دانشگاه صنعتی شاهرود

درس ریاضی 1

مدرّس : دکترمغاری

$$\int \frac{1}{\sqrt{x+۳}}\,dx = ?$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$ می‌دانیم

$$\int (x+۳)^{-۱/۲}\,dx \quad \frac{x+۳=T}{dx=dT}$$

$$= \int T^{-۱/۲}\,dT = \frac{T^{-۱/۲+۱}}{-۱/۲+۱}$$

$$= \frac{(x+۳)^{۱/۲}}{۱/۲} = ۲\,(x+۳)^{۱/۲}$$

$$\int (x-1)^{15}\, x\, dx \qquad \frac{x-1=T}{dx=dT} \Rightarrow x = T+1$$

$$\int T^{15}(T+1)\,dT$$

$$= \int (T^{16} + T^{15})\,dT = \int T^{16}\,dT + \int T^{15}\,dT$$

$$= \frac{T^{17}}{17} + \frac{T^{16}}{16} = \frac{(x-1)^{17}}{17} + \frac{(x-1)^{16}}{16}$$

$$\int \frac{\sqrt[۳]{\ln x + 1}}{x}\,dx \qquad \ln x + 1 = T \qquad \frac{1}{x}\,dx = dT$$

$$= \int \frac{(\ln x+1)^{\frac{1}{۳}}}{x}\,dx = \int T^{1/۳}\,dT$$

$$= \frac{T^{\frac{1}{۳}+1}}{1/۳+1} = \frac{T^{٤/۳}}{٤/۳} = \frac{۳}{٤}(\ln x+1)^{٤/۳}$$

$$\int_{-1/۲}^{۰} \frac{dx}{۲x+۳} = ؟$$

میدانیم $\int \frac{du}{u} = \ln u$ و

$$\int_a^b f(x)\, dx = F\Big]_a^b = F(b) - F(a)$$

$$\frac{۲x+۳ = T}{۲dx = dT} \quad \frac{1}{2}\int \frac{2\, dx}{۲x+۳} = \frac{1}{۲}\int \frac{dT}{T}$$

$$= \frac{1}{۲} \ln T = \frac{1}{۲} \ln(۲x+۳)\Big]_{-1/۲}^{۰}$$

$$= \left(\tfrac{1}{۲}\ln ۳\right) - \left(\tfrac{1}{۲}\ln ۲\right) = \tfrac{1}{۲}\ln\frac{۳}{۲}$$

$$\int_{e}^{e^{۲}} \frac{dx}{x(\ln x)^{۳}}$$

$$\ln x = T$$

$$\frac{۱}{x}dx = dT$$

x	T
$e^{۲}$	$\ln e^{۲} = ۲$
e	$\ln e = ۱$

$$\int \frac{dT}{T^{۳}} = \int T^{-۳}\, dT = \frac{T^{-۳+۱}}{-۳+۱}$$

$$= \frac{T^{-۲}}{-۲} = \frac{-۱}{۲\, T^{۲}} = \left.\frac{-۱}{۲(\ln x)^{۲}}\right]_{e}^{e^{۲}}$$

$$= \left(\frac{-۱}{۲(\ln e^{۲})^{۲}}\right) - \left(\frac{-۱}{۲(\ln e)^{۲}}\right)$$

$$= \frac{-۱}{۸} + \frac{۱}{۲} = \frac{۳}{۸}$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

$\int e^{u}\,du = e^{u}$ می‌دانیم

$\sqrt{x} = T$

$\frac{1}{۲\sqrt{x}}\,dx = dT$

$$2\int \boxed{\frac{1}{2\sqrt{x}}\,e^{\sqrt{x}}\,dx} = 2\int e^{T}\,dT = ۲e^{T} = ۲e^{(\sqrt{x})}$$

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int -2x\,(1-x^2)^{-1/2}\,dx$$

$$\frac{1-x^2 = T}{-2x\,dx = dT} \Longrightarrow -\frac{1}{2}\int T^{-1/2}\,dT$$

$$= -\frac{1}{2}\left(\frac{T^{-1/2+1}}{-1/2+1}\right) = -\frac{1}{2}\left(2T^{1/2}\right) = -(1-x^2)^{1/2}\Big]_0^1$$

$$= \left(-(1-1)^{1/2}\right) - \left(-(1-0)^{1/2}\right) = +1$$

مثال: اگر $\int_1^2 f(x)\,dx = 3$ آنگاه $\int_{1/2}^{1} \frac{1}{x^2} f\left(\frac{1}{x}\right)dx$ را محاسبه کنید. کافی است

$$\frac{1}{x} = T$$

$$-\frac{1}{x^2}\,dx = dT$$

x	T
1	1
$\frac{1}{2}$	2

$$-\int_{1/2}^{1} \boxed{-\frac{1}{x^2} f\left(\frac{1}{x}\right) dx} = -\int_{2}^{1} f(T)\,dT$$

$$= -\left(-\int_1^2 f(T)\,dT\right) = +\int_1^2 f(T)\,dT = +3$$

$$\int \cos 2x \, e^{\sin 2x} \, dx$$

$$\int e^{u} du = e^{u}$$ می‌دانیم

$$\frac{1}{2}\int \boxed{2\cos 2x \; e^{\sin 2x} \; dx} \qquad \sin 2x = T \qquad 2\cos 2x = dT$$

$$\frac{1}{2}\int e^{T} dT = \frac{1}{2} e^{T} = \frac{1}{2} e^{\sin 2x}$$

$$\int \frac{dx}{x\sqrt{\ln x}} \xrightarrow[\frac{1}{x}dx = dT]{\ln x = T} \int \frac{dT}{T^{۱/۲}}$$

$$= \int T^{-۱/۲}\,dT = \frac{T^{-۱/۲+۱}}{-۱/۲+۱} = ۲\sqrt{(\ln x)}$$

$$-\int \frac{-\sin x}{1+\cos^2 x}\,dx = ؟$$

می دانیم $\int \frac{du}{1+u^2} = tg^{-1} u$

$$\cos x = u$$

$$-\sin x\,dx = du$$

$$-\int \frac{du}{1+u^2} = -\,tg^{-1} u = -\,tg^{-1}(\cos x)$$

$$\int x^{3}\sqrt{x^{2}-1}\,dx = \int x^{3}(x^{2}-1)^{1/2}\,dx$$

$$\frac{\;x^{2}-1 = T\;}{2x\,dx = dT} \qquad \frac{1}{2}\int \boxed{2}\,\boxed{x^{2}}\,x\,(x^{2}-1)^{1/2}\,dx$$

$$= \frac{1}{2}\int (T+1)\,T^{1/2}\,dT$$

$$\frac{1}{2}\int T^{\frac{3}{2}} + T^{1/2}\,dT = \frac{1}{2}\left(\frac{2}{5}T^{5/2} + \frac{2}{3}T^{\frac{3}{2}}\right)$$

$$= \frac{1}{5}(x^{2}-1)^{5/2} + \frac{1}{3}(x^{2}-1)^{\frac{3}{2}}$$

$$\int \sec x \, dx = \ln(\sec x + \operatorname{tg} x)$$

چون:

$$\int \sec x \circ \frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x} \, dx$$

$$= \int \frac{\sec^{۲} x + \sec x \operatorname{tg} x}{\sec x + \operatorname{tg} x} \, dx$$

$$= \ln(\sec x + \operatorname{tg} x)$$

$$\int \csc x \, dx = \ln(\csc x - \cot x)$$

چون

$$\int \csc x \, d \frac{\csc x - \cot x}{\csc x - \cot x} =$$

$$= \int \frac{\csc^{2} x - \csc x \cot x}{\csc x - \cot x} dx$$

$$= \ln(\csc x - \cot x)$$

$$\int \frac{x+2}{x^2+2x+2}\,dx = ?$$

چون مشتق مخرج در صورت ایجاد نمی‌شود پس مخرج کسر را مربع سازی انجام دهید

$$\left(x^2+ax = \left(x+\frac{a}{2}\right)^2-\frac{a^2}{4}\right)$$

$$\int \frac{x+2}{(x+1)^2+1}\,dx = \int \frac{x+1+1}{(x+1)^2+1}\,dx$$

$$= \underbrace{\int \frac{x+1}{(x+1)^2+1}\,dx}_{1} + \underbrace{\int \frac{1}{(x+1)^2+1}\,dx}_{2}$$

(1) $$\int \frac{x+1}{(x+1)^2+1}\,dx \xrightarrow[dx=du]{x+1=u} \frac{1}{2}\int \frac{2u}{u^2+1}\,du$$

$$= \frac{1}{2}\ln(u^2+1)$$

(2) $$\int \frac{1}{(x+1)^2+1}\,dx \xrightarrow[dx=du]{x+1=u} \int \frac{du}{u^2+1}$$

$$= tg^{-1}u = tg^{-1}(x+1)$$

فرمول انتگرال گیری جزء به جزء :

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\,\underbrace{e^{x}\,dx}_{dv} = xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x}$$

$$\begin{cases} e^{x}\,dx = dv \\ u = x \end{cases} \Rightarrow \begin{cases} e^{x} = v \\ du = dx \end{cases}$$

نکته: هرگاه در محاسبه انتگرال جزء به جزء تابعی که به عنوان u انتخاب می‌شود

چند جمله‌ای باشد می‌توان از جدول زیر برای محاسبه انتگرال کمک گرفت.

مشتق		انتگرال
x	$+$	e^x
1	$-$	e^x
0		e^x

$$\int x e^x \, dx = x e^x - e^x$$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$$\begin{cases} e^x \, dx = dv \\ x^2 = u \end{cases} \Rightarrow \begin{cases} e^x = v \\ 2x \, dx = du \end{cases}$$

$$= x^2 e^x - 2\left(x e^x \quad - e^x \qquad \right)$$

$$\int x e^x \, dx = \!=$$

$$\begin{cases} e^x \, dx = dv \\ u = x \end{cases} \quad \begin{matrix} e^x = v \\ \longrightarrow du = dx \end{matrix}$$

$$\int x^2 e^x \, dx =$$

$$= x^2 e^x - 2x e^x + 2e^x$$

مشتق		انتگرال
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - x$$

$$\begin{cases} \ln x = u \\ dx = dv \end{cases} \Rightarrow \begin{cases} \frac{1}{x}\,dx = du \\ v = x \end{cases}$$

$$\int u\,dv = uv - \int v\,du$$ میدانیم

$$\int x \ln x \, dx$$

$$\begin{cases} \ln x = u \\ x\,dx = dv \end{cases} \Rightarrow \begin{array}{l} \frac{1}{x}dx = du \\ v = \frac{x^2}{2} \end{array}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \int \frac{1}{2} x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2$$

$$\int \underbrace{\operatorname{arctag} x}_{u}\; \underbrace{dx}_{dv}$$

$$\begin{cases} dx = dv \\ \operatorname{arctag} x = u \end{cases} \Rightarrow \begin{cases} v = x \\ du = \dfrac{1}{1+x^{2}}\, dx \end{cases}$$

$$\int \tan^{-1} x \; dx = x \tan^{-1} x - \int \frac{x}{1+x^{2}}\, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^{2})$$

$$\int \operatorname{Arcsin} x \, dx$$

$$\begin{cases} dx = dv \\ \operatorname{Arcsin} x = u \end{cases} \Rightarrow \quad v = x \qquad du = \frac{1}{\sqrt{1-x^2}} dx$$

$$\int \sin^{-1} x \, dx = x \operatorname{Arcsin} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x . \sin^{-1} x - \frac{-1}{2} \int -2x \, (1-x^2)^{-1/2} dx$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$\int \sin(\ln x)\,dx$$

$$\begin{cases} \sin(\ln x) = u \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\cos(\ln x) \\ v = x \end{cases}$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int x \cdot \frac{1}{x}\cos(\ln x)\,dx$$

پس مجبوریم $\int \cos(\ln x)\,dx$ را محاسبه کنیم.

$$\int \cos(\ln x)\,dx = x\cos\ln x - \int \left(\frac{-1}{x}\sin\ln x\right)(x)\,dx$$

$$\begin{cases} dx = dv \\ \cos(\ln x) = u \end{cases} \Rightarrow \begin{cases} v = x \\ du = -\frac{1}{x}\sin(\ln x)\,dx \end{cases}$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int x \cdot \frac{1}{x}\cos(\ln x)\,dx$$

پس :

$$\int \sin \ln x\,dx = x\sin \ln x - \left(x\cos \ln x + \int \sin \ln x\,dx\right)$$

$$\int \sin \ln x\,dx + \int \sin \ln x\,dx = x\sin \ln x - x\cos \ln x$$

$$\Rightarrow \int \sin(\ln x)\,dx = \frac{x\sin \ln x - x\cos \ln x}{\text{۲}}$$

$$\int \sec^{۳} x \, dx = ؟$$

$$\int \underbrace{\sec x}_{u} \; \underbrace{\sec^{۲} x \, dx}_{dv}$$

$$\begin{cases} \sec x = u \\ \sec^{۲} x \, dx = dv \end{cases} \Rightarrow \begin{cases} du = \sec x \, \mathrm{tag}\, x \, dx \\ v = \mathrm{tag}\, x \end{cases}$$

$$\int \sec^{۳} x \, dx = \sec x \, \mathrm{tag}\, x - \int \sec x \, \overset{(\sec^{۲} x - 1)}{\mathrm{tag}^{۲} x} \, dx$$

$$\boxed{1 + \mathrm{tag}^{۲} x = \sec^{۲} x}$$

$$\int \sec^{۳} x\,dx = \sec x \,\mathrm{tog}\, x - \int \sec x\,(\sec^{۲} x - ۱)\,dx$$

$$\int \sec^{۳} x\,dx = \sec x\,\mathrm{tag}\,x - \int \sec^{۳} dx + \int \sec x\,dx$$

$$۲\int \sec^{۳} x\,dx = \sec x\,\mathrm{tag}\,x + \ln(\mathrm{se}x + \mathrm{tag}\,x)$$

$$\Rightarrow \int \sec^{۳} x\,dx = \frac{\sec x\,\mathrm{tag}\,x + \ln(\sec x + \mathrm{tg}\,x)}{۲}$$

$$\int \ln(x^2+a^2)\,dx$$

$$\begin{cases} u = \ln(x^2+a^2) \\ dx = dv \end{cases} \Rightarrow \begin{array}{l} du = \dfrac{2x}{x^2+a^2}\,dx \\ x = v \end{array}$$

$$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - \int \frac{2x^2}{x^2+a^2}\,dx$$

$$= x\ln(x^2+a^2) - 2\int \frac{x^2+a^2-a^2}{x^2+a^2}\,dx$$

$$= x\ln(x^2+a^2) - 2\left(\int dx + \int \frac{-a^2}{x^2+a^2}\,dx\right)$$

$$= x\ln(x^2+a^2) - 2x + 2a^2\left(\frac{1}{a}\operatorname{tag}^{-1}\frac{x}{a}\right) + c$$

$$\int \mathrm{tag}^{2}\, dx = ?$$

$$\int \left\{\mathrm{tag}^{2} x + 1\right\} - 1 \; dx = \int (1+\mathrm{tag}^{2} x)\, dx - \int dx$$

$$= \mathrm{tag}\, x - x$$

$$\int \operatorname{tag}^4 x \, dx = \int \operatorname{tag}^2 x \operatorname{tag}^2 x \, dx$$

$$= \int \operatorname{tag}^2 x \left(\left(\operatorname{tg}^2 x + 1 \right) - 1 \right)$$

$$= \int \underbrace{\operatorname{tag}^2 x}_{u} \underbrace{\left(1 + \operatorname{tg}^2 x \right) dx}_{du} - \int \left(\operatorname{tag}^2 x \, dx \right.$$

$+1) - 1$

$$= \frac{(\operatorname{tag} x)^3}{3} - \operatorname{tag} x + x$$

$$\int \frac{dx}{\sqrt{x^2-4x+1}} = \int \frac{dx}{\sqrt{(x-2)^2-4+1}}$$

$$= \int \frac{dx}{\sqrt{(x-2)^2-3}} = \int \frac{dx}{\sqrt{(x-2)^2-(\sqrt{3})^2}}$$

$$\frac{x-2=T}{dx=dT} \qquad \frac{dT}{\sqrt{T^2-(\sqrt{3})^2}} = \cosh^{-1}\frac{T}{\sqrt{3}}$$

$$\boxed{\int \frac{dT}{\sqrt{T^2-a^2}} = \cosh^{-1}\frac{T}{a}} \qquad = \cosh^{-1}\frac{(x-2)}{\sqrt{3}}$$

بررسی توابعی به صورت

$$\int \sin^{2n+1}x \, dx \quad \text{یا} \quad \int \cos^{2n+1}x \, dx$$

$$\int \sin^{2n+1}x \, dx = \int \sin^{2n}x \, \sin x \, dx$$

$$= \int (1-\cos^2 x)^n \sin x \, dx$$

کافی است با تغییر متغیر زیر مسئله را حل کنید.

$$\cos x = u$$

$$-\sin x \, dx = du$$

$$= -\int (1-u^2)^n \, du$$

$$\int \cos^{2n+1}x \, dx = \int \cos^{2n}x \cos x \, dx$$

$$= \int (1-\boxed{\sin^2 x})^n \cos x \, dx$$

$\sin x = u$

$\cos x \, dx = du$

$$= \int (1-u^2)^n \, du$$

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \sin x \, dx \xrightarrow[-\sin x \, dx = dT]{\cos x = T}$$

$$= -\int (1-T^2)^2 \, dT$$

$$1 + T^4 - 2T^2$$

$$= -T - \frac{T^5}{5} + \frac{2T^3}{3} =$$

بجای $T = \cos x$ قرار دهید.

$$\int \cos^{3}(\ln x)\,\frac{dx}{x} \xrightarrow[\frac{1}{x}dx = dT]{\ln x = T} \int \cos^{3} T\, dT$$

$$= \int (\cos^{2} T) \cos T\, dT = \int (1 - \sin^{2} T) \cos T\, dT$$

$$\xrightarrow[\cos T\, dT = du]{\sin T = u} \int (1 - u^{2})\, du = u - \frac{u^{3}}{3}$$

$$= \sin(\ln x) - \frac{\sin^{3}(\ln x)}{3}$$

بررسی توابعی به صورت $\int \sin^{2n}x\,dx$ یا $\int \cos^{2n}x\,dx$

از فرمول طلایی استفاده می‌کنیم:

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$2\int \frac{\sin^{۴}(\sqrt{x})}{2\sqrt{x}}\,dx = ? \xrightarrow[\frac{۱\,dx}{۲\sqrt{x}} = dT]{\sqrt{x} = T}$$

$$= 2\int \sin^{۴} T\, dT = ۲\int \left(\frac{۱-\cos ۲T}{۲}\right)^{۲} dT$$

$$= ۱/۲ \int \left(۱ - ۲\cos ۲T + \cos^{۲} ۲T\right) dT$$

$$= \frac{۱}{۲}T - \frac{۱}{۲}\sin ۲T + \frac{۱}{۲}\int \frac{۱+\cos ۴T}{۲}\, dT$$

$$= \frac{T}{۲} - \frac{۱}{۲}\sin ۲T + \frac{۱}{۴}\left(T + \frac{۱}{۴}\sin ۴T\right)$$

بجای $T=\sqrt{x}$ قرار بدهید.

نکته :

$$\sin ax \sin bx = \frac{1}{٢}\left(\cos(b-a)x - \cos(a+b)x\right)$$

$$\cos ax \cos bx = \frac{1}{٢}\left(\cos(a+b)x + \cos(a-b)x\right)$$

$$\sin ax \cos bx = \frac{1}{٢}\left(\sin(a-b)x - \sin(a+b)x\right)$$

نکته :

$$\int \sin^n x \cos^m x \, dx$$

که یکی از m یا n فرد باشد مثلا n فرد

$n = 2n' + 1$

$$\int (\sin^2 x)^{n'} \sin x \cos^m x \, dx$$

$$= \int (1 - \cos^2 x)^{n'} \sin x \cos^m x \, dx$$

$\cos x = u$

$-\sin x \, dx = du$

$$= -\int (1 - u^2) u^m \, du$$

موفق باشید