دانشگاه صنعتی شاهرود

درس ریاضی 1

مدرّس : دکترمغاری

تعریف :

حدراست $\lim_{x\to x_0^+} f(x) = a_1$

$(x > x_0)$

حدچپ $\lim_{x\to x_0^-} f(x) = a_2$

$(x < x_0)$

یک تابع حد دارد هرگاه : حدراست = حدچپ

مثال : تابع $y=[x]$ در $x \in \mathbb{Z}$ حد ندارد

$$\lim_{x \to k^+} [x] = k \quad ; \quad \lim_{x \to k^-} [x] = k-1$$

$k-1 < x < k$

$k \neq k-1$ حد ندارد

مثال : مقدار a را طوری بیابید که تابع زیر در $x=1$ حد داشته باشد

$x>1$

$-x<-1$

$[-x]=-2$

$$f(x)=\begin{cases} [-x]+a & x>1 \\ 2^{[x]}+x & x<1 \end{cases}$$

$$\lim_{x\to 1^+} [-x]+a = -2+a \quad (x>1)$$

$$\Rightarrow -2+a=2 \qquad \boxed{a=4}$$

$$\lim_{x\to 1^-} [x] = 2^{[x]} + x = 2^0 + (1) = 2$$

$x<1 \to (0<x<1)$

تعریف : $\lim_{x\to\infty} f(x) = L$ هرگاه ،

$$\forall \varepsilon>0 \quad \exists N>0 \quad s.t \quad x>N \quad |f(x)-L|<\varepsilon$$

تعریف : $\lim_{x\to -\infty} f(x) = L$ هرگاه ؛

$$\forall \varepsilon>0 \quad \exists N>0 \quad ؛ s.t \quad x<-N \quad |f(x)-L|<\varepsilon$$

تعریف : $\lim\limits_{x \to a} f(x) = \infty$

$$\forall M > 0 \quad \exists \delta > 0 \quad 0 < |x-a| < \delta \longrightarrow f(x) > M$$

تعریف : $\lim\limits_{x \to a} f(x) = -\infty$

$$\forall M > 0 \quad \exists \delta > 0 \quad 0 < |x-a| < \delta \longrightarrow f(x) < -M$$

قضية :

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

$$= \lim_{x\to\infty} \frac{a_n x^n \left(1 + \cancel{\frac{a_{n-1} x^{n-1}}{a_n x^n}} + \cdots + \cancel{\frac{a_0}{a_n x^n}}\right)}{b_m x^m \left(1 + \cancel{\frac{b_{m-1} x^{m-1}}{b_m x^m}} + \cdots + \cancel{\frac{b_0}{b_m x^m}}\right)}$$

$$= \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \dfrac{a_n}{b_m} & n = m \\ \infty & n > m \\ 0 & n < m \end{cases}$$

مثال :

$$\lim_{x\to\infty} \frac{[x]}{x}$$

$$x-1 < [x] \le x \qquad x>0$$

$$1-\frac{1}{x} = \frac{x-1}{x} < \frac{[x]}{x} \le \frac{x}{x} = 1$$

$$\lim_{x\to\infty} 1-\frac{1}{x} = 1-0 = 1 \qquad \text{؛} \quad \lim_{x\to\infty} 1 = 1$$

$$\xrightarrow[\text{فشردگی}]{} \lim_{x\to\infty} \frac{[x]}{x} = 1$$

مثال:

$$\lim_{x\to+\infty} x\left(\sqrt{x^2+3}-x\right)$$

$$\lim_{x\to\infty} x\left(\sqrt{x^2+3}-x\right)\cdot\frac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x}=$$

$$=\lim_{x\to\infty}\frac{x\left(x^2+3-x^2\right)}{\sqrt{x^2+3}+x}=\lim_{x\to\infty}\frac{3x}{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+x}$$

$$\sqrt{x^2}=|x| \xrightarrow{x=+\infty} =x$$

$$=\lim_{x\to\infty}\frac{3x}{x+x}=\frac{3}{2}$$

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \frac{1\sin\frac{1}{x}}{\frac{1}{x}}$$

$$\frac{1}{x}=t$$

$$x\to\infty \Rightarrow t\to 0 \qquad = \lim_{t\to 0}\frac{\sin t}{t}=1$$

$$\lim_{x \to \infty} \left[\frac{x}{[x]} \right]$$

می‌دانیم: $[x] \le x < [x]+1$ پس

$$1 \le \frac{x}{[x]} < \underbrace{1 + \frac{1}{[x]}}$$ ★

ازطرفی

$$x-1 < [x] \le x$$

$$\frac{1}{x} \le \frac{1}{[x]} < \frac{1}{x-1} =$$

$$1 \le \frac{x}{[x]} < 1 + \frac{1}{[x]} \quad * \Leftarrow$$

$$\lim_{x \to \infty} (1) = 1 \qquad \lim_{x \to \infty} 1 + \frac{1}{[x]} = 1$$

$$\longrightarrow \lim_{x \to \infty} \left[\frac{x}{[x]} \right] = 1$$

$$\lim_{x \to ۲^-} \frac{[x^۲]-۴}{x^۲-۴} = \frac{۳-۴}{0^-} = \frac{-۱}{0^-} = +\infty$$

$$x < ۲ \to x^۲ < ۴ \to ۳ < x^۲ < ۴$$

$$[x^۲] = ۳$$

دستور هوپیتال :

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \quad \text{یا} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

$$\xrightarrow[H]{\text{هوپیتال}} \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

$$\text{یا} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{\ln(x+1)} = \frac{0}{0} \xrightarrow{H} \lim_{x \to 0} \frac{(\cos x)\, e^{\sin x}}{\frac{1}{x+1}}$$

$$= \frac{1 \cdot e^{0}}{\frac{1}{0+1}} = \frac{1}{1} = 1$$

$$\lim_{x\to\infty} \frac{x^{۱۳۸۹}}{e^{x}} = \frac{\infty}{\infty} \xrightarrow{H} \frac{۱۳۸۹\, e^{۱۳۸۸}}{e^{x}} \xrightarrow{H} \cdots$$

$$= \lim_{x\to\infty} \frac{(۱۳۸۹)!}{e^{x}} = \frac{\text{عدد}}{\infty} = ۰$$

$$\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right)=\infty\cdot 0$$

$x=\frac{1}{\frac{1}{x}}$

$$=\lim \frac{\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}}{\frac{1}{x}} \overset{H}{=} \frac{\frac{\frac{1}{(x+2)^2}}{1+\left(\frac{x+1}{x+2}\right)^2}}{\frac{-1}{x^2}}$$

$(\tan^{-1}u)'=\frac{u'}{1+u^2}$

$$=\frac{-x^2}{(x+2)^2\left(1+\left(\frac{x+1}{x+2}\right)^2\right)} \cong \frac{-x^2}{x^2(1+(1)^2)}=\frac{-1}{2}$$

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$$\lim_{x \to 0} \frac{(1-\cos x)\sin 2x}{x^3 \cos x}$$

$$1-\cos x = 2\sin^2\frac{x}{2}$$

$$= \lim_{x \to 0} \frac{2\sin^2\frac{x}{2}}{x^2} \cdot \lim_{x \to 0} \frac{\sin 2x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \lim 2\left(\overset{1}{\frac{\sin\frac{x}{2}}{2\,\frac{x}{2}}}\right)^2 \cdot \lim \overset{1}{\frac{\sin 2x}{\frac{1}{2}\,2x}} \cdot \lim_{x \to 0} \frac{1}{\cos 0}$$

$$= 2 \cdot \frac{1}{4} \cdot 2 \cdot 1 = 1$$

$$\lim_{x\to 1}\frac{(1-\sqrt{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^{n-1}}$$

$$=\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\cdot\frac{1-\sqrt[3]{x}}{1-x}\cdots\frac{1-\sqrt[n]{x}}{1-x}$$

$$=\lim_{x\to 1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\cdot\frac{1-\sqrt[3]{x}}{(1-\sqrt[3]{x})(1+\sqrt[3]{x}+\sqrt[3]{x^2})}\cdot$$

$$\frac{1-\sqrt[n]{x}}{(1-\sqrt[n]{x})(1+\sqrt[n]{x}+\sqrt[n]{x^2}+\cdots+\sqrt[n]{x^{n-1}})}$$

$$=\lim_{x\to 1}\frac{1}{1+\sqrt{x}}\cdot\lim\frac{1}{1+\sqrt[3]{x}+\sqrt[3]{x^2}}\cdots\lim\frac{1}{1+\sqrt[n]{x}+\sqrt[n]{x^2}+\cdots+\sqrt[n]{x^{n-1}}}$$

$$=\frac{1}{2}\cdot\frac{1}{3}\cdot\cdots\cdot\frac{1}{n}=\frac{1}{n!}$$

$$\lim_{n\to\infty} \cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}$$

$$\left.\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ \frac{1}{2}\sin\alpha &= \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\end{aligned}\right\}$$

$$\cos\frac{x}{2}\cos\frac{x}{4}\cdot\cdots\frac{\cos x}{2^n} = \frac{\sin\frac{x}{2}\sin\frac{x}{4}\cdots\sin\frac{x}{2^n}\ \frac{1}{2^n}}{\sin\frac{x}{2}\sin\frac{x}{4}\cdots\sin\frac{x}{2^n}}$$

$$\lim_{n\to\infty} \frac{\left(\sin\frac{x}{۲}\cos x/_{۲}\right)\left(\sin\frac{x}{۴}\cos\frac{x}{۴}\right)\cdots\left(\sin\frac{x}{۲^n}\cos\frac{x}{۲^n}\right)}{\sin\frac{x}{۲}\cdot\sin\frac{x}{۴}\cdots\sin\frac{x}{۲^n}}$$

$$=\lim_{n\to\infty} \frac{\left(\frac{1}{۲}\sin x\right)\left(1/_{۲}\sin\frac{x}{۲}\right)\cdots\left(\frac{1}{۲}\sin\frac{x}{۲^{n-1}}\right)}{\sin\frac{x}{۲}\sin\frac{x}{۴}\cdots\sin\frac{x}{۲^n}}$$

$$= \lim_{n \to \infty} \frac{\frac{1}{2^n} \sin x}{\sin \frac{x}{2^n}} = \lim \frac{\frac{1}{2^n} \sin x}{\frac{x}{2^n}} = \frac{\sin x}{x}$$

$$\sin\left(\frac{x}{2^n}\right) \xrightarrow{\text{کمان صفر}} \frac{x}{2^n}$$

موفق باشید