

بسم الله الرحمن الرحيم

فصل پنجم

زبان‌های مستقل از متن (۱)

Context-Free Languages (1)

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Context-Free Languages

$$\{a^n b^n\}$$

$$\{ww^R\}$$

Regular Languages

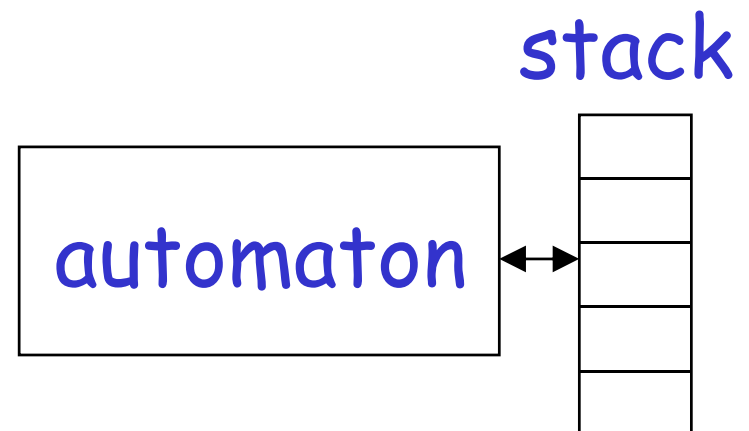
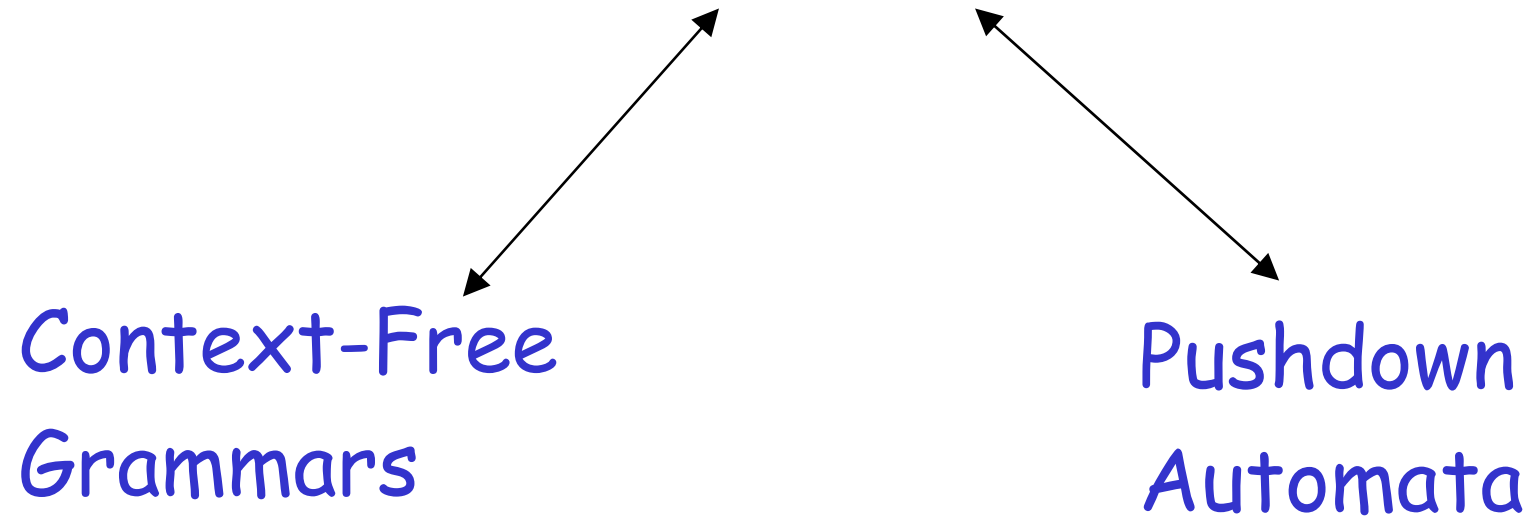
Context-Free Languages

$$\{a^n b^n\}$$

$$\{ww^R\}$$

Regular Languages

Context-Free Languages



Context-Free Grammars

Example

A context-free grammar G :

$$S \rightarrow aSb$$
$$S \rightarrow \lambda$$

A derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

A context-free grammar G :

$$S \rightarrow aSb$$
$$S \rightarrow \lambda$$

Another derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbbb$$

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

It remembers us to: (((()))

Example

A context-free grammar G :

$$S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \lambda$$

A derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

A context-free grammar G :

$$S \rightarrow aSa$$
$$S \rightarrow bSb$$
$$S \rightarrow \lambda$$

Another derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \lambda$$

$$L(G) = \{ww^R : w \in \{a,b\}^*\}$$

Example

A context-free grammar G : $S \rightarrow aSb$

$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

A context-free grammar G : $S \rightarrow aSb$

$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$S \rightarrow aSb$$

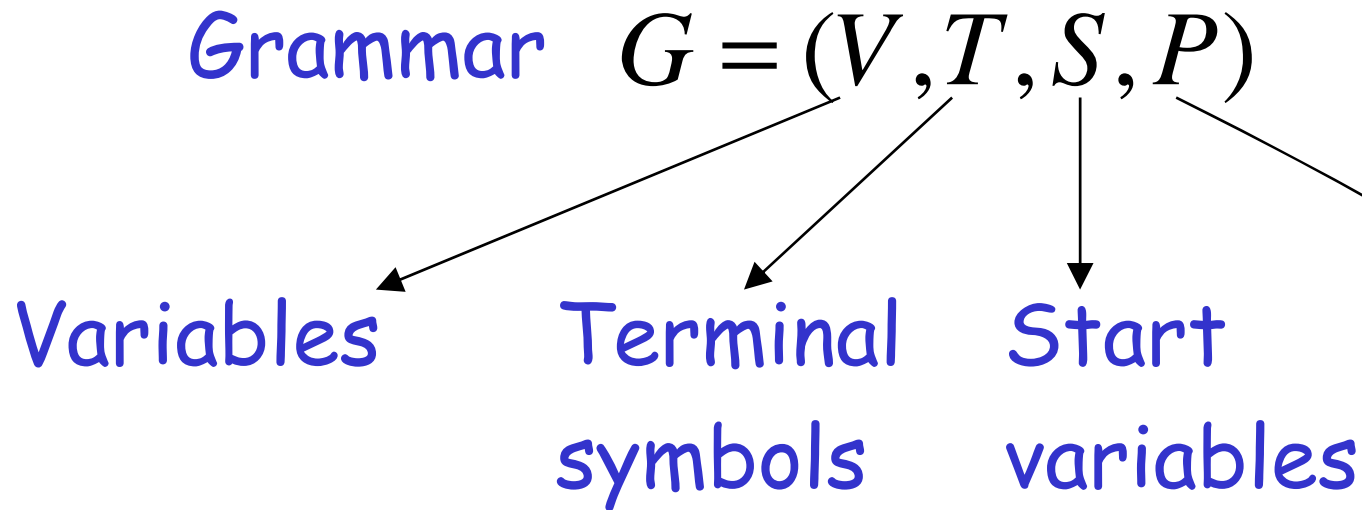
$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

$$L(G) = \{w : n_a(w) = n_b(w), \\ \text{and } n_a(v) \geq n_b(v) \\ \text{in any prefix } v\}$$

() ((())) (())

Definition: Context-Free Grammars



Productions of the form:

$$A \rightarrow x$$

x is string of variables and terminals

Definition: Context-Free Languages

A language L is context-free

if and only if

there is a grammar G with $L = L(G)$

Derivation Order

- | | | |
|-----------------------|----------------------------|----------------------------|
| 1. $S \rightarrow AB$ | 2. $A \rightarrow aaA$ | 4. $B \rightarrow Bb$ |
| | 3. $A \rightarrow \lambda$ | 5. $B \rightarrow \lambda$ |

Leftmost derivation:

$$\begin{array}{ccccccccc} & 1 & & 2 & & 3 & & 4 & & 5 \\ S & \Rightarrow & AB & \Rightarrow & aaAB & \Rightarrow & aaB & \Rightarrow & aaBb & \Rightarrow & aab \end{array}$$

Rightmost derivation:

$$\begin{array}{ccccccccc} & 1 & & 4 & & 5 & & 2 & & 3 \\ S & \Rightarrow & AB & \Rightarrow & ABb & \Rightarrow & Ab & \Rightarrow & aaAb & \Rightarrow & aab \end{array}$$

$$S \rightarrow aAB$$

$$A \rightarrow bBb$$

$$B \rightarrow A \mid \lambda$$

Leftmost derivation:

$$\begin{aligned} S &\Rightarrow aAB \Rightarrow abBbB \Rightarrow abAbB \Rightarrow abbBbbB \\ &\Rightarrow abbbbB \Rightarrow abbbb \end{aligned}$$

Rightmost derivation:

$$\begin{aligned} S &\Rightarrow aAB \Rightarrow aA \Rightarrow abBb \Rightarrow abAb \\ &\Rightarrow abbBbb \Rightarrow abbbb \end{aligned}$$

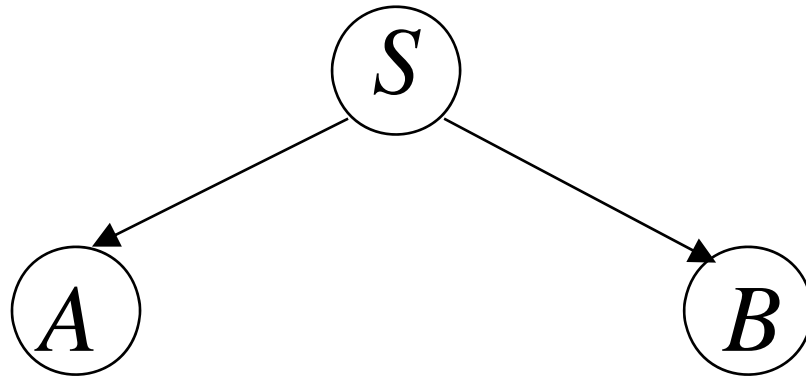
Derivation Trees

$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \lambda$$

$$B \rightarrow Bb \mid \lambda$$

$$S \Rightarrow AB$$

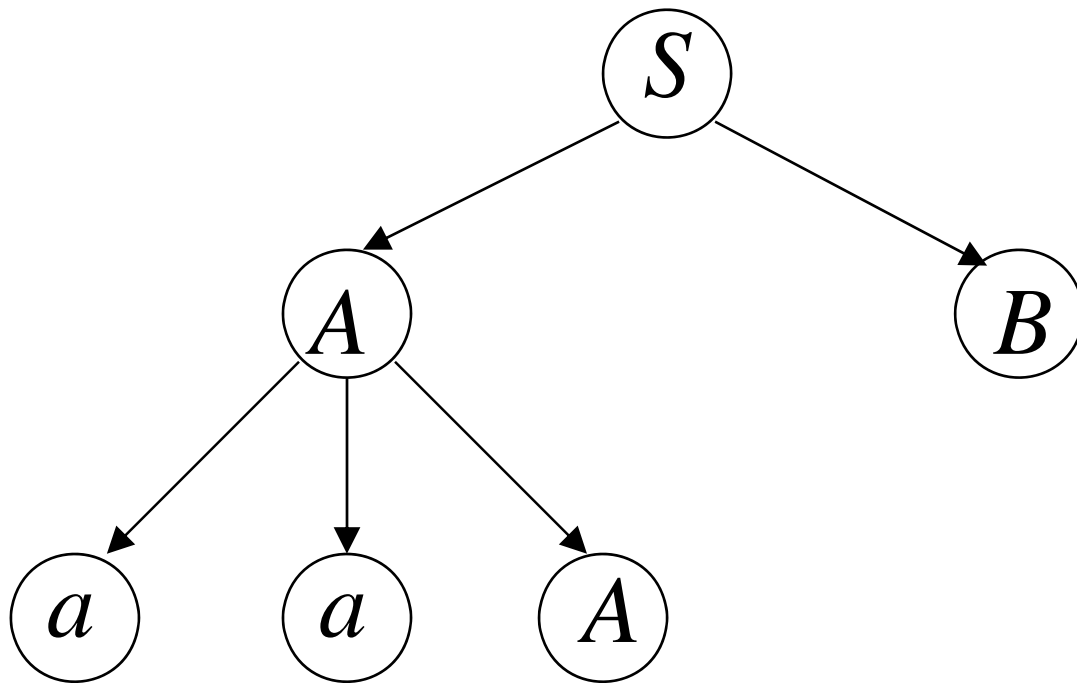


$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \lambda$$

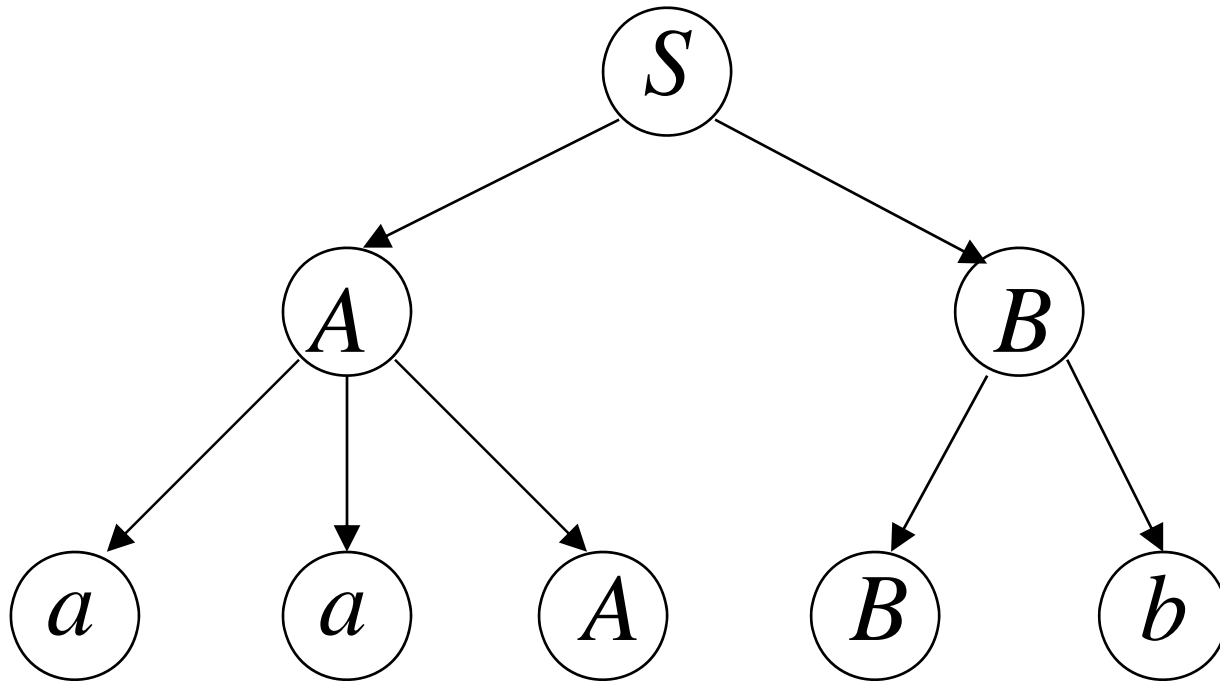
$$B \rightarrow Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB$$



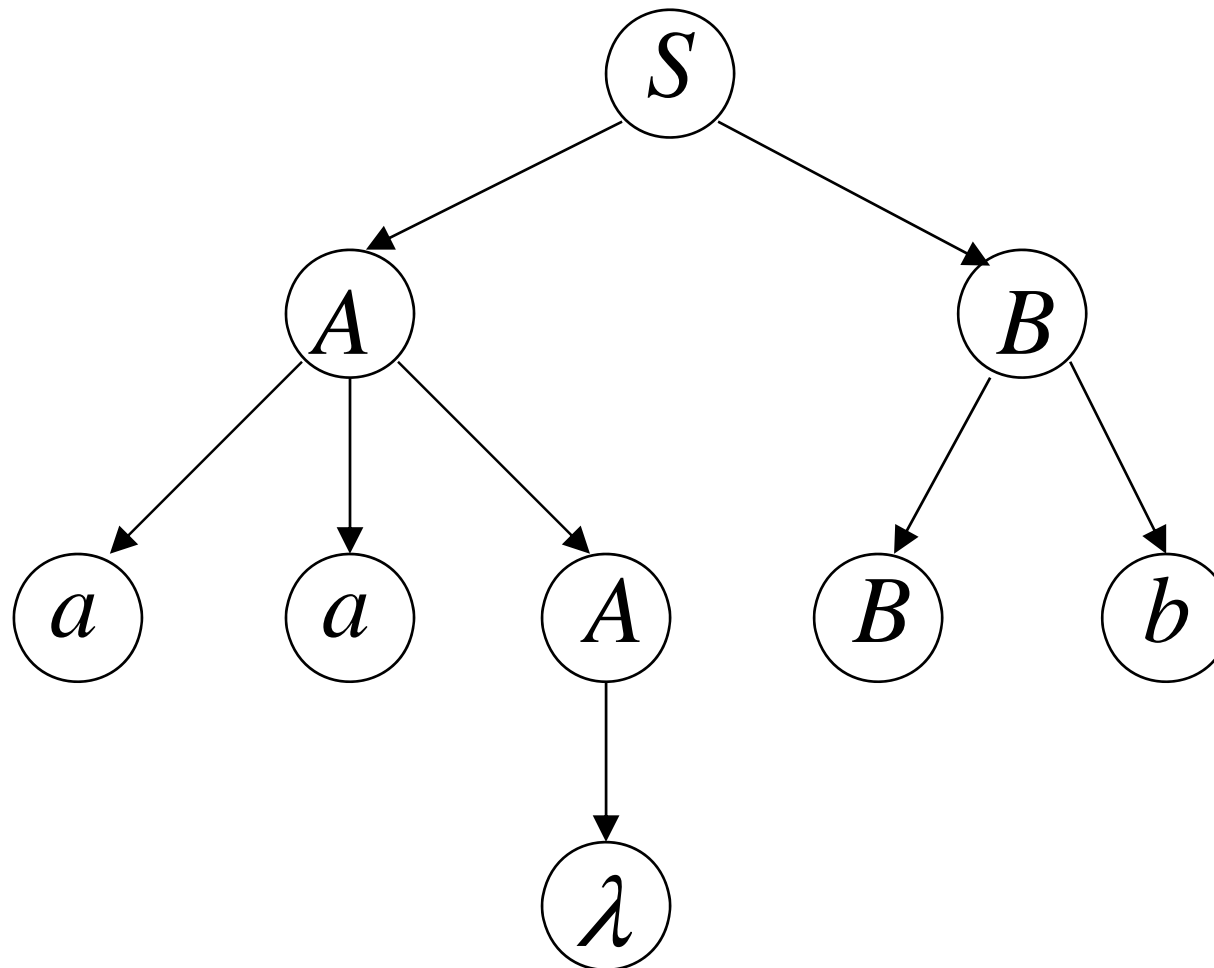
$$S \rightarrow AB \qquad A \rightarrow aaA \mid \lambda \qquad B \rightarrow Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \rightarrow AB \qquad A \rightarrow aaA \mid \lambda \qquad B \rightarrow Bb \mid \lambda$$

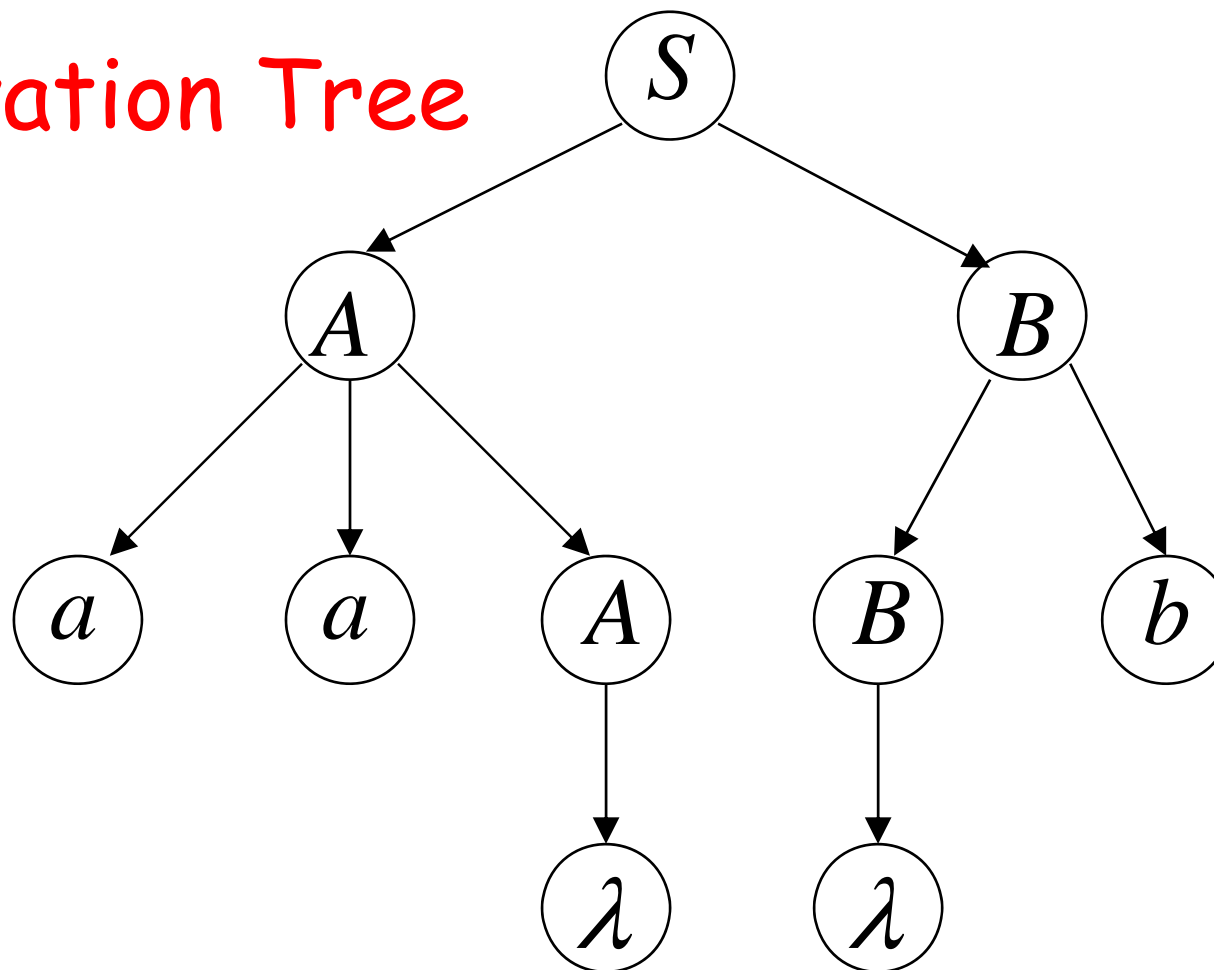
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



$$S \rightarrow AB \qquad A \rightarrow aaA \mid \lambda \qquad B \rightarrow Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree



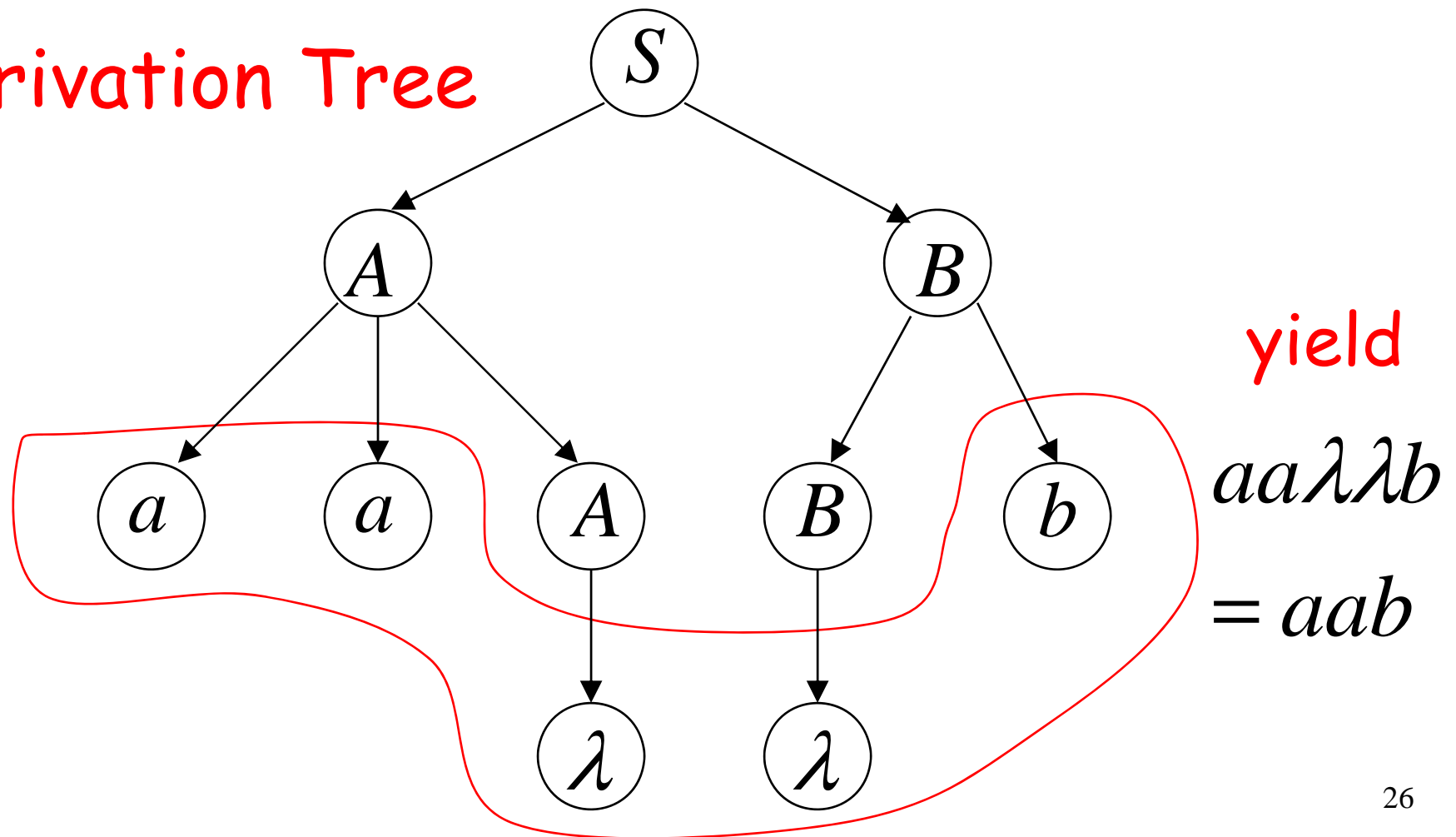
$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \lambda$$

$$B \rightarrow Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree



Partial Derivation Trees

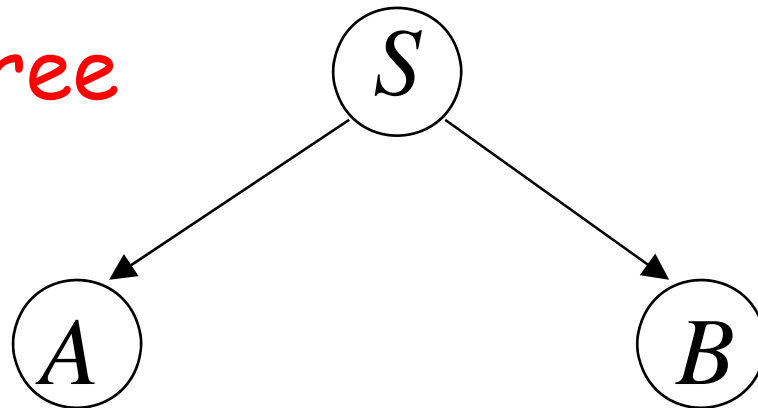
$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \lambda$$

$$B \rightarrow Bb \mid \lambda$$

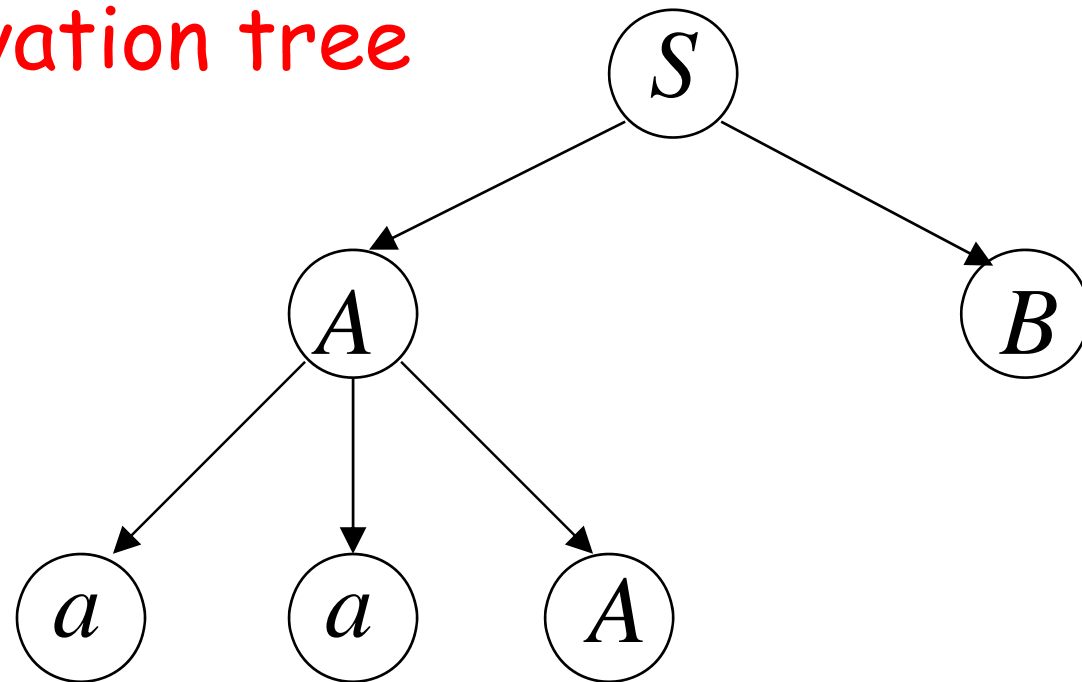
$$S \Rightarrow AB$$

Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

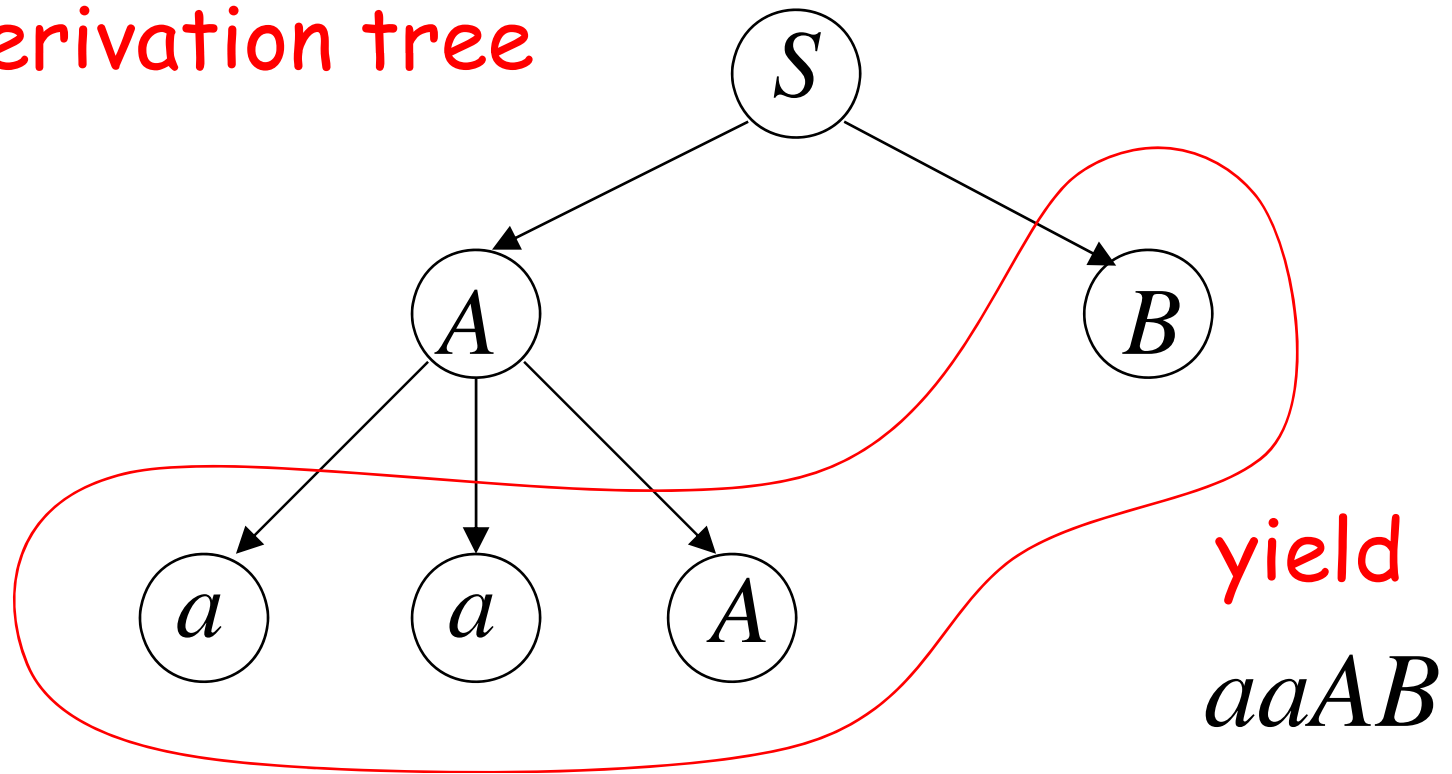
Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

sentential
form

Partial derivation tree



Sometimes, derivation order doesn't matter

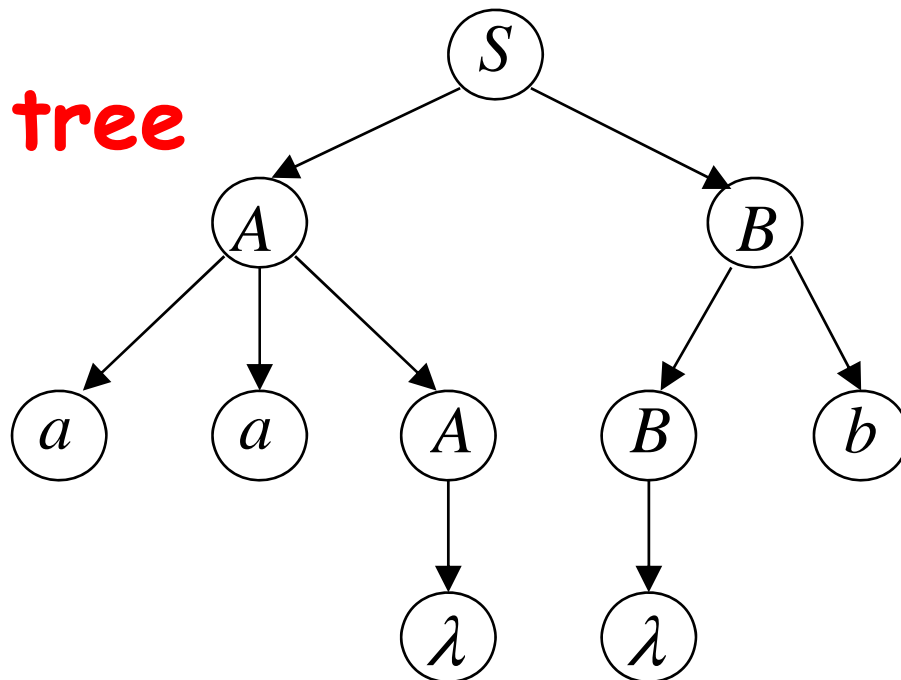
Leftmost:

$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$

Rightmost:

$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$

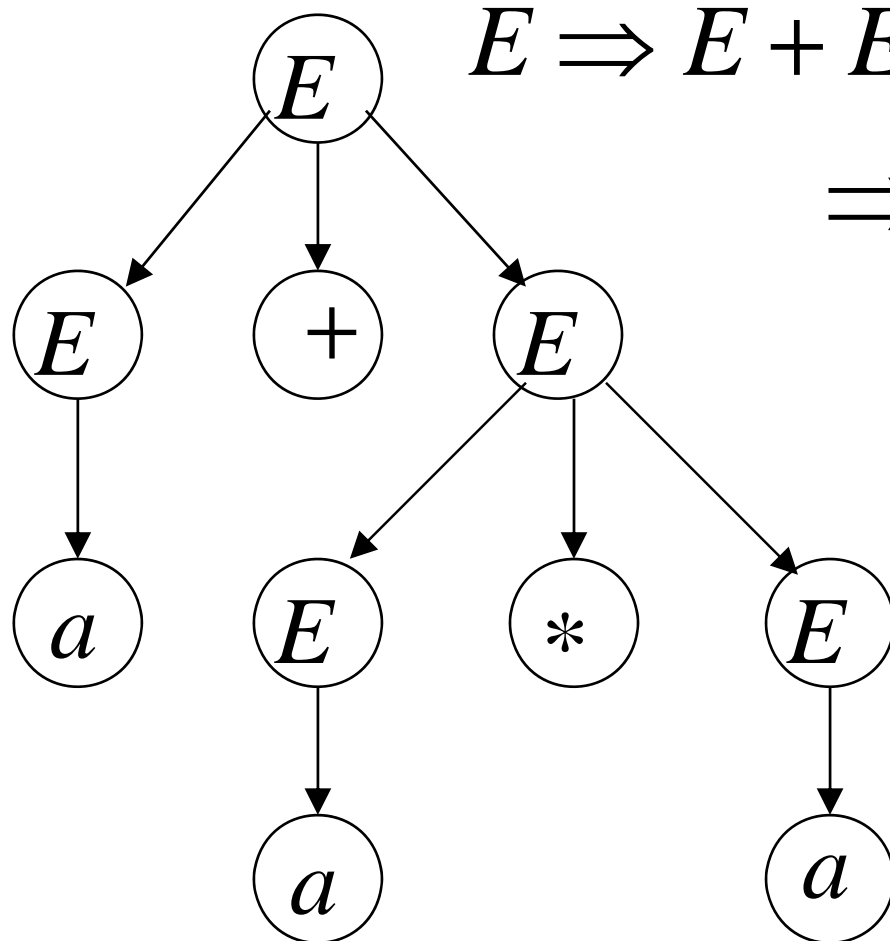
Same derivation tree



Ambiguity

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

$$a + a * a$$



$$\begin{aligned}
 E &\Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E \\
 &\Rightarrow a + a * E \Rightarrow a + a * a
 \end{aligned}$$

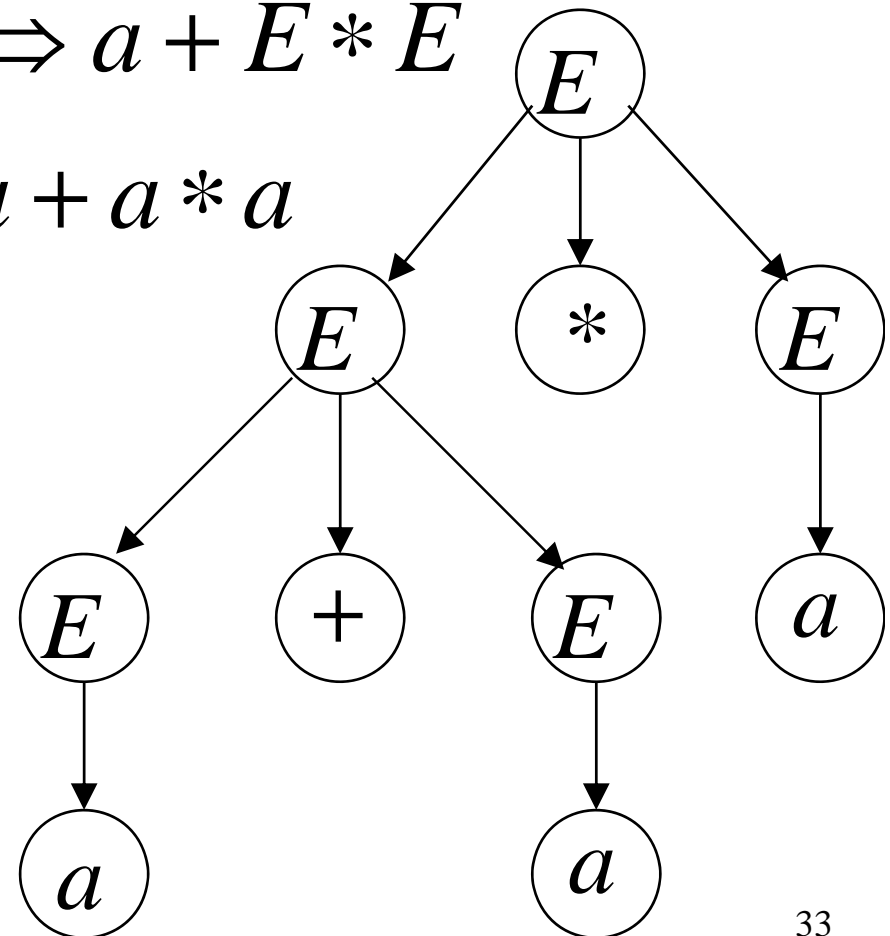
leftmost derivation

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

$$a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E \\ \Rightarrow a + a * E \Rightarrow a + a * a$$

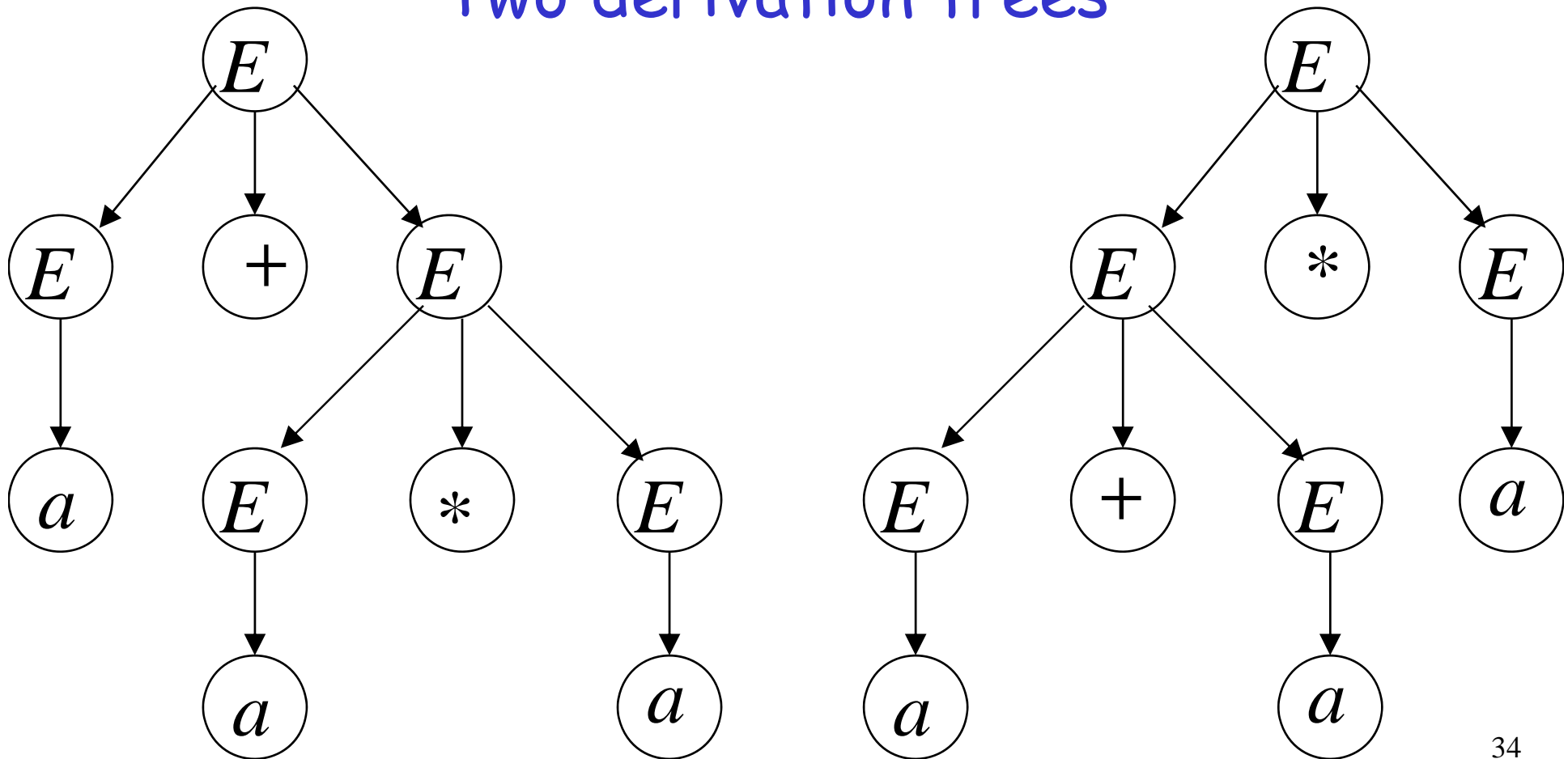
leftmost derivation



$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

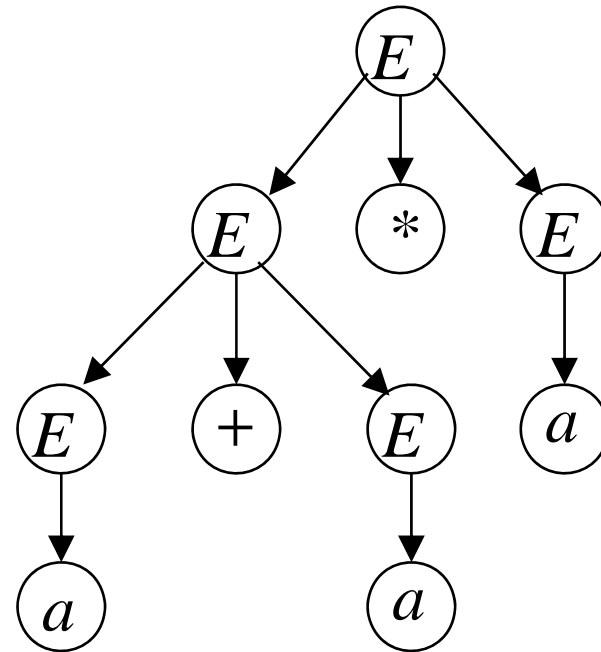
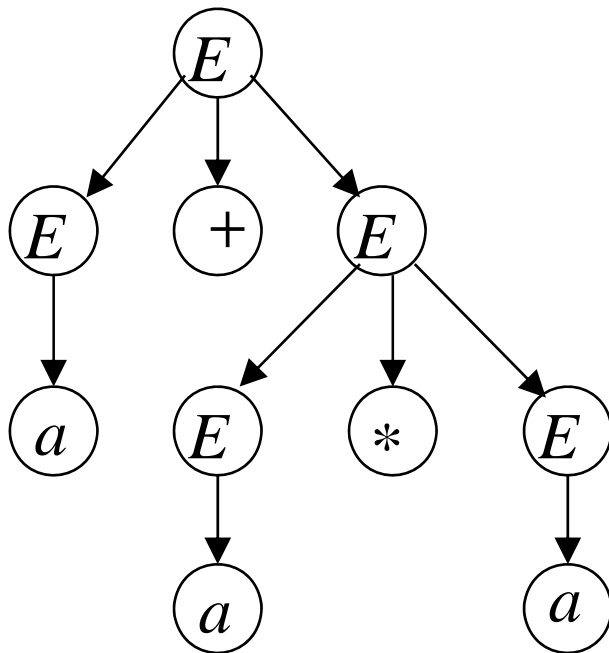
$$a + a * a$$

Two derivation trees



The grammar $E \rightarrow E + E \mid E * E \mid (E) \mid a$
is ambiguous:

string $a + a * a$ has two derivation trees



The grammar $E \rightarrow E + E \mid E * E \mid (E) \mid a$
is ambiguous:

string $a + a * a$ has two leftmost derivations

$$\begin{aligned} E &\Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E \\ &\Rightarrow a + a * E \Rightarrow a + a * a \end{aligned}$$

$$\begin{aligned} E &\Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E \\ &\Rightarrow a + a * E \Rightarrow a + a * a \end{aligned}$$

Definition:

A context-free grammar G is **ambiguous**

if some string $w \in L(G)$ has:

two or more derivation trees

In other words:

A context-free grammar G is **ambiguous**

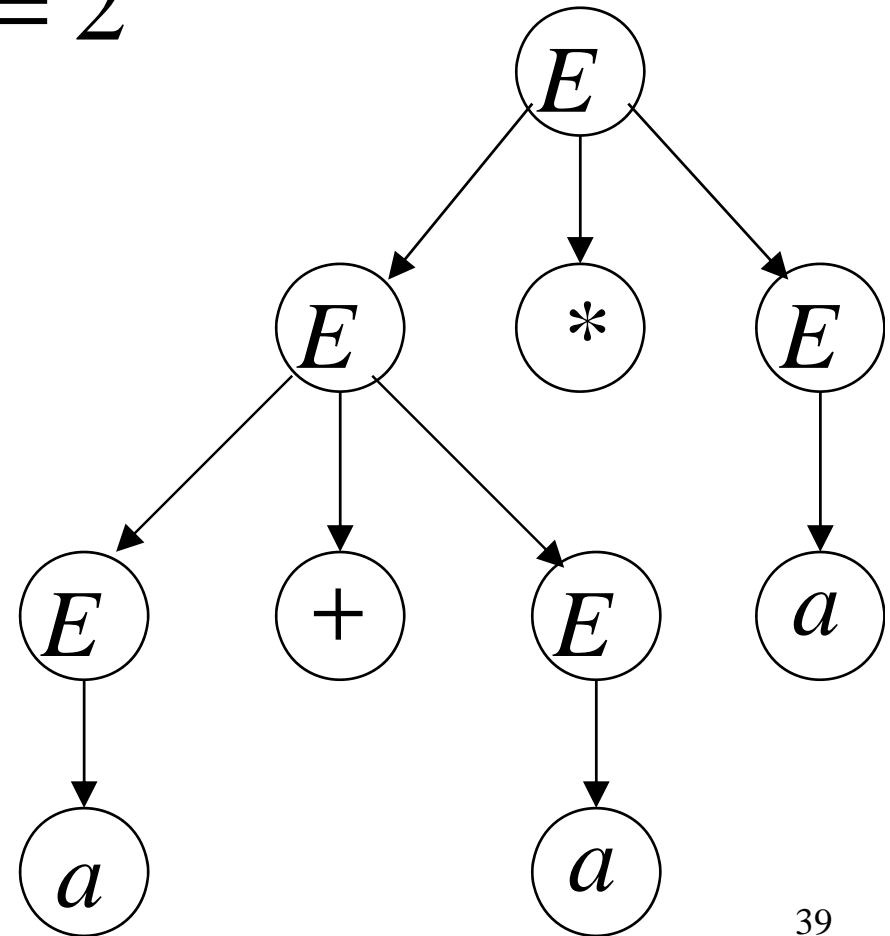
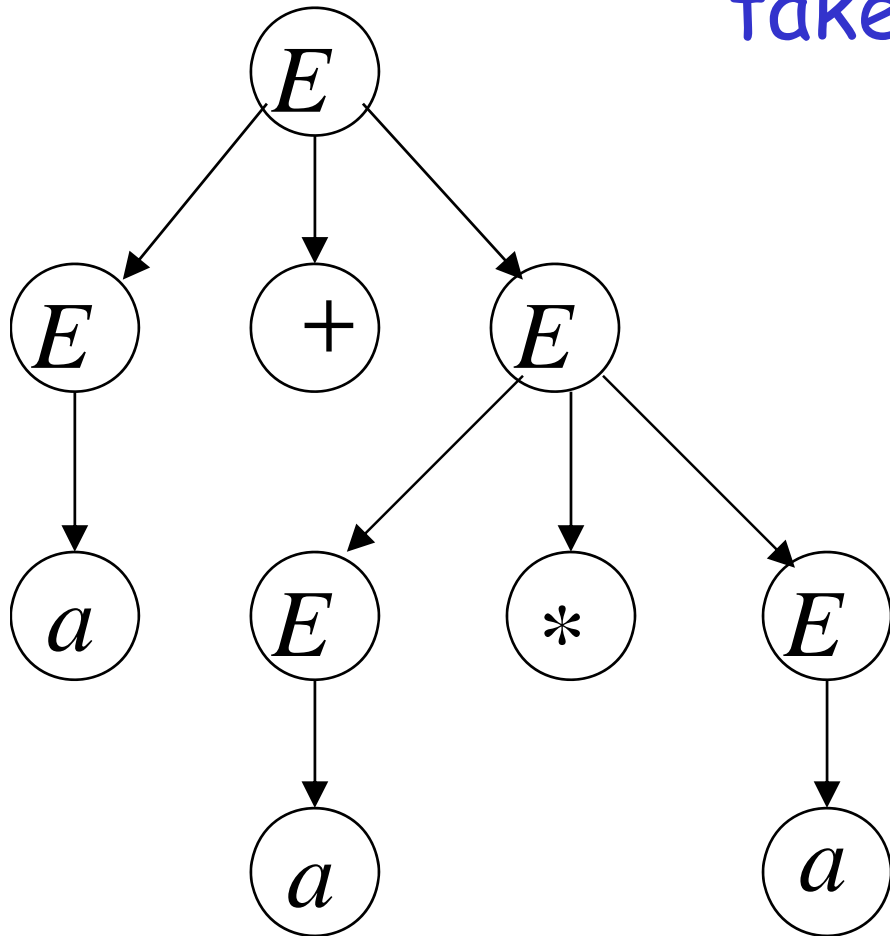
if some string $w \in L(G)$ has:

two or more leftmost derivations
(or rightmost)

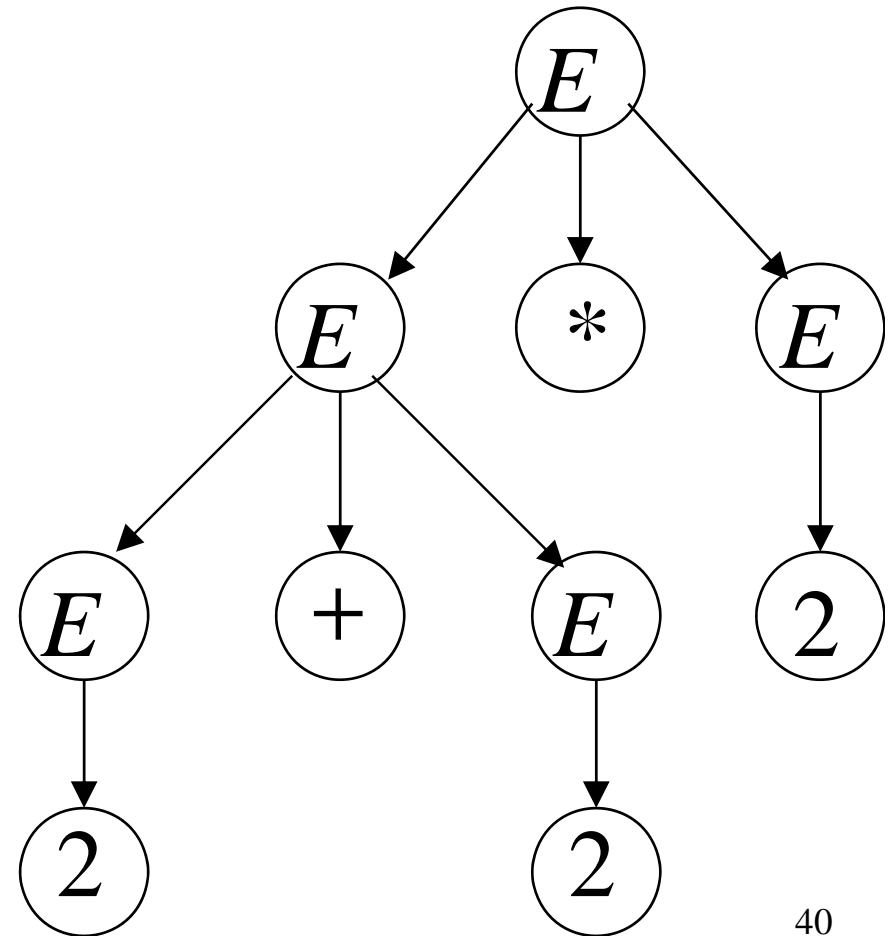
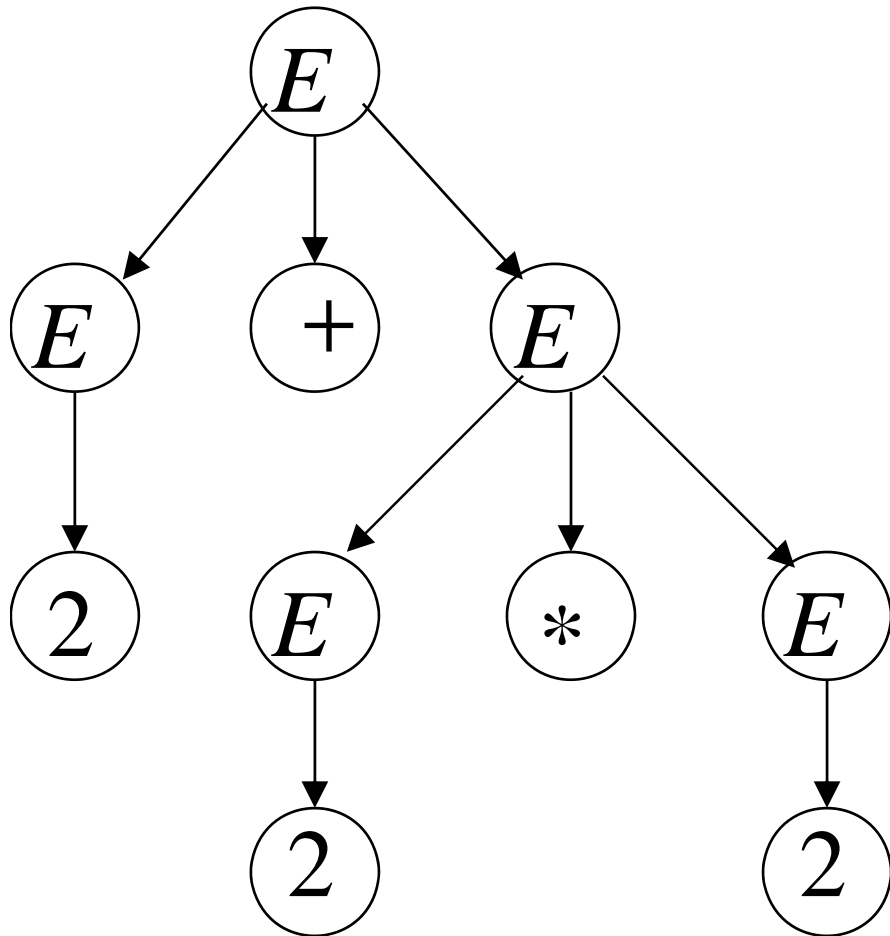
Why do we care about ambiguity?

$$a + a * a$$

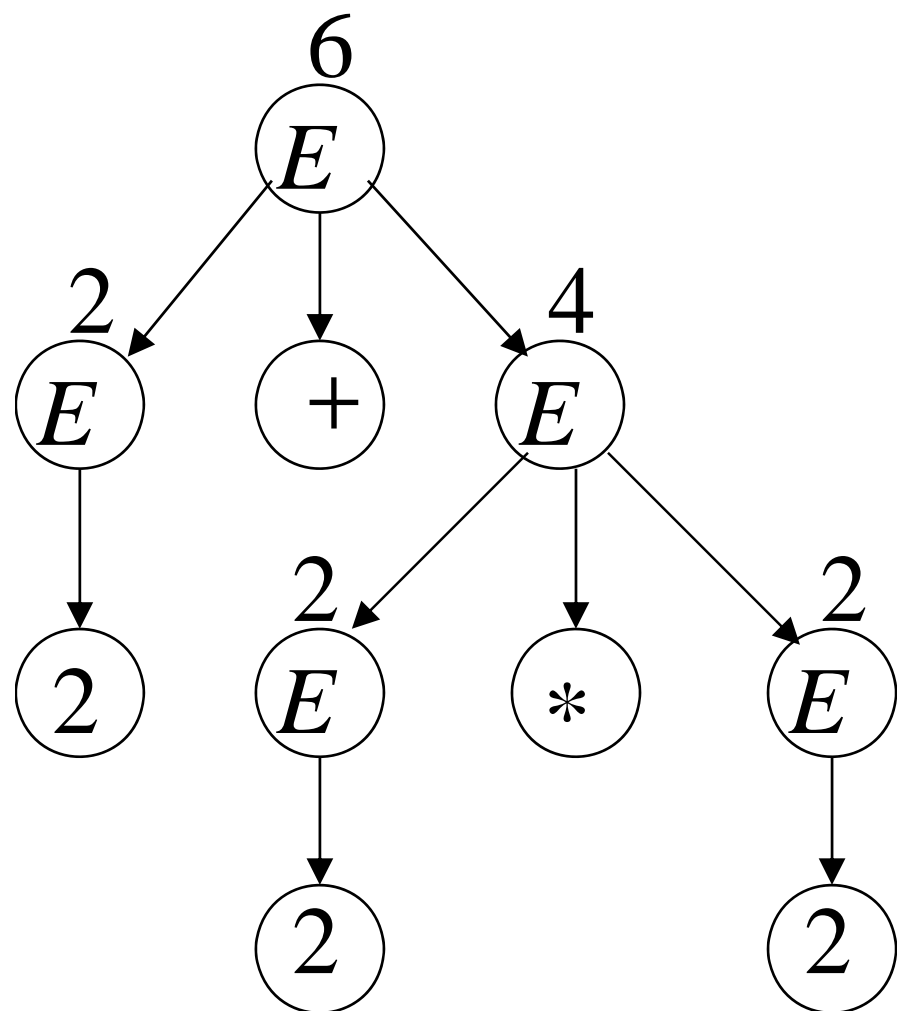
take $a = 2$



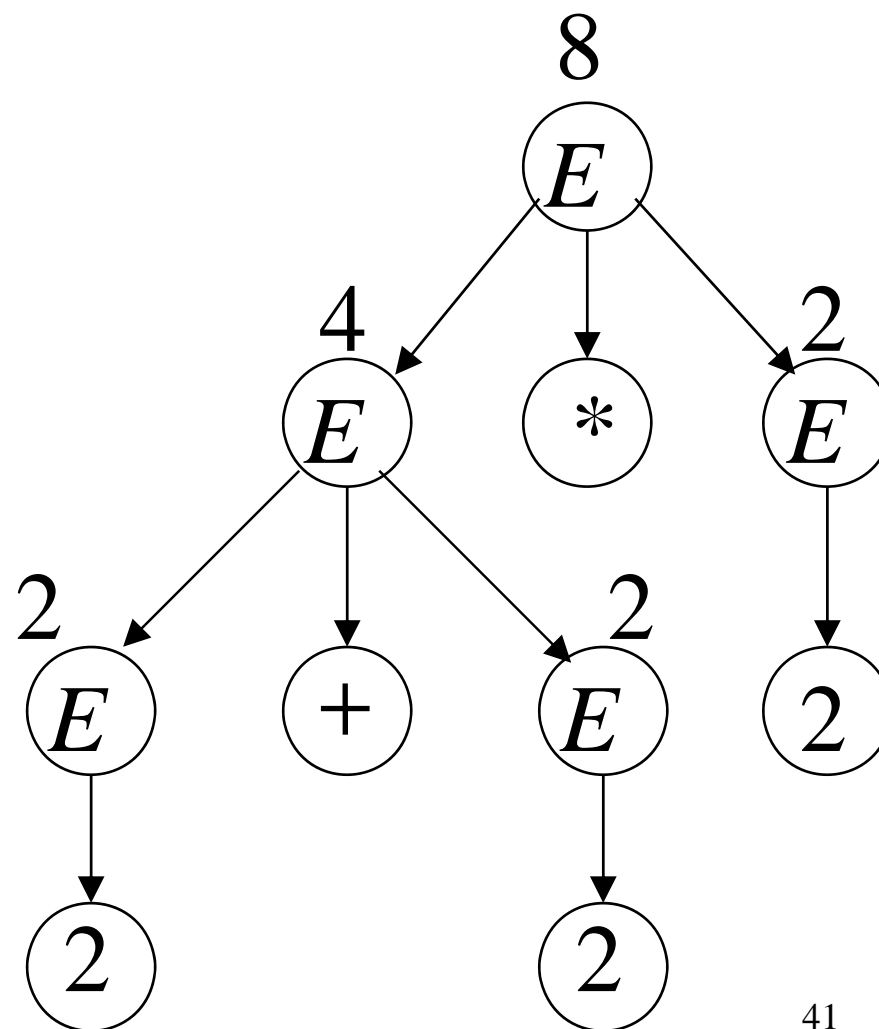
$$2 + 2 * 2$$



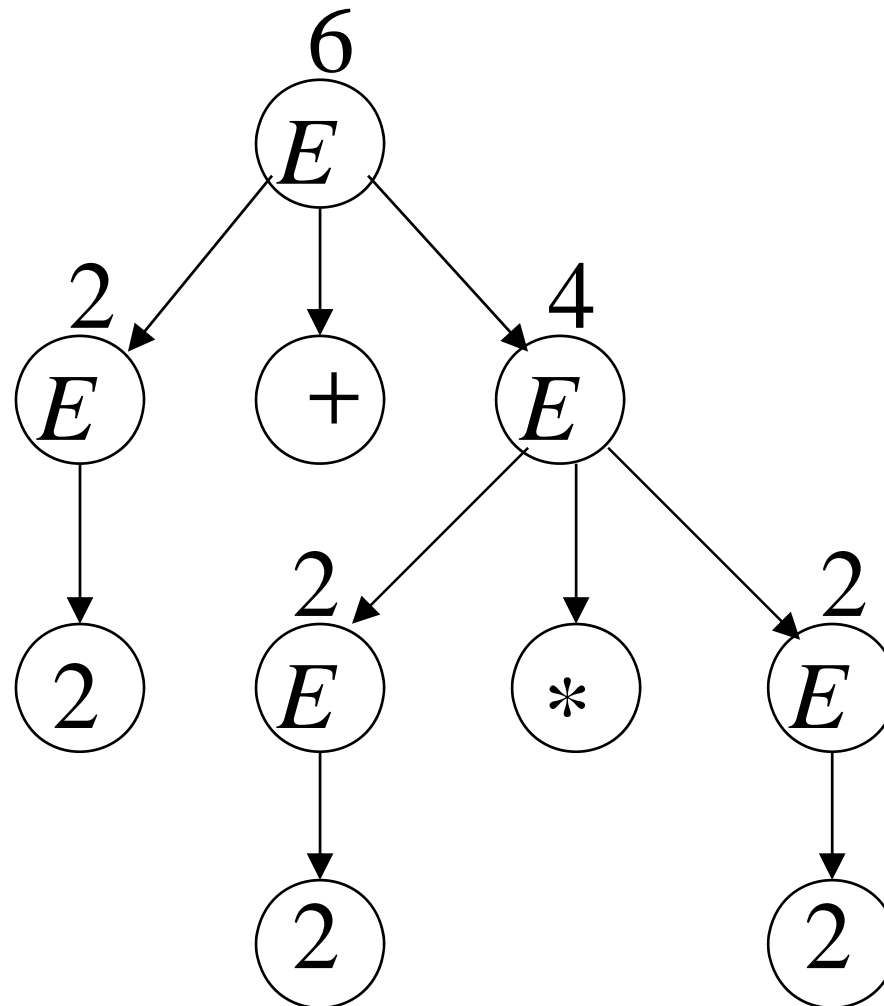
$$2 + 2 * 2 = 6$$



$$2 + 2 * 2 = 8$$



Correct result: $2 + 2 * 2 = 6$



- Ambiguity is **bad** for programming languages
- We want to remove ambiguity
 - Usually, by adding some new non-terminals and productions.

We fix the ambiguous grammar:

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

New non-ambiguous grammar: $E \rightarrow E + T$

$$E \rightarrow T$$

$$T \rightarrow T * F$$

$$T \rightarrow F$$

$$F \rightarrow (E)$$

$$F \rightarrow a$$

$$\begin{aligned}
 E &\Rightarrow E + T \Rightarrow T + T \Rightarrow F + T \Rightarrow a + T \Rightarrow a + T * F \\
 &\Rightarrow a + F * F \Rightarrow a + a * F \Rightarrow a + a * a
 \end{aligned}$$

$$E \rightarrow E + T$$

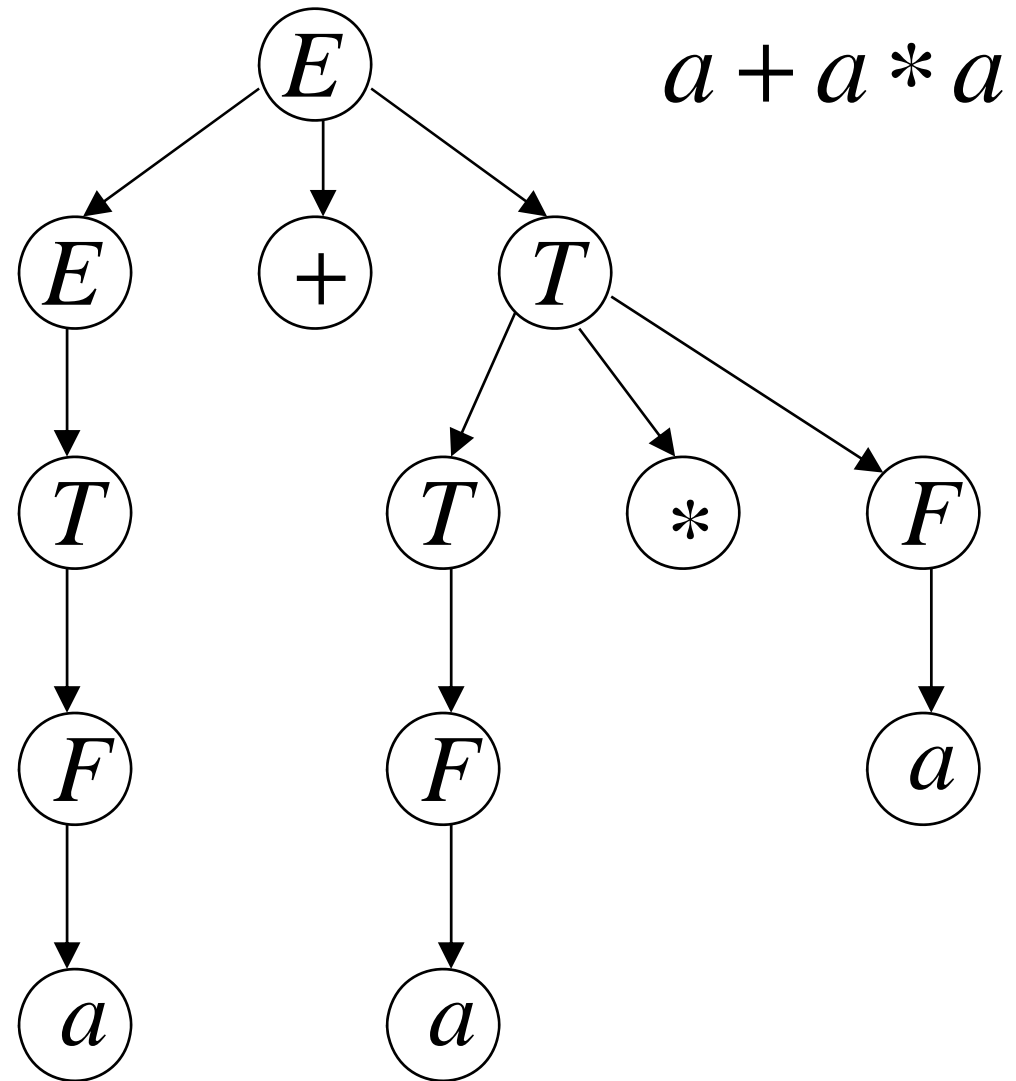
$$E \rightarrow T$$

$$T \rightarrow T * F$$

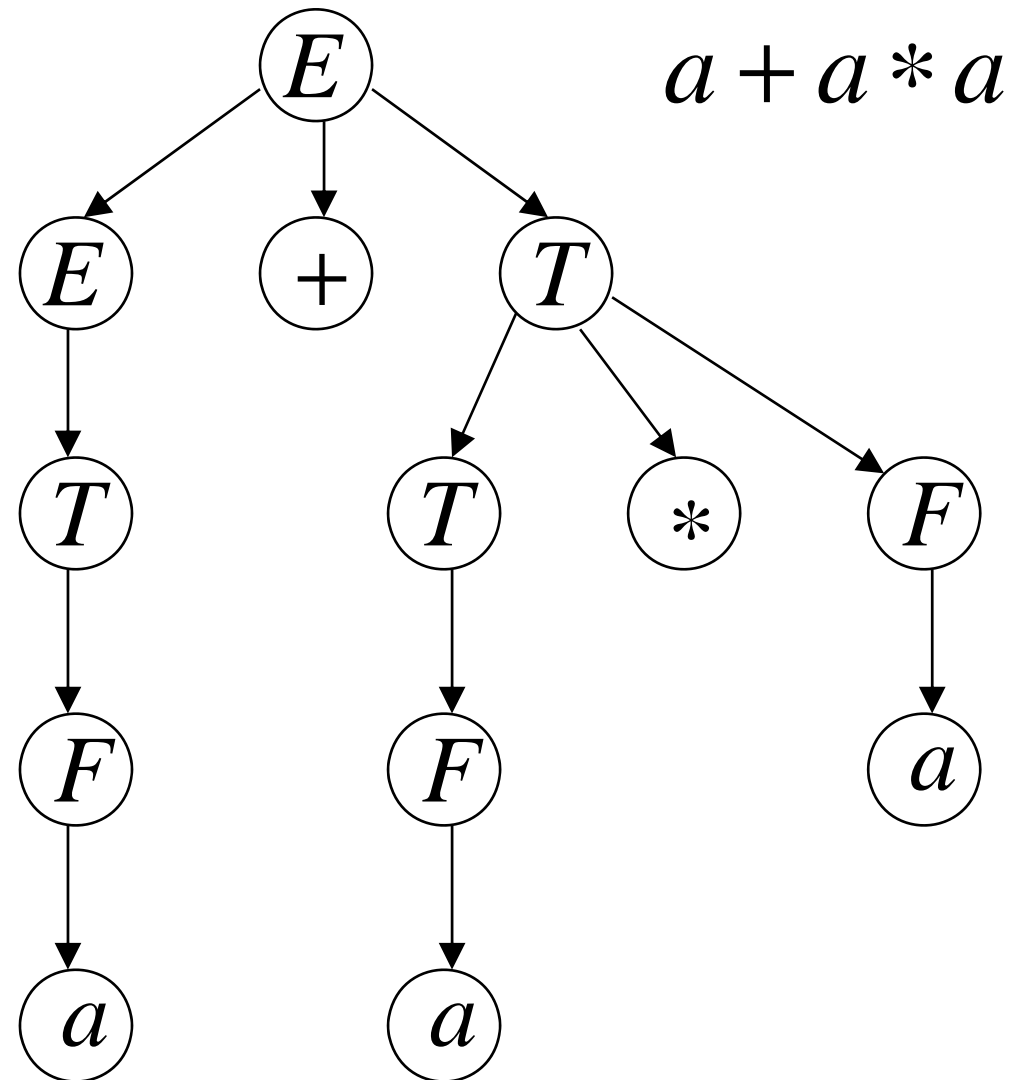
$$T \rightarrow F$$

$$F \rightarrow (E)$$

$$F \rightarrow a$$



Unique derivation tree



The grammar G :

$$E \rightarrow E + T$$

$$E \rightarrow T$$

$$T \rightarrow T * F$$

$$T \rightarrow F$$

$$F \rightarrow (E)$$

$$F \rightarrow a$$

is non-ambiguous:

Every string $w \in L(G)$ has
a unique derivation tree

Inherent Ambiguity

Some context free languages
have only ambiguous grammars

Example: $L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$


$$S \rightarrow S_1 \mid S_2$$


$$S_1 \rightarrow S_1 c \mid A$$

$$A \rightarrow aAb \mid \lambda$$


$$S_2 \rightarrow aS_2 \mid B$$

$$B \rightarrow bBc \mid \lambda$$

The string $a^n b^n c^n$

has two derivation trees

