to use Cartesian xyz-coordinates and for $\mathbf{b}$ a unit vector. Then the line L is given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} = \mathbf{p}_0 + s\mathbf{b} \qquad (|\mathbf{b}| = 1) \tag{3}$$

where $\mathbf{p}_0$ the position vector of P. Equation (2) now shows that $D_{\mathbf{b}}f = df/ds$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to the arc length s of L. Hence, assuming that f has continuous partial derivatives and applying the chain rule [formula (3) in the previous section], we obtain

(4) $$D_{\mathbf{b}}f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'$$

where primes denote derivatives with respect to s (which are taken at $s = 0$). But here, differentiating (3) gives $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = \mathbf{b}$. Hence (4) is simply the inner product of grad f and $\mathbf{b}$ [see (2), Sec. 9.2]; that is,

(5) $$D_{\mathbf{b}}f = \frac{df}{ds} = \mathbf{b} \bullet \operatorname{grad} f \qquad (|\mathbf{b}| = 1).$$

ATTENTION! If the direction is given by a vector $\mathbf{a}$ of any length ($\neq 0$), then

(5*) $$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \bullet \operatorname{grad} f.$$

EXAMPLE 1 Gradient. Directional Derivative

Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at P: (2, 1, 3) in the direction of $\mathbf{a} = [1, 0, -2]$.

Solution. grad $f = [4x, 6y, 2z]$ gives at P the vector grad $f(P) = [8, 6, 6]$. From this and (5*) we obtain, since $|\mathbf{a}| = \sqrt{5}$,

$$D_{\mathbf{a}} f(P) = \frac{1}{\sqrt{5}} [1, 0, -2] \cdot [8, 6, 6] = \frac{1}{\sqrt{5}} (8 + 0 - 12) = -\frac{4}{\sqrt{5}} = -1.789.$$

The minus sign indicates that at P the function f is decreasing in the direction of $\mathbf{a}$.

Gradient Is a Vector. Maximum Increase

THEOREM 1

Use of Gradient: Direction of Maximum Increase

Let $f(P) = f(x, y, z)$ be a scalar function having continuous first partial derivatives in some domain B in space. Then grad f *exists in B and is a vector, that is, its length and direction are independent of the particular choice of Cartesian coordinates. If* grad $f(P) \neq \mathbf{0}$ *at some point P, it has the direction of maximum increase of f at P.*

Gradient as Surface Normal Vector

Gradients have an important application in connection with surfaces, namely, as surface normal vectors, as follows. Let S be a surface represented by $f(x, y, z) = c = \text{const}$, where f is differentiable. Such a surface is called a **level surface** of f, and for different c we get different level surfaces. Now let C be a curve on S through a point P of S. As a curve in space, C has a representation $\mathbf{r}(t) = [x(t), y(t), z(t)]$. For C to lie on the surface S, the components of $\mathbf{r}(t)$ must satisfy $f(x, y, z) = c$, that is,

$$f(x(t), y(t), z(t)) = c. \tag{7}$$

Now a tangent vector of C is $\mathbf{r}'(t) = [x'(t), y'(t), z'(t)]$. And the tangent vectors of all curves on S passing through P will generally form a plane, called the **tangent plane** of S at P. (Exceptions occur at edges or cusps of S, for instance, at the apex of the cone in Fig. 217.) The normal of this plane (the straight line through P perpendicular to the tangent plane) is called the **surface normal** to S at P. A vector in the direction of the surface normal is called a **surface normal vector** of S at P. We can obtain such a vector quite simply by differentiating (7) with respect to t. By the chain rule,

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = (\operatorname{grad} f) \bullet \mathbf{r}' = 0.$$

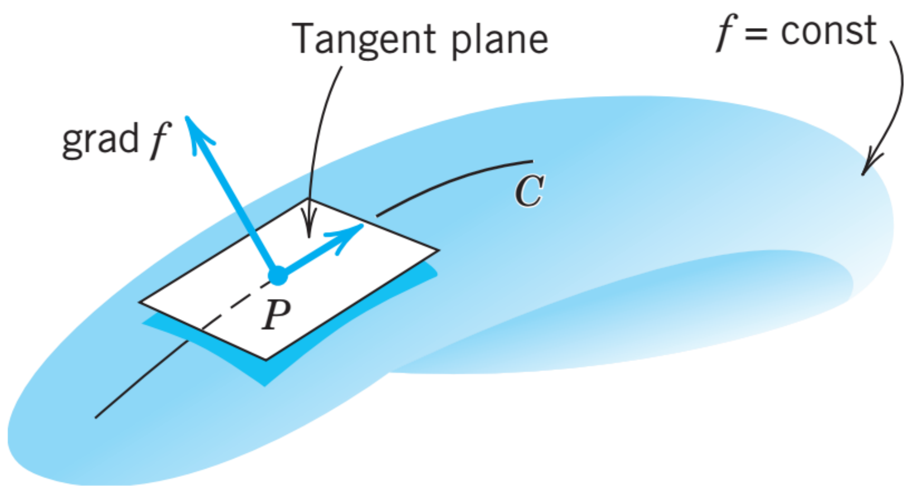


Fig. 216. Gradient as surface normal vector

THEOREM 2

Gradient as Surface Normal Vector

Let f be a differentiable scalar function in space. Let $f(x, y, z) = c = const$ represent a surface S. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

EXAMPLE 2 Gradient as Surface Normal Vector. Cone

Find a unit normal vector **n** of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: (1, 0, 2).

Solution. The cone is the level surface $f = 0$ of $f(x, y, z) = 4(x^2 + y^2) - z^2$. Thus (Fig. 217)

$$\text{grad} f = [8x, \quad 8y, \quad -2z], \qquad \text{grad} f(P) = [8, \quad 0, \quad -4]$$

$$\mathbf{n} = \frac{1}{|\text{grad} f(P)|} \text{grad} f(P) = \left[\frac{2}{\sqrt{5}}, \quad 0, \quad -\frac{1}{\sqrt{5}} \right].$$

n points downward since it has a negative z-component. The other unit normal vector of the cone at P is $-\mathbf{n}$. ■

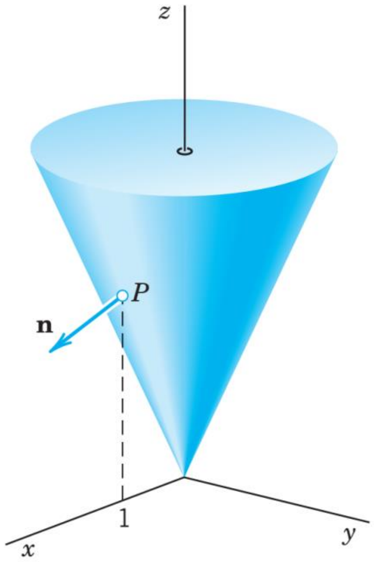


Fig. 217. Cone and unit normal vector **n**

9.8. Divergence of a Vector Field

Vector calculus owes much of its importance in engineering and physics to the gradient, divergence, and curl. From a scalar field we can obtain a vector field by the gradient (Sec. 9.7). Conversely, from a vector field we can obtain a scalar field by the divergence or another vector field by the curl (to be discussed in Sec. 9.9).

To begin, let $\mathbf{v}(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of $\mathbf{v}$. Then the function

(1) $$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the **divergence** of $\mathbf{v}$ or the *divergence of the vector field defined by* $\mathbf{v}$. For example, if

$$\mathbf{v} = [3xz,\ 2xy,\ -yz^2] = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}, \qquad \text{then} \qquad \operatorname{div} \mathbf{v} = 3z + 2x - 2yz.$$

$$\text{div}\,\mathbf{v} = \nabla \bullet \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}}\right] \bullet [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

with the understanding that the "product" $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that $\nabla \bullet \mathbf{v}$ means the scalar div **v**, whereas ∇f means the vector grad f defined in Sec. 9.7.

THEOREM 1

Invariance of the Divergence

The divergence div **v** *is a scalar function, that is, its values depend only on the points in space (and, of course, on* **v***) but not on the choice of the coordinates in* (1), *so that with respect to other Cartesian coordinates x^*, y^*, z^* and corresponding components $v_1{}^*$, $v_2{}^*$, $v_3{}^*$ of* **v**,

$$\text{(2)} \qquad \text{div}\,\mathbf{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*}.$$

Presently, let us turn to the more immediate practical task of gaining a feel for the significance of the divergence. Let $f(x, y, z)$ be a twice differentiable scalar function. Then its gradient exists,

$$\mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and we can differentiate once more, the first component with respect to x, the second with respect to y, the third with respect to z, and then form the divergence,

$$\operatorname{div} \mathbf{v} = \operatorname{div} (\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Hence we have the basic result that *the divergence of the gradient is the Laplacian* (Sec. 9.7),

(3) $$\operatorname{div} (\operatorname{grad} f) = \nabla^2 f.$$

9.10. Curl of a Vector Field

Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function of the Cartesian coordinates x, y, z. Then the **curl** *of the vector function* $\mathbf{v}$ or *of the vector field given by* $\mathbf{v}$ is defined by the "symbolic" determinant

(1)
$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

Instead of curl $\mathbf{v}$ one also uses the notation rot $\mathbf{v}$. This is suggested by "rotation," an application explored in Example 2. Note that curl $\mathbf{v}$ is a vector, as shown in Theorem 3.

EXAMPLE 1 Curl of a Vector Function

Let $\mathbf{v} = [yz, \quad 3zx, \quad z] = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$ with right-handed x, y, z. Then (1) gives

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}. \quad \blacksquare$$

EXAMPLE 2 Rotation of a Rigid Body. Relation to the Curl

We have seen in Example 5, Sec. 9.3, that a rotation of a rigid body B about a fixed axis in space can be described by a vector $\mathbf{w}$ of magnitude ω in the direction of the axis of rotation, where $\omega\ (>0)$ is the angular speed of the rotation, and $\mathbf{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\mathbf{w}$. According to (9), Sec. 9.3, the velocity field of the rotation can be represented in the form

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}$$

where $\mathbf{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system *having the origin on the axis of rotation*. Let us choose right-handed Cartesian coordinates such that the axis of rotation is the z-axis. Then (see Example 2 in Sec. 9.4)

$$\mathbf{w} = [0, \quad 0, \quad \omega] = \omega\mathbf{k}, \qquad \mathbf{v} = \mathbf{w} \times \mathbf{r} = [-\omega y, \quad \omega x, \quad 0] = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

Hence

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = [0, \quad 0, \quad 2\omega] = 2\omega\mathbf{k} = 2\mathbf{w}.$$

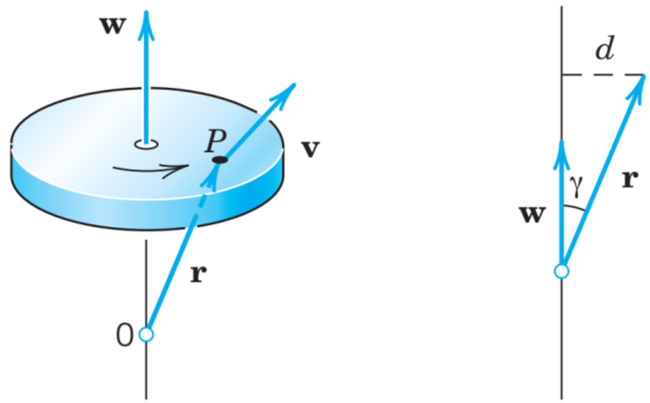


Fig. 192. Rotation of a rigid body

THEOREM 1

Rotating Body and Curl

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

THEOREM 2

Grad, Div, Curl

Gradient fields are ***irrotational***. *That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,*

$$\text{curl}\,(\text{grad}\, f) = \mathbf{0}. \tag{2}$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function $\mathbf{v}$ *is zero,*

$$\text{div}\,(\text{curl}\,\mathbf{v}) = 0. \tag{3}$$