

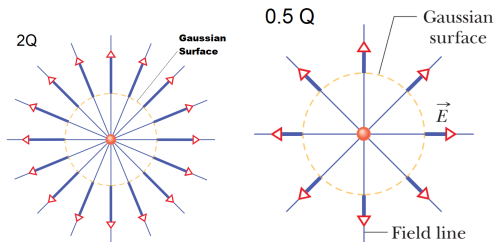
23. Gauss' law

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Guass' law

Guass' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

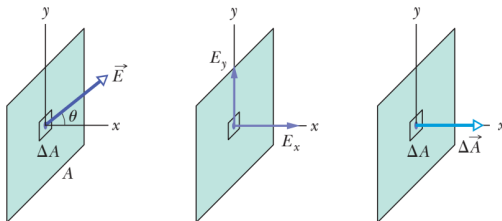


Electric Flux I

The amount of electric field piercing the patch is defined to be the **electric flux** $\Delta\Phi$ through it:

$$\Delta\Phi = (E \cos \theta) \Delta A,$$

$$\Delta\Phi = \vec{E} \cdot \Delta\vec{A}.$$



Electric Flux II

To find the total flux through the surface, we sum the flux through every patch on the surface

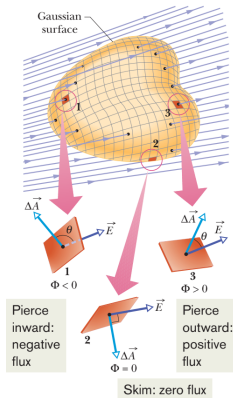
$$\Phi = \sum \vec{E} \cdot \Delta \vec{A}.$$

However, because we do not want to sum hundreds (or more) flux values, we transform the summation into an integral by shrinking the patches from small squares with area ΔA to patch elements (or area elements) with area dA . The total flux is then

$$\Phi = \int \vec{E} \cdot d\vec{A}.$$



Electric Flux III



Electric Flux IV

An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.

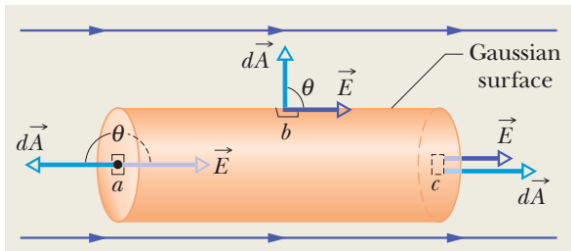
Net Flux. In principle, to find the net flux through the surface in we find the flux at every patch and then sum the results (with the algebraic signs included). However, we are not about to do that much work. Instead, we shrink the squares to patch elements with area vectors $d\vec{A}$ and then integrate:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad (\text{net flux}).$$



Example 1 I

Figure 23-6 shows a Gaussian surface in the form of a closed cylinder (a Gaussian cylinder or G-cylinder) of radius R . It lies in a uniform electric field with the cylinder's central axis (along the length of the cylinder) parallel to the field. What is the net flux Φ of the electric field through the cylinder?



Solution. We break the integral of Eq. 23-4 into three terms: integrals over the left cylinder cap a , the curved cylindrical surface b , and the right cap c :

$$\begin{aligned}\Phi &= \oint \vec{E} \cdot d\vec{A}, \\ &= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}.\end{aligned}$$

Example 1 II

We can write the flux through the left cap as

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180^\circ) dA = -E \int dA = -EA,$$

where $\int dA = \pi R^2$. Similarly, for the right cap, where $\theta = 0^\circ$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 0^\circ) dA = E \int dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90^\circ) dA = 0.$$

Thus we have

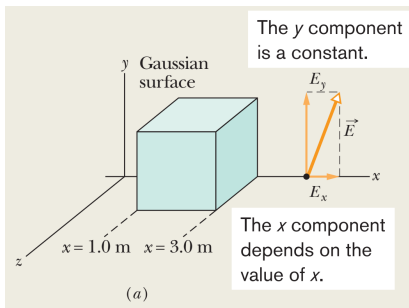
$$\Phi = \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A},$$

$$\Phi = -EA + 0 + EA = 0. \quad (\text{Answer}).$$



Example 2 I

A nonuniform electric field given by $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ pierces the Gaussian cube shown in Fig. 23-7a. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)



Example 2 II

Solution.

Right face: An area vector is always perpendicular to its surface and always points away from the interior of a Gaussian surface. The most convenient way to express the vector is in unit-vector notation,

$$d\vec{A} = dA\hat{i}.$$

The flux Φ_r through the right face is then

$$\begin{aligned}\Phi_r &= \int \vec{E} \cdot \vec{A} = \int (3.0x\hat{i} + 4.0\hat{j}) \cdot (dA\hat{i}), \\ &= \int [(3.0x)(dA)\hat{i} \cdot \hat{i} + (4.0)(dA)\hat{j} \cdot \hat{i}], \\ &= \int (3.0x dA + 0) \stackrel{x=3}{=} 9.0 \int dA.\end{aligned}$$

The integral $\int dA$ merely gives us the area $A = 4.0\text{m}^2$ of the right face, so

$$\Phi_r = (9.0\text{N/C})(4.0\text{m}^2) = 36\text{N} \cdot \text{m}^2/\text{C}, \quad (\text{Answer}).$$

Example 2 III

Left face: For the surface element in the left face we have

$$d\vec{A} = -dA\hat{i}.$$

The flux Φ_l through the left face is then

$$\begin{aligned}\Phi_l &= \int \vec{E} \cdot \vec{A} = \int (3.0x\hat{i} + 4.0\hat{j}) \cdot (-dA\hat{i}), \\ &= \int \left[(3.0x)(-dA)\hat{i} \cdot \hat{i} + (4.0)(-dA)\hat{j} \cdot \hat{i} \right], \\ &= - \int (3.0x dA + 0) \stackrel{x=1}{=} -3.0 \int dA.\end{aligned}$$

The integral $\int dA$ merely gives us the area $A = 4.0\text{m}^2$ of the right face, so

$$\Phi_l = (3.0\text{N/C})(-4.0\text{m}^2) = -12\text{N} \cdot \text{m}^2/\text{C}, \quad (\text{Answer}).$$

Example 2 IV

Top face: Now $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = dA \hat{j}$. The flux Φ_t is

$$\begin{aligned}\Phi_t &= \int \left(3.0x \hat{i} + 4.0\hat{j} \right) \cdot (dA \hat{j}), \\ &= \int \left[(3.0x)(dA) \hat{i} \cdot \hat{j} + (4.0)(dA) \hat{j} \cdot \hat{j} \right], \\ &= \int (0 + 4.0 dA) = 4.0 \int dA, \\ &= (4.0 N/C)(4.0 m^2) = 16 N \cdot m^2/C, \quad (\text{Answer}).\end{aligned}$$

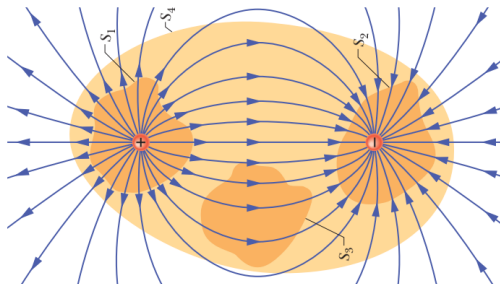
Gauss' Law I

Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is enclosed by that surface. It tells us that

$$\epsilon_0 \Phi = q_{\text{enc}}, \quad (\text{Gauss' law}).$$

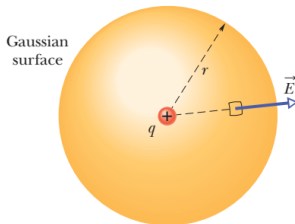
where q_{enc} is the algebraic sum of all the enclosed positive and negative charges, and it can be positive, negative, or zero. By substituting the definition of flux, we can also write Gauss' law as

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}}, \quad (\text{Gauss' law}).$$



Gauss' Law and Coulomb's Law I

One of the situations in which we can apply Gauss' law is in finding the electric field of a charged particle. That field has spherical symmetry (the field depends on the distance r from the particle but not the direction). So, to make use of that symmetry, we enclose the particle in a Gaussian sphere that is centered on the particle, as shown in Fig. 23-9 for a particle with positive charge q . Then the electric field has the same magnitude E at any point on the sphere (all points are at the same distance r). That feature will simplify the integration.

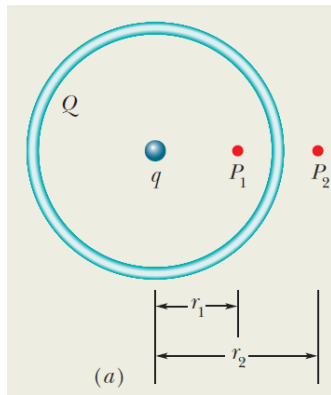


$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} \stackrel{\theta=0}{\implies} \epsilon_0 \oint E dA = q_{\text{enc}} = q,$$

$$\epsilon_0 E (4\pi r^2) = q_{\text{enc}} \implies E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}.$$

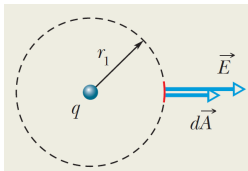
Example 23.03 I

Figure 23-10a shows, in cross section, a plastic, spherical shell with uniform charge $Q = -16e$ and radius $R = 10$ cm. A particle with charge $q = +5e$ is at the center. What is the electric field (magnitude and direction) at (a) point P_1 at radial distance $r_1 = 6.00$ cm and (b) point P_2 at radial distance $r_2 = 12.0$ cm?



Example 23.03 II

Solution: a)

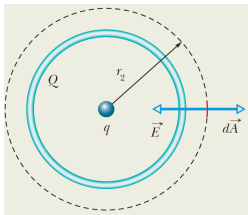


$q_{enc} = 5e$ and $r = r_1 = 6.00 \times 10^{-2}m$. Thus we have

$$\begin{aligned} E &= \frac{q_{enc}}{4\pi\epsilon_0 r^2}, \\ &= \frac{5(1.60 \times 10^{-19}C)}{4\pi(8.85 \times 10^{-12}C^2/N \cdot m^2)(0.0600m)^2}, \\ &= 2.00 \times 10^{-6}N/C. \end{aligned}$$

Example 23.03 III

Solution: b)



In this case we have $q_{enc} = q + Q = 5e + () - 16e = -11e$. Because the net charge is negative, the electric field vectors on the sphere's surface pierce inward the angle θ between and dA is 180° . With $r_2 = 12.00 \times 10^{-2}m$ we obtain

$$\begin{aligned} E &= \frac{-q_{enc}}{4\pi\epsilon_0 r^2}, \\ &= \frac{-[-11(1.60 \times 10^{-19}C)]}{4\pi(8.85 \times 10^{-12}C^2/N \cdot m^2)(0.120m)^2}, \\ &= 1.10 \times 10^{-6}N/C. \end{aligned}$$

Example 23.04

What is the net charge enclosed by the Gaussian cube of Sample Problem 23.02?



Example 23.04

What is the net charge enclosed by the Gaussian cube of Sample Problem 23.02?

- **Solution:** We already know the flux through the right face ($\Phi_r = 36N \cdot m^2/C$), the left face ($\Phi_l = -12N \cdot m^2/C$), and the top face ($\Phi_t = 16N \cdot m^2/C$). For the bottom face, our calculation is just like that for the top face except that the element area vector $d\vec{A}$ is now directed downward along the y axis (recall, it must be outward from the Gaussian enclosure). Thus, we have $d\vec{A} = -dA\hat{i}$, and we find

$$\Phi_b = -16N \cdot m^2/C. \quad (1)$$

For the front face we have $d\vec{A} = dA\hat{k}$, and for the back face, $d\vec{A} = -dA\hat{k}$. When we take the dot product of the given electric field with either of these expressions for $d\vec{A}$, we get 0 and thus there is no flux through those faces. We can now find the total flux through the six sides of the cube:

$$\begin{aligned}\Phi &= (36 - 12 + 16 - 16 + 0 + 0)N \cdot m^2/C, \\ &= 24N \cdot m^2/C.\end{aligned}$$

Next, we use Gauss' law to find the charge q_{enc} enclosed by the cube:

$$\begin{aligned}q_{enc} &= \epsilon_0 \Phi = (8.85 \times 10^{-12} C^3/N \cdot m^2)(24N \cdot m^2/C), \\ &= 2.1 \times 10^{-12} C.\end{aligned}$$

Thus, the cube encloses a net positive charge.



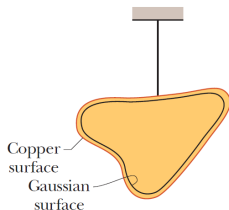
A Charged Isolated Conductor I

Gauss' law permits us to prove an important theorem about conductors:

Theorem

If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

A Charged Isolated Conductor II



The electric field inside this conductor must be zero. If this were not so, the field would exert forces on the conduction (free) electrons, which are always present in a conductor, and thus current would always exist within a conductor. (That is, charge would flow from place to place within the conductor.) Of course, there is no such perpetual current in an isolated conductor, and so the internal electric field is zero. If $\vec{E}$ is zero everywhere inside our copper conductor, it must be zero for all points on the Gaussian surface because that surface, though close to the surface of the conductor, is definitely inside the conductor. This means that the flux through the Gaussian surface must be zero. Gauss' law then tells us that the net charge inside the Gaussian surface must also be zero. Then because the excess charge is not inside the Gaussian surface, it must be outside that surface, which means it must lie on the actual surface of the conductor.

A Charged Isolated Conductor III

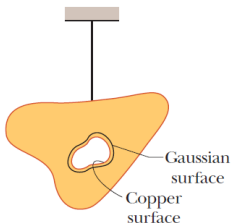
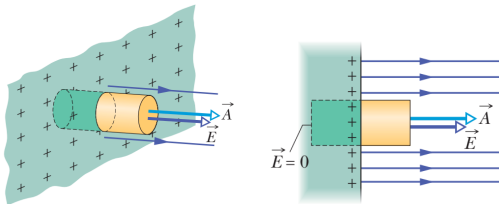
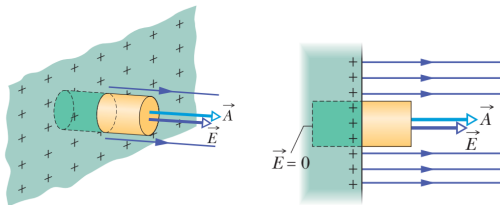


Figure 23-11b shows the same hanging conductor, but now with a cavity that is totally within the conductor. It is perhaps reasonable to suppose that when we scoop out the electrically neutral material to form the cavity, we do not change the distribution of charge or the pattern of the electric field that exists in Fig. 23-11a. Again, we must turn to Gauss' law for a quantitative proof. We draw a Gaussian surface surrounding the cavity, close to its surface but inside the conducting body. Because inside the conductor, there can be no flux through this new Gaussian surface. Therefore, from Gauss' law, that surface can enclose no net charge. We conclude that there is no net charge on the cavity walls; all the excess charge remains on the outer surface of the conductor, as in Fig. 23-11a.

Consider one side of a conductor plan like



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For its electric field with the aim of Gauss' law we have

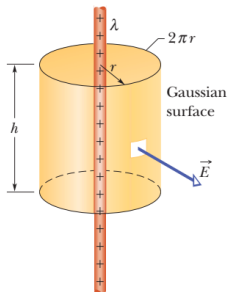
$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = \oint EdA \cos 0^\circ = \oint EdA = q_{enc}.$$

Considering a constant electric field results in

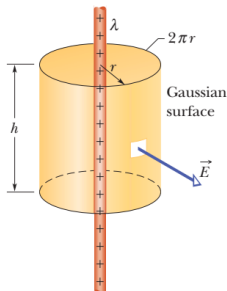
$$E \oint dA = EA = q_{enc} = \sigma A,$$

$$\Rightarrow E = \frac{\sigma}{\epsilon_0}.$$

Applying Gauss' Law: Cylindrical Symmetry



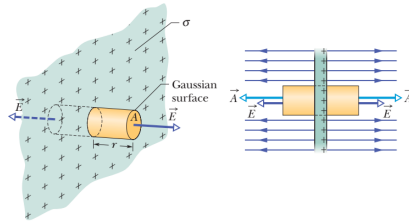
Applying Gauss' Law: Cylindrical Symmetry



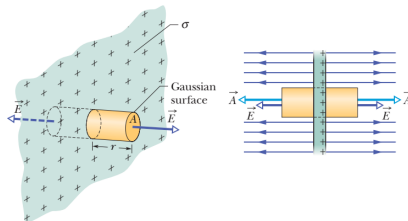
$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = \oint E dA \cos 0^\circ = E \oint dA = E(2\pi rh) = q_{enc} = \lambda h,$$

$$E = \frac{\lambda}{2\pi \epsilon_0 r}.$$

Applying Gauss' Law: Planar Symmetry



Applying Gauss' Law: Planar Symmetry

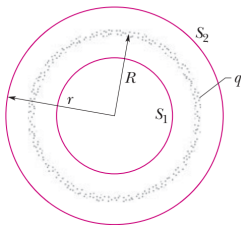


$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc},$$

$$\epsilon_0 (EA + EA) = \sigma A,$$

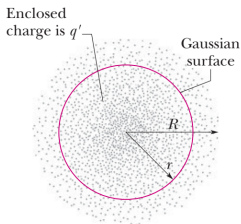
$$\Rightarrow E = \frac{\sigma}{2\epsilon_0}.$$

Applying Gauss' Law: Spherical Symmetry I



$$E = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}, & r \geq R, \\ 0, & r < R. \end{cases}$$

Applying Gauss' Law: Spherical Symmetry II



$$E = \frac{1}{4\pi\epsilon_0} \frac{q'}{r^2}, \quad r \leq R$$

If the full charge q enclosed within radius R is uniform, then q' enclosed within radius r in Fig. 23-21b is proportional to q :

$$\frac{\text{charge enclosed by sphere of radius } r}{\text{volume enclosed by sphere of radius } r} = \frac{\text{full charge}}{\text{full volume}}$$

$$\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{q}{q'} \Rightarrow q' = q \frac{r^3}{R^3}.$$

$$\Rightarrow E = \left(\frac{q}{4\pi\epsilon_0 R^3} \right) r, \quad r \leq R.$$

The End