درس ریاضی 1

مدرّس : دکترمغاری

انتگرال گیری از توابع گویا (روش تفکیک کسر)

فرض کنید $\int \frac{P(x)}{Q(x)}dx$ را محاسبه کنیم به طوریکه:

درجه مخرج کسر $>$ درجه صورت

درحالتی که درجه مخرج $\leq$ درجه صورت باشد:

$$\begin{array}{l|l} P(x) & Q(x) \\ \cline{2-2} & q(x) \\ \hline r(x) & \end{array} \qquad \int \frac{P}{Q} = \int q + \int \frac{r}{Q}$$

حالت اول : $Q(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$

$$\frac{P}{Q} = \frac{A}{x-\alpha_1} + \frac{B}{x-\alpha_2} + \cdots + \frac{C}{x-\alpha_k}$$

حالت دوم : $Q(x) = (x-\alpha_1)^k (x-\alpha_2)$

$$\frac{P}{Q} = \frac{A}{(x-\alpha_1)^1} + \frac{B}{(x-\alpha_1)^2} + \cdots + \frac{E}{(x-\alpha_1)^k} + \frac{F}{x-\alpha_2}$$

حالت سوم :

$$Q=\underbrace{(ax^2+bx+c)}_{\Delta<0}\underbrace{(ex^2+fx+k)}_{\Delta<0}$$

$$\frac{P}{Q}=\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{ex^2+fx+k}$$

حالت چهارم :

$$Q=(ax^2+bx+c)^k$$

$$\frac{P}{Q}=\frac{A_1x+B_1}{(ax^2+bx+c)^1}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

$$\int \frac{5x-2}{x^2-4}\,dx$$

$$\frac{5x-2}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$5x-2 = A(x+2) + B(x-2)$$

$$\begin{cases} x=2 \\ x=-2 \end{cases} \Rightarrow \begin{array}{ll} 8 = 4A & \boxed{A=2} \\ -12 = -4B & \boxed{B=3} \end{array}$$

$$\int \frac{۵x-۲}{x^{۲}-۴}\,dx = \int \frac{۲}{x-۲}\,dx + \int \frac{۳}{x+۲}\,dx$$

$$= ۲\ln(x-۲) + ۳\ln(x+۲) + c$$

$$= \ln (x-۲)^{۲}(x+۲)^{۳} + c$$

$$\int \frac{x+۱}{x^۳+x^۲-۶x}\,dx$$

$$\int \frac{x+۱}{x(x-۲)(x+۳)}\,dx=\int \frac{\overset{-\frac{۱}{۶}}{A}}{x}\,dx+\int \frac{\overset{+\frac{۳}{۱۵}}{B}}{x-۲}\,dx+\int \frac{\overset{-\frac{۲}{۱۵}}{C}}{x+۳}\,dx$$

$$x+۱=A(x-۲)(x+۳)+B(x)(x+۳)+C(x)(x-۲)$$

$x=۲ \qquad ۳=۱۰B \qquad B=\frac{۳}{۱۰}$

$x=-۳ \qquad -۲=۱۵C \qquad C=-\frac{۲}{۱۵}$

$x=۰ \rightarrow ۱=-۶A \qquad A=\frac{-۱}{۶}$

$$=\frac{-۱}{۶}\ln x+\frac{۳}{۱۰}\ln(x-۲)-\frac{۲}{۱۵}\ln(x+۳)+c$$

$$\int \frac{dx}{x^3+3x^2} = \int \frac{dx}{x^2(x+3)}$$

$$= \int \left(\frac{A \to -\frac{1}{9}}{x} + \frac{B \to \frac{1}{3}}{x^2} + \frac{C \to \frac{1}{9}}{x+3} \right) dx$$

$$1 = Ax(x+3) + B(x+3) + Cx^2$$

اجباری

$x = 0 \qquad 1 = 3B \rightarrow B = \frac{1}{3}$

$x = -3 \qquad 1 = 9C \rightarrow C = \frac{1}{9}$

اختیاری $x = 1 \qquad 1 = 4A + 4\left(\frac{1}{3}\right) + \frac{1}{9}(1)$

$$A = -\frac{1}{9}$$

$$= \frac{-1}{9} \ln x + \frac{1}{3} \frac{x^{-2+1}}{-2+1} + \frac{1}{9} \ln(x+3)$$

$$\int \frac{dx}{x(x+1)^{2}} = \int \left(\frac{\overset{1}{A}}{x} + \frac{\overset{-1}{B}}{(x+1)^{1}} + \frac{\overset{-1}{\gamma}}{(x+1)^{2}} \right) dx$$

$$= \ln x - \ln(x+1) + 1\,\frac{1}{(x+1)}$$

$$1 = A(x+1)^{2} + Bx(x+1) + Cx$$

$$\begin{cases} x = 0 \;\rightarrow\; \underline{A = 1} \\ x = -1 \;\rightarrow\; 1 = -C \quad \underline{C = -1} \end{cases}$$

$$x = 1 \quad \text{---} \quad B = -1$$

$$\int \frac{u^2+2}{u^3+1}\,du$$

$$\int \frac{u^2+2}{u^3+1}\,dx = \int \left(\frac{\overset{1}{A}}{u+1} + \frac{\overset{0}{B}u+\overset{1}{C}}{u^2-u+1} \right) du$$

$$\Delta < 0$$

$$u^2+2 = A(u^2-u+1) + Bu(u+1) + C(u+1)$$

$$\begin{cases} u=0 \rightarrow & 2 = A + C & C=1 \\ u=-1 & 3 = 3A & \boxed{A=1} \end{cases}$$

$$u=1 \qquad 3 = 1(1) + 2B + 2(1)$$

$$\boxed{B=0}$$

$$\int \frac{1}{u+1}\,du + \int \frac{1}{\underbrace{u^2-u}_{}+1}\,du$$

$$= \ln(u+1) + \int \frac{1}{\underbrace{\left(u-\frac{1}{2}\right)^2-\frac{1}{4}}_{}+1}\,du$$

$$= \ln(u+1) + \int \frac{du}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \ln(u+1) + \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\int \frac{dx}{(x-1)(x^2+1)} = \int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1} \right) dx$$

$$1 = A(x^2+1) + Bx(x+1) + C(x+1)$$

$$\begin{cases} x=0 & \rightarrow 1 = A + C \qquad C = 1 - \frac{1}{2} = \frac{1}{2} \\ x=-1 & \rightarrow 1 = 2A \qquad A = \frac{1}{2} \end{cases}$$

$$x=1 \rightarrow \quad 1 = \frac{1}{2}(2) + B(2) + \frac{1}{2}(2)$$

$$-1 = 2B \qquad B = \frac{-1}{2}$$

$$\int \frac{1/2}{x+1} + \frac{-1/2\,x + 1/2}{x^2+1}\, dx$$

$$= \frac{1}{2}\ln(x+1) + -1/2 \int \frac{x}{x^2+1}\,dx + 1/2 \int \frac{dx}{x^2+1}$$

$$'' \qquad + -1/2\left(\frac{1}{2}\ln(x^2+1)\right) + 1/2\,\tan^{-1} x$$

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \int\left(\frac{Ax+B}{(x^2+1)^1} + \frac{Cx+D}{(x^2+1)^2}\right)dx$$

$$2x^2+3 = (Ax+B)(x^2+1) + Cx + D$$

$$2x^2+3 = Ax^3 + (B)x^2 + (A+C)x + (B+D)$$

ضریب x^3 دو طرف: $A=0$

ضریب x^2 دو طرف: $2=B$

" x ": $A+C=0 \rightarrow C=0$ ✓

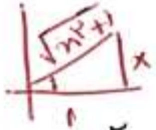
ثابت دو طرف: $3=B+D \rightarrow D=1$ ✓

$$\int \frac{2}{x^2+1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$

$\bigstar$

$$= 2\tan^{-1}x + \bigstar$$

$$\bigstar \xrightarrow[dx=\sec^2\theta]{x=\tan\theta} \int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}$$



$$= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \frac{1}{\frac{1}{\cos^2\theta}}\,d\theta = \int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right) + c$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{4}\left(2\,\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right) + c$$

انتگرال گیری از توابع گویایی از $\sin\theta$ ، $\cos\theta$

در این توابع از تغییر متغیر $z=\tan\frac{\theta}{2}$ ، $-\pi<\theta<\pi$ استفاده می‌شود.

$$\sin\theta=\frac{2z}{1+z^2} \qquad dz=\frac{1}{2}\left(1+tg^2\frac{\theta}{2}\right)d\theta$$

$$\cos\theta=\frac{1-z^2}{1+z^2} \qquad d\theta=\frac{2dz}{1+z^2}$$

$$\int \frac{dx}{2\sin x + 2\cos x + 3}$$

$$z = \tan \frac{x}{2}, \qquad \begin{cases} \sin x = \dfrac{2z}{1+z^2} \\[2ex] \cos x = \dfrac{1-z^2}{1+z^2} \end{cases}, \qquad dx = \frac{2dz}{1+z^2}$$

$$= \int \frac{\dfrac{2dz}{1+z^2}}{2\left(\dfrac{2z}{1+z^2}\right) + 2\left(\dfrac{1-z^2}{1+z^2}\right) + 3}$$

$$= \int \frac{2dz/1+z^2}{\dfrac{4z+2-2z^2+3+3z^2}{1+z^2}} = \int \frac{2dz}{z^2+4z+5}$$

$$= \int \frac{2dz}{(z+2)^2+1} = 2\,\mathrm{tag}^{-1}(z+2) + c$$

$$= 2\,\mathrm{tag}^{-1}\left(\tan \frac{x}{2} + 2\right)$$

$$\int \frac{dx}{۳\sec x+۲}=\int \frac{dx}{۳\left(\frac{1}{\cos x}\right)+۲}=\int \frac{\cos x\,dx}{۳+۲\cos x}$$

$$=\int \frac{\left(\frac{1-z^۲}{1+z^۲}\right)\left(\frac{۲dz}{1+z^۲}\right)}{۲\left(\frac{1-z^۲}{1+z^۲}\right)+۳}=\int \frac{\frac{۲(1-z^۲)}{(1+z^۲)^۲}dz}{\frac{۲-۲z^۲+۳+۳z^۲}{z^۲+1}}$$

$$= ۲\int \frac{۱ - z^{۲}}{(۵+z^{۲})(z^{۲}+۱)} dz = \int \frac{Az+B}{۵+z^{۲}} + \frac{Cz+D}{z^{۲}+۱} dz$$

$$۲ - ۲z^{۲} = (Az+B)(z^{۲}+۱) + (Cz+D)(۵+z^{۲})$$

$$۲ - ۲z^{۲} = (A+C)z^{۳} + (B+D)z^{۲}$$

$$+ (A+۵C)z + (B+۵D)$$

$$\Rightarrow A = ۰ \quad \rightarrow B = -۳ \;,\; C = ۰ \;,\; D = ۱$$

$$\int \frac{-۳}{۵ + z^{۲}}\, dz + \int \frac{dz}{z^{۲} + ۱}$$

$$= \frac{-۳}{۱} \cdot \frac{۱}{\sqrt{۵}}\, \tan^{-1}\left(\frac{\overset{tg\, x/۲}{z}}{\sqrt{۵}}\right) + tg^{-1}\left(\overset{tg\, x/۲}{z}\right)$$

$$\int \operatorname{sech} x \, dx = \int \frac{1}{\cosh x} dx = \int \frac{2}{e^{x} + e^{-x}} dx$$

$$= \int \frac{2e^{x} \, dx}{e^{2x} + 1} \xrightarrow[e^{x} dx = du]{e^{x} = u} 2 \int \frac{du}{u^{2} + 1}$$

$$= 2 \, tg^{-1} u = 2 \tan^{-1}(e^{x}) + c$$

$$\int \operatorname{csch} x\,dx = \int \frac{1}{\sinh x}\,dx$$

$$= \int \frac{2}{e^{x} - e^{-x}}\,dx = \int \frac{2e^{x}\,dx}{e^{2x} - 1}$$

$$\xrightarrow[e^{x}dx = du]{e^{x} = u} \int \frac{2\,du}{u^{2}-1} = \int \frac{\overset{1}{\cancel{A}}}{u-1} + \frac{\overset{-1}{\cancel{B}}}{u+1}\,du$$

$$= \ln u-1 - \ln(u+1) = \ln\left(\frac{\overset{e^{x}}{u}-1}{\underset{e^{x}}{u}+1}\right)$$

$z = tg\, x/2$

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx \overset{*}{=} \int \frac{\frac{2\,dz}{1+z^2}}{\frac{1-z^2}{1+z^2}}$$

$$\int \frac{2\,dz}{1-z^2} = \int \frac{\overset{1}{\cancel{A}}}{1-z} + \frac{\overset{+1}{\cancel{B}}}{1+z}\, dz$$

$$2 = A(1+z) + B(1-z)$$

$z = 1 \qquad A = 1$

$z = -1 \qquad B = +1$

$$= -\ln(1-z) + \ln(1+z)$$

$tg\frac{x}{2}$

$$= \ln\left(\frac{z+1}{1-z}\right)$$

$$\int \frac{x^4+1}{x^3+2x^2-x-2}\,dx = \int \left(x-2 + \frac{5x^2-3}{x^3+2x^2-x-2}\right)dx$$

$$\begin{array}{l|l}
\cancel{x^4}+1 & x^3+2x^2-x-2 \\ \hline
-\cancel{x^4} - 2x^3 + x^2 + 2x & x-2 \\ \hline
-2\cancel{x^3}+1+x^2+\cancel{2x} & \\
+\cancel{2x^3}+4x^2-\cancel{2x}-4 & \\ \hline
5x^2-3 &
\end{array}$$

$$\frac{P}{Q} = \text{خارج قسمت} + \frac{\text{باقی مانده}}{\text{مقسوم علیه}}$$

$$= \frac{x^2}{2} - 2x + \int \frac{5x^3 - 3}{(x-1)(x+1)(x+2)}$$

$$*$$

$$* = \int \left(\frac{\overset{-1}{\cancel{A}}}{x+1} + \frac{\overset{\frac{1}{3}}{\cancel{B}}}{x-1} + \frac{C \rightarrow \frac{17}{3}}{x+2} \right) dx$$

$$-\ln(x+1) + \frac{1}{3}\ln(x-1) + \frac{17}{3}\ln(x+2)$$

تعریف انتگرال ناسره :

$$\int_{0}^{\infty} f(t)\,dt = \lim_{b\to\infty} \int_{0}^{b} f(t)\,dt$$

مثال : اگر $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$ آنگاه $\int_{0}^{\infty} x^2 e^{-x^2}\,dx$ را بیابید.

می‌دانیم : $f(x) = e^{-x^2}$ تابع زوج است.

از طرفی طبق ویژگی انتگرال ها داریم :

$$\int_{-a}^{a} \text{زوج}\, dx = 2\int_{0}^{a} \text{زوج}\, dx$$

$$\int_{-a}^{a} \text{فرد}\, dx = 0$$

پس داریم :

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = 2\int_{0}^{\infty} e^{-x^2} dx$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^{۲}}\,dx = \sqrt{\pi}$$

$$۲\int_{۰}^{\infty} e^{-x^{۲}}\,dx = \sqrt{\pi} \Longrightarrow$$

$$\boxed{\int_{۰}^{\infty} e^{-x^{۲}}\,dx = \sqrt{\pi}/۲}$$

$$\int_{۰}^{\infty} x^{۲} e^{-x^{۲}}\,dx = \lim_{b\to\infty} \int_{۰}^{b} x^{۲} e^{-x^{۲}}\,dx$$

$$= \lim_{b\to\infty} \int \underbrace{x}_{u} \cdot \underbrace{x\, e^{-x^{۲}}\, dx}_{dv} \xrightarrow{\begin{cases} u = x \\ dv = x e^{-x^{۲}} dx \end{cases}}$$

$$du = dx$$

$$v = -\frac{۱}{۲} e^{-x^{۲}}$$

$$= \lim_{b\to\infty} \left(-\frac{x}{۲} e^{-x^{۲}} \Big]_{۰}^{b} + \frac{۱}{۲} \int_{۰}^{b} e^{-x^{۲}}\, dx \right)$$

$$= \left(\underbrace{-\frac{b}{۲}\, e^{-b^{۲}}}_{۰} + {}^{۱}\!/_{۲} \int_{۰}^{\infty} e^{-x^{۲}}\, dx \right) - (\,۰\,)$$

$$= \frac{۱}{۲} \left(\frac{\sqrt{\pi}}{۲} \right) = \frac{\sqrt{\pi}}{۴}$$

$$\int \frac{dx}{\sqrt[3]{x}+1}$$

ک م م فرجه

$x = u$

$x = u^3$

$dx = 3u^2\,du$

$$\int \frac{3u^2\,du}{u+1} = 3\int \frac{(u^2-1)+1}{u+1}\,du$$

$$= 3\int (u-1) + 3\int \frac{1}{u+1}\,du$$

$$= 3\,\frac{\overset{\sqrt[3]{x}}{u}{}^2}{2} - 3\overset{\sqrt[3]{x}}{u} + 3\ln\left(\overset{\sqrt[3]{x}}{u}+1\right)$$

$$\int \frac{dx}{\sqrt{x}\,(\sqrt[3]{x})(1+\sqrt[3]{x})^{2}}$$

$$x = z^{\text{ک.م.م}(2,3)} = z^{6}$$

$$x = z^{6}$$

$$dx = 6z^{5}\,dz$$

$$\int \frac{6z^{5}\,dz}{z^{3}\,z^{2}\,(1+z^{2})^{2}} = \int \frac{6\,dz}{(z^{2}+1)^{2}}$$

$$\xrightarrow[dz=\sec^{2}x\,dx]{z=\tan x} 6\int \cos^{2}x\,dx = 6\int \frac{1+\cos 2x}{2}\,dx$$

$$= 3x + \frac{3}{2}\left(2\cdot \frac{x}{\sqrt{x^{2}+1}}\cdot\frac{1}{\sqrt{x^{2}+1}}\right)$$

$x = \overset{6}{z}$ قرار دهید.

موفق باشید