درس ریاضی 1

مدرّس : دکترمغاری

$$\int \frac{\sin x \, dx}{\sin^{۲} x + \cos x}$$

$$= \int \frac{\sin x \, dx}{۱ - \cos^{۲} x + \cos x} \qquad \cos x = u$$

$$-\sin x \, dx = du$$

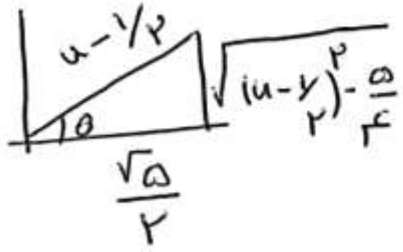
$$= \int \frac{-du}{۱ - u^{۲} + u} = \int \frac{du}{u^{۲} - u - ۱} = \int \frac{du}{(u - \frac{۱}{۲})^{۲} - \frac{۵}{۴}}$$

$$u - {}^{۱}\!/_{۲} = \frac{\sqrt{۵}}{۲} \sec\theta \quad \rightarrow du = \frac{\sqrt{۵}}{۲} \sec\theta \tan\theta \, d\theta$$

$$\int \frac{\frac{\sqrt{۵}}{۲} \sec\theta \tan\theta \, d\theta}{{}^{۵}\!/_{۴} \sec^{۲}\theta - {}^{۵}\!/_{۴}} = \frac{۲\sqrt{۵}}{۵} \int \frac{\sec\theta \tan\theta}{\tan^{۲}\theta} \, d\theta$$

$$= \frac{۲\sqrt{۵}}{۵}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta = \frac{۲\sqrt{۵}}{۵}\int \csc\theta\, d\theta$$

$$= \frac{۲\sqrt{۵}}{۵}\ln(\csc\theta - \cot\theta)$$



$$= \frac{۲\sqrt{۵}}{۵}\ln\left(\frac{u-۱/۲}{\sqrt{(u-۱/۲)^۲-\frac{۵}{۴}}} - \frac{\frac{\sqrt{۵}}{۲}}{\sqrt{(u-۱/۲)^۲-\frac{۵}{۴}}}\right)$$

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{\frac{1}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}}$$

$$= \int \frac{(\sec^2\theta)\,d\theta}{1+tg^2\theta+tg^2\theta} \overset{\tan\theta = t}{=} \int \frac{dt}{1+2t^2}$$

$$= \int \frac{dt}{2t^2+1} = \frac{1}{2}\int \frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}\,t) + c$$

$t \to \tan\theta$

$$\int \frac{\sqrt{x} + \sqrt[3]{x}}{\sqrt[4]{x^5} + \sqrt[6]{x^7}}\,dx$$

$x = t^{12}$

$dx = 12\,t^{11}\,dt$

$$\int \frac{t^6 + t^4}{t^{15} + t^{14}}\left(12\,t^{11}\,dt\right)$$

$$= 12 \int \frac{t^4(1+t^2)}{t^{14}(t+1)}\,t^{11}\,dt = 12 \int \frac{t^3+t}{1+t}\,dt$$

$$= 12 \int t^2 - t + 2 + \frac{-2}{t+1}\,dt$$

$$= 12\left(t^3/3 - t^2/2 + 2t - 2\ln(t+1)\right)$$

$$\int \frac{dx}{e^{x}+e^{2x}} \qquad e^{x}=t$$

$$e^{x}dx=dt \rightarrow dx=\frac{dt}{e^{x}}=\frac{dt}{t}$$

$$\int \frac{\frac{dt}{t}}{t+t^{2}}=\int \frac{dt}{t^{2}+t^{3}}=\int \frac{dt}{t^{2}(t+1)}$$

$$=\int \left(\frac{A}{t}+\frac{B}{t^{2}}+\frac{C}{t+1}\right)dt$$

$$1 = At(t+1) + B(t+1) + Ct^2$$

$$\begin{cases} t=0 & \to B=1 \\ t=-1 & \to C=1 \end{cases}$$

$$t=1 \quad \to A=-1$$

e^x e^x

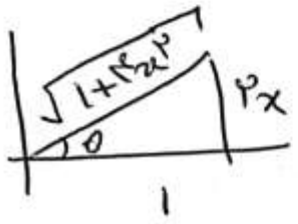
$$= -\ln(t) - \frac{1}{t} + \ln(t+1) + C$$

e^x

$$\int \sqrt{1+4x^2}\,dx$$

$$2\,dx = (\sec^2\theta)\,d\theta$$

$$2x = \tan\theta$$



$$= \int \sqrt{1+(2x)^2}\,dx$$

$$= \int \sqrt{1+tg^2\theta}\left(\frac{1}{2}\sec^2\theta\,d\theta\right)$$

$$= \frac{1}{2}\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\frac{\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)}{2}\right)$$

$$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left(\sqrt{1+4x^2}+2x\right)\right)$$

$$\int \frac{e^{x}\,dx}{\sqrt{e^{۲x}+۹}} \qquad e^{x} = ۳\sinh t$$

$$e^{x}\,dx = ۳\cosh t\,dt$$

$$\int \frac{۳\cosh t\,dt}{\sqrt{۹\sinh^{۲} + ۹}} = \int \frac{۳\cosh t\,dt}{۳\cosh t}$$

$$= t + c = \sinh^{-1}\left(\frac{e^{x}}{۳}\right)$$

$$\int \frac{dx}{\cos^2 x - \sin^2 x + 3\sin x\cos x}$$

$$\int \frac{\frac{dx}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} + \frac{3\sin x\cos x}{\cos^2 x}}$$

$$= \int \frac{(\sec^2\theta)\, d\theta}{1 - \tan^2\theta + 3\tan\theta} \xrightarrow[\sec^2\theta\, d\theta = du]{\tan\theta = u}$$

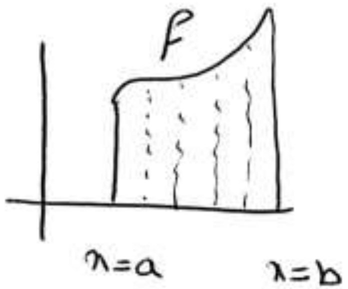
$$\int \frac{du}{-u^{2}+3u+1} = \int \frac{du}{-\left(u-\frac{3}{2}\right)^{2}+\frac{13}{4}}$$

$$= \int \frac{du}{\left(\frac{\sqrt{13}}{2}-u+\frac{3}{2}\right)\left(\frac{\sqrt{13}}{2}+u-\frac{3}{2}\right)}$$

$$= \frac{1}{\sqrt{13}} \int \frac{1}{\left(\frac{\sqrt{13}+3}{2}\right)-u} + \frac{1}{\left(\frac{\sqrt{13}-3}{2}\right)+u} \, du$$

$$= \frac{1}{\sqrt{14}} \ln \frac{u+\frac{\sqrt{13}-3}{2}}{\frac{\sqrt{13}+3}{2}-u}$$

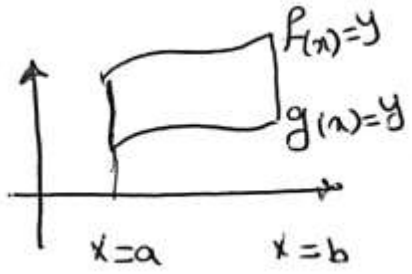
کاربرد انتگرال :



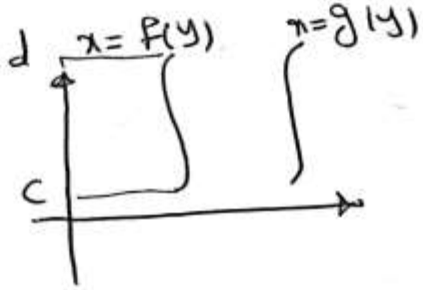
$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

$$= \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$$

$$= \int_a^b f(x)\,dx$$



$$A = \int_a^b f(x) - g(x)\, dx$$



$$A = \int_c^d \left(g(y) - f(y)\right) dy$$

مساحت محصور بین $x=y^2$ و $x=2y^2-y-2$ را بدست آورید.

$$2y^2-y-2=y^2 \longrightarrow y^2-y-2=0$$

$$(y-2)(y+1)=0$$

$$\int_{-1}^{2} y^2-(2y^2-y-2)\,dy=\int_{-1}^{2} -y^2+y-2\,dy$$

$$=\dots$$

سطح محصور بین $x-y=۷$ و $۲y^{۲}-y+۳-x=۰$ را بدست آورید

$$\begin{cases} x = ۷ + y \\ x = ۲y^{۲} - y + ۳ \end{cases}$$

$$۷ + y = ۲y^{۲} - y + ۳ \Rightarrow ۲y^{۲} - ۲y - ۴ = ۰$$

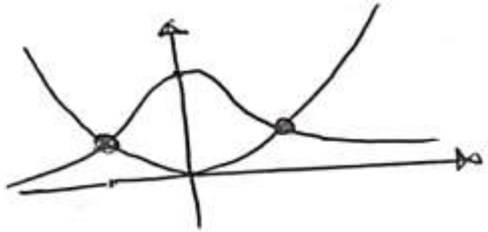
$$۲(y^{۲} - y - ۲) = ۰$$

$$y = ۲ , y = -۱$$

$$A = \int_{-۱}^{۲} (۷ + y) - (۲y^{۲} - y + ۳)\, dy = \cdots$$

مساحت محصور بین $x^2=4y$ و $y=\dfrac{8}{x^2+4}$ را بدست آورید.

$$\begin{cases} y=\dfrac{x^2}{4} \\ y=\dfrac{8}{x^2+4} \end{cases}$$



$$\frac{x^2}{4}=\frac{8}{x^2+4}$$

$$x^4+4x^2-32=0$$

$$x^2=\frac{-4\pm\sqrt{16+4(1)(-32)}}{2}$$

$$x^2=\frac{-4\pm 12}{2}=\begin{cases} x^2=4 & \text{قق} \\ x^2=-8 & \text{غقق} \end{cases}$$

تابع زوج

$$A = \int_{-۲}^{۲} \left(\frac{۸}{x^{۲}+۴} - \frac{x^{۲}}{۴} \right) dx$$

$$= ۲ \int_{۰}^{۲} \frac{۸}{x^{۲}+۴} - \frac{x^{۲}}{۴} \, dx$$

$$= ۲ \left(۸ \cdot \frac{۱}{۲} \tan^{-1}\left(\frac{x}{۲}\right) - \frac{۱}{۴} \frac{x^{۳}}{۳} \right) \Bigg]_{۰}^{۲} = \cdots$$

مساحت بیضی به معادله $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ را پیدا کنید.

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad 0 \le t \le 2\pi$$

نکته: در حالت

$$A = \int_a^b |f(t)g'(t)|\,dt \; : \begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad a \le t \le b$$

$$= \int_0^{2\pi} (a\cos t)(b\cos t)\,dt$$

$$= ab\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = ab\left(\frac{t}{2} + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \pi ab$$

تعریف : فرض کنید $r = f(\theta)$ معادله منحنی در حالت قطبی باشد

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

مثال : مساحت ناحیه محصور به $r = e^{2\theta}$ بدست آورید $(0 < \theta < 2\pi)$

$$A = \frac{1}{2} \int_{0}^{2\pi} (e^{2\theta})^2 d\theta = \frac{1}{2} \left(\frac{1}{4} e^{4\theta} \right)_{0}^{2\pi}$$

$$= \frac{1}{8} (e^{8\pi} - 1)$$

تعریف: طول قوس که با L نمایش داده می‌شود.

$$L=\int_a^b \sqrt{1+y'^2}\,dx \qquad y=f(x) \quad (1$$

$$L=\int \sqrt{1+x'^2}\,dy \qquad x=g(y) \quad (2$$

$$L=\int_a^b \sqrt{f'^2_{(t)}+g'^2_{(t)}}\,dt \qquad \begin{cases} x=f(t) \\ y=g(t) \end{cases} \quad a\le t\le b \quad (3$$

محاسبه طول قوس در حالت قطبی

$$r = f(\theta)$$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(dr)^2 + r^2 (d\theta)^2}$$

مثال: طول قوس (طول منحنی) $y=\ln \cos x$ در $0 \leq x \leq \frac{\pi}{4}$ را بیابید.

$$y' = \frac{-\sin x}{\cos x}$$

$$1+y'^2 = 1+tg^2 x = \sec^2 x$$

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx =$$

$$= \ln(\sec x + \tan x)\Big]_0^{\pi/4} = \cdots$$

طول قوس

را بیابید . $\begin{cases} x = e^{t}\cos t \\ y = e^{t}\sin t \end{cases}$ $0 \le t \le ۳\pi$

$$x'(t) = e^{t}\cos t - \sin t\, e^{t} = e^{t}(\cos t - \sin t)$$

$$y'(t) = e^{t}\sin t + \cos t\, e^{t} = e^{t}(\sin t + \cos t)$$

$$x'^{2}(t) + y'^{2}(t) = e^{۲t}\big(\cos^{۲} t + \sin^{۲} t - ۲\sin t\cos t$$

$$\sin^{۲} t + \cos^{۲} t + ۲\sin t\cos t\big)$$

$$= ۲e^{۲t}$$

$$x'^{2}(t)+y'^{2}(t)=e^{۲t}\Big(\cos^{۲}t+\sin^{۲}t-۲\sin t\cos t$$

$$\sin^{۲}t+\cos^{۲}t+۲\sin t\cos t\Big)$$

$$=۲e^{۲t}$$

$$L=\int_{۰}^{۴}\sqrt{۲e^{۲t}}\,dt=\sqrt{۲}\,e^{t}\Big)_{۰}^{۴}$$

$$=\sqrt{۲}\,(e^{۴}-1)$$

طول قوس $r = a\sin^3\frac{\theta}{3}$ که $0 \le \theta \le \pi$ است را محاسبه کنید.

$$dr = 3a\sin^2\frac{\theta}{3}\left(\frac{1}{3}\cos\frac{\theta}{3}\right)d\theta$$

$$(dr)^2 = a^2 \sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}(d\theta)^2$$

$$(dr)^{۲} + r^{۲}(d\theta)^{۲} =$$

$$= a^{۲} \sin^{۴}\frac{\theta}{۳} \cos^{۲}\frac{\theta}{۳}(d\theta)^{۲} + a^{۲} \sin^{۶}\frac{\theta}{۳}(d\theta)^{۲}$$

$$= a^{۲} \sin^{۴}\frac{\theta}{۳}\left(\cos^{۲}\frac{\theta}{۳} + \sin^{۲}\frac{\theta}{۳}\right)(d\theta)^{۲}$$

$$\sqrt{(dr)^{۲} + r^{۲}(d\theta)^{۲}} = a \sin^{۲}\frac{\theta}{۳}\, d\theta$$

$$L = \int_{۰}^{\pi} a \sin^{۲}\frac{\theta}{۳}\, d\theta$$

$$L = \int_{0}^{\pi} a \sin^{2}\frac{\theta}{3}\, d\theta$$

$$= \int_{0}^{\pi} \frac{a\left(1 - \cos\frac{2\theta}{3}\right)}{2}\, d\theta$$

$$= \frac{a}{2}\left(\theta - \frac{3}{2}\sin\frac{2\theta}{3}\right)\Big|_{0}^{\pi}$$

$$= \frac{a}{2}\left(\left(\pi - \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\right) - \left(0 - 0 \right)\right) = \cdot$$

موفق باشید