



Non-Newtonian Fluid Mechanics

(Part - VII)

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Conservation Laws of Newtonian Flows



Governing Equations of Incompressible Newtonian Flows

For incompressible Newtonian flows, the governing equations are consisting of continuity, momentum and energy equations:

$$\nabla \cdot \mathbf{V} = 0$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V} \quad (1)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right) = k \nabla^2 T + \frac{1}{2} \mu \{\dot{\gamma} : \dot{\gamma}\}$$

where \mathbf{V} is the velocity vector, p is pressure, T is temperature, ρ is density, \mathbf{g} is the gravitational acceleration, μ is viscosity c_p is the specific heat and k is the thermal conductivity. The term $\dot{\gamma}$ is shear strain tensor which is defined as:

$$\dot{\gamma} = \nabla \mathbf{V} + \nabla \mathbf{V}^T \quad (2)$$

where superscript T denotes the transposed. For Newtonian fluids, the last term of energy equation is always positive and therefore represents an irreversible degradation of mechanical work into internal (thermal) energy. It is known as the *viscous dissipation*. The operator of this term is also defined as:

$$\dot{\gamma} : \dot{\gamma} = \text{tr} \{ \dot{\gamma} \cdot \dot{\gamma} \} \quad (3)$$



Conservation Laws of Newtonian Flows



Dimensionless Form of Governing Equations of Incompressible Newtonian Flows

The dimensionless form of this equations for low Reynolds flows are usually expressed as follows:

$$\begin{aligned}\nabla^* \cdot \mathbf{V}^* &= 0 \\ \frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* \mathbf{V}^* &= \frac{l}{Fr} \frac{\mathbf{g}}{|\mathbf{g}|} + \frac{l}{Re} \left\{ -\nabla^* p^* + \nabla^{*2} \mathbf{V}^* \right\} \\ \frac{\partial T^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* T^* &= \frac{l}{Re Pr} \left\{ \nabla^{*2} T^* + \frac{Br}{2} \left\{ \dot{\gamma}^* : \dot{\gamma}^* \right\} \right\}\end{aligned}\tag{4}$$

where the dimensionless groups are:

$$\begin{aligned}\mathbf{x}^* &= \frac{\mathbf{x}}{L}, \quad \mathbf{V}^* = \frac{\mathbf{V}}{U}, \quad t^* = \frac{tU}{L}, \quad \dot{\gamma}^* = \frac{L}{U} \dot{\gamma}, \quad p^* = \frac{L}{\mu U} p, \quad \nabla^* = L \nabla \\ T^* &= \frac{T - T_0}{\Delta T_0}, \quad Re = \frac{\rho UL}{\mu}, \quad Fr = \frac{U^2}{Lg}, \quad Pr = \frac{\mu c_p}{k}, \quad Br = \frac{\mu U^2}{k \Delta T_0}\end{aligned}\tag{5}$$

where U is reference velocity, L is reference length, T_0 is reference temperature, ΔT_0 is reference temperature reference, Re is Reynolds number, Fr is Froude number, Pr is Prandtl number and Br is Brinkman number.



Conservation Laws of Non-Newtonian Flows



Governing Equations of Non-Newtonian Flows

The non-Newtonian flows is mostly incompressible and their governing equations can be expressed as:

$$\nabla \cdot \mathbf{V} = 0$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau} \quad (6)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right) = k \nabla^2 T + \frac{1}{2} \{ \boldsymbol{\tau} : \dot{\boldsymbol{\gamma}} \}$$

where $\boldsymbol{\tau}$ is stress tensor that should be determined using a suitable constitutive equation. For viscoelastic fluids flows, the term $\{ \boldsymbol{\tau} : \dot{\boldsymbol{\gamma}} \} / 2$ represents the work of stress field and it may be negative, since some amount of energy may be stored as elastic energy. The dimensionless form of Eq. (6) is:

$$\nabla^* \cdot \mathbf{V}^* = 0$$

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = \frac{1}{Fr} \frac{\mathbf{g}}{|\mathbf{g}|} + \frac{1}{Re} \left\{ -\nabla^* p^* + \nabla^* \cdot \boldsymbol{\tau}^* \right\} \quad (7)$$

$$\frac{\partial T^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* T^* = \frac{1}{RePr} \left\{ \nabla^{*2} T^* + \frac{Br}{2} \{ \boldsymbol{\tau}^* : \dot{\boldsymbol{\gamma}}^* \} \right\}$$

where $\boldsymbol{\tau}^*$ is dimensionless stress and defined as $\boldsymbol{\tau}^* = (L / \mu U) \boldsymbol{\tau}$.



Boundary Condition at Walls



Boundary Condition at Walls

The **no-slip boundary conditions** are almost used as the boundary condition for **polymeric solutions**. There is considerable experimental evidence that the classical no-slip boundary condition of fluid mechanics is not always a valid assumption for the flow of high molecular weight molten polymers. **Wall slip** occurs in many industrial applications, such as in **polymer extrusion processes**, thus affecting the throughput and the quality of the final product. In fact, **molten polymers slip macroscopically at solid surfaces** when the wall shear stress exceeds a critical value. Moreover, for linear polymers there exists a second critical wall shear stress value at which a transition from **a weak to a strong slip occurs**. These two modes of slip (weak and strong) are due to flow-induced chain detachment/desorption at the polymer/wall interface and to chain disentanglement of the polymer chains in the bulk from a monolayer of polymer chains adsorbed at the interface.

1. Flow-induced chain detachment/desorption (weak slip)

An important fundamental question is related to the origin of slip i.e. is slip a result of an adhesive failure of the interface or of a cohesive one at a layer within the polymer although close to the interface. If slip is due to an adhesive failure (direct detachment of adsorbed polymer chains from the wall), then the critical shear stress for the onset of slip should be related to a proper measure of the interfacial characteristics of the corresponding polymer/wall interface. If the failure is cohesive, such a correlation is not expected. Instead, the critical shear stress for the onset of slip should be directly related to molecular characteristics of the polymer chains.

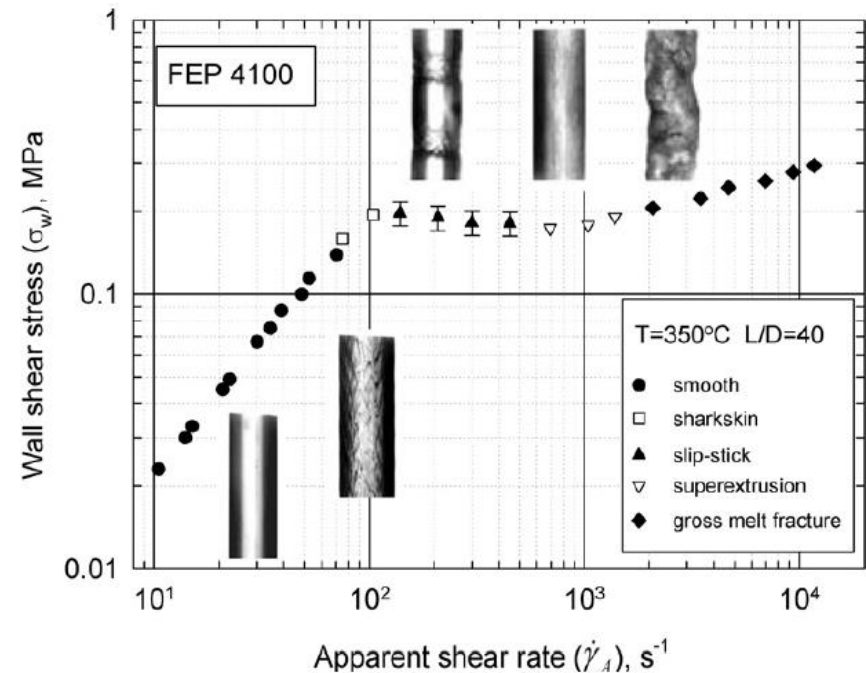
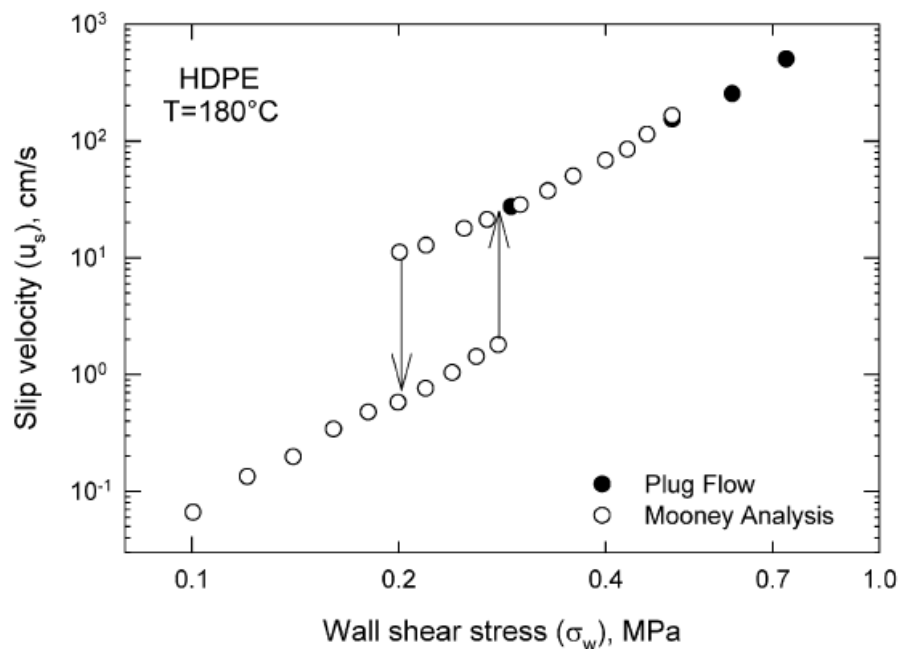


Boundary Condition at Walls



2. Chain disentanglement (strong slip)

Bergem first proposed that the onset of pressure oscillations in a capillary rheometer is due to sudden disentanglement of the chains in the bulk from those adsorbed/attached to the wall. This disentanglement is accompanied by a transition from a weak slip to a strong one. The latter corresponds to a jump in the flow curve at a second critical shear stress value, τ_{c2} from the low flow rate branch into a high flow rate one. This jump was explained convincingly by Park et al. using their sliding plate rheometer experiments at ambient and high pressure.





Boundary Condition at Walls

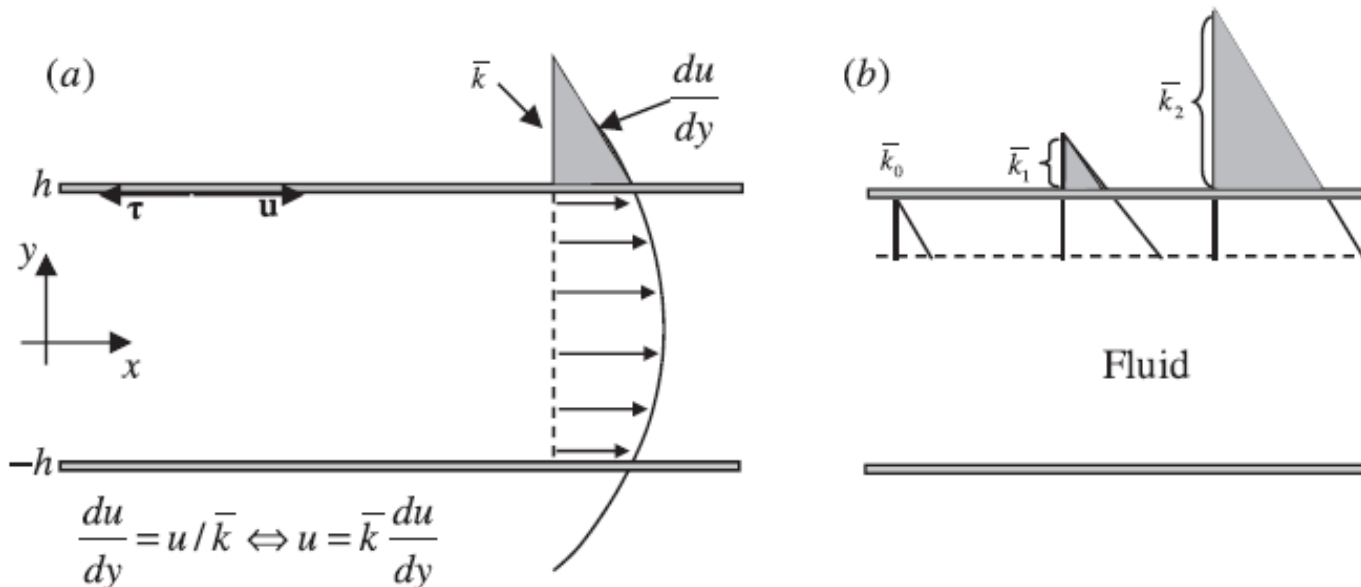


Modeling of Wall Slip

Navier argued that in the presence of slip the liquid motion must be opposed by a force proportional to the relative velocity between the first liquid layer and the solid wall.

$$u_{ws} = \bar{k} \left. \frac{\partial u}{\partial y} \right|_{at\ wall} \quad (8)$$

where \bar{k} is named slip length or friction coefficient. The slip length can take any positive value ($\bar{k} \geq 0$), with no-slip at wall for $\bar{k} = 0$, and increasingly large slip velocity as \bar{k} increases to infinity in which case the velocity profile becomes a plug with zero velocity gradient.





Boundary Condition at Walls



Eq. (8) can now be rewritten for a **Generalized Newtonian fluid** as

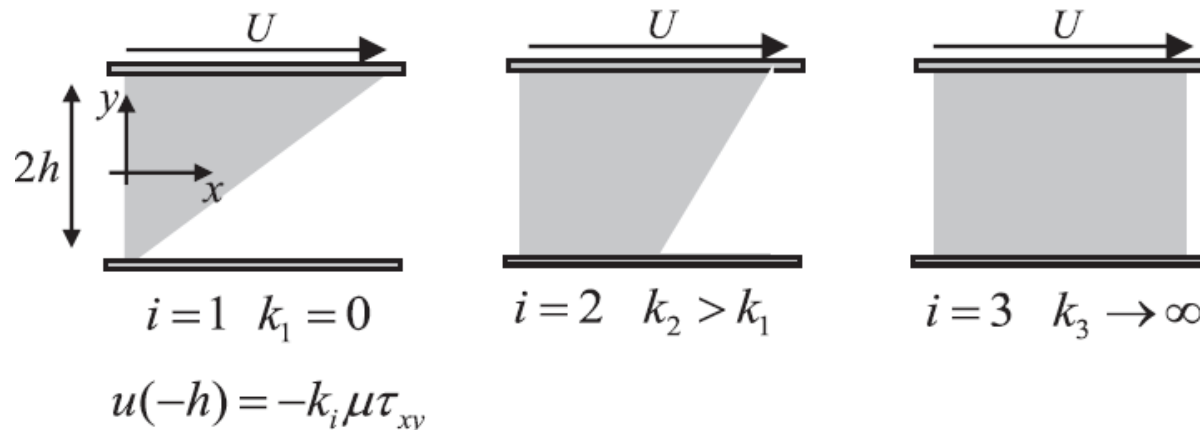
$$u_{ws} = \text{sing} \left(\frac{\partial u}{\partial y} \right) k \tau_w \quad (9)$$

where $k = \bar{k} / \eta(\dot{\gamma})$. This linear relationship between slip velocity at the wall u_{ws} and shear stress at the wall τ_w is called **the linear Navier slip law** or simply **the Navier slip law**.

The **non-linear Navier slip law** assumes that the friction coefficient is a function of the shear stress τ_w , thus providing a non-linear power function,

$$u_{ws} = \text{sing} \left(\frac{\partial u}{\partial y} \right) k |\tau_w|^{m-1} \tau_w \quad (10)$$

where $m > 0$. For $m = 1$ **the Navier slip law** is recovered.





Boundary Condition at Walls



It provides a good approximation for several conditions, but it fails to describe the slip velocity in the neighborhood of the critical stress at which the slip starts. To eliminate this discrepancy, Hatzikiriakos proposed an alternative slip law based on the Eyring theory of liquid viscosity in order to provide a smooth transition from no-slip to slip flow at the critical shear stress. The argument goes as follows:

Let τ_c be the positive critical stress at which slip starts and k_1 & $k_2 \geq 0$. Then, the Hatzikiriakos slip law is given by,

$$u_{ws} = \begin{cases} k_1 \sinh \left\{ k_2 \sinh \left(\frac{\partial u}{\partial y} \right) \tau_w - \tau_c \right\} & \tau_w \geq \tau_c \\ 0 & \tau_w < \tau_c \end{cases} \quad (11)$$

The asymptotic slip law, is given by

$$u_{ws} = -\frac{1}{k_2} \left[1 - \exp \left(\frac{u}{k_1} \right) \right] \quad (12)$$

For both the Hatzikiriakos and the asymptotic slip models, the coefficients k_1 and k_2 allow controlling the amount of slip and the shape of the curve of τ vs u_{ws} that is obtained by experimental measurements.



Pressure!



Thermodynamic Pressure versus Mechanical Pressure (Newtonian Fluids)

In general, the total stress of Newtonian fluids can be defined as:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\dot{\boldsymbol{\gamma}} + (\zeta \nabla \cdot \mathbf{V})\mathbf{I} \quad (13)$$

where $\boldsymbol{\sigma}$ is total stress (arisen from pressure, shear deformation and compression of fluid), \mathbf{I} is identity tensor, \mathbf{V} is velocity and μ is viscosity of fluid. In above equation, ζ is named as *the second coefficient of viscosity*. Since ζ is associated only with volume expansion, it is customary to call it *the coefficient of bulk viscosity*. The **mechanical pressure**, \bar{p} , is defined as the negative one-third of three normal stresses:

$$\bar{p} = -\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \quad (14)$$

Substituting Eq. (13) into the Eq. (14), we have:

$$\bar{p} = p - \left(\zeta + \frac{2}{3} \mu \right) \nabla \cdot \mathbf{V} \quad (15)$$

Thus, the **mechanical pressure** in a deforming viscous fluid is not equal to the **thermodynamic** property called **pressure**. This distinction is rarely important, since $\nabla \cdot \mathbf{V}$ is usually very small in typical flow problems, but the exact meaning of Eq. (15) has been a controversial subject for more than a century. Stokes himself simply resolved the issue by an assumption:

$$\zeta + \frac{2}{3} \mu = 0 \longrightarrow \zeta = -\frac{2}{3} \mu \longrightarrow p = \bar{p} \quad (16)$$

However, the available experimental evidence from the measurement of soundwave attenuation, as reviewed by Karim and Rosenhead (1952), indicates that ζ for most liquids is actually positive, rather than $-2\mu/3$, and often is much larger than μ . The experiments themselves are a matter of some controversy.



Pressure!



Thermodynamic Pressure versus Mechanical Pressure (Non-Newtonian Fluids)

Based on Eq. (15), the **thermodynamic pressure** is identical with **mechanical pressure** for incompressible Newtonian fluid ($\nabla \cdot \mathbf{V} = 0$). For incompressible viscoelastic flows, the total stress can be defined as:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad (17)$$

where $\boldsymbol{\tau}$ is shear stress which is determined using the constitutive equation. For shear flows of viscoelastic liquids, the **thermodynamic pressure** and **mechanical pressure** are not equal due to the effect of normal stress differences:

$$\left. \begin{aligned} \bar{p} &= -\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = p - \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} \\ N_1 &= \tau_{11} - \tau_{22} \quad \& \quad N_2 = \tau_{22} - \tau_{33} \end{aligned} \right\} \rightarrow \bar{p} = p - \tau_{22} + \frac{1}{3}(N_2 - N_1) \quad (18)$$

For flow inside a pipe or slit, what is actually indicated by pressure transducer mounted flush with the wall of the flow channel is $-\sigma_{22}$ which is known as the **measured pressure** (p_M).

$$\bar{p} = p - \tau_{22} + \frac{1}{3}(N_2 - N_1) = -\sigma_{22} + \frac{1}{3}(N_2 - N_1) \rightarrow \bar{p} = p_M + \frac{1}{3}(N_2 - N_1) \quad (19)$$

For other flows, we may reach to different correlations. Therefore, for any specific problem, we should do some analysis to distinguish the **thermodynamic pressure**, **mechanical pressure** and **measured pressure**.



Dimensionless Groups



Prof. Karl Weissenberg illustrating a lecture in Moscow (1965). The Weissenberg number has been named for his honor.

$$We = \frac{\text{Elastic Force}}{\text{Viscous Force}} = \lambda \dot{\gamma}$$

$$De = \frac{\text{Time of Relaxation}}{\text{Time of Observation}} = \frac{\lambda}{T} = \lambda \omega$$

$$El = \frac{\text{Elastic Force}}{\text{Inertia Force}} = \frac{We}{Re}$$



Amazing what can happen in 250 years!
(An estimation, based on nearby tombstones.)





Time of Observation!

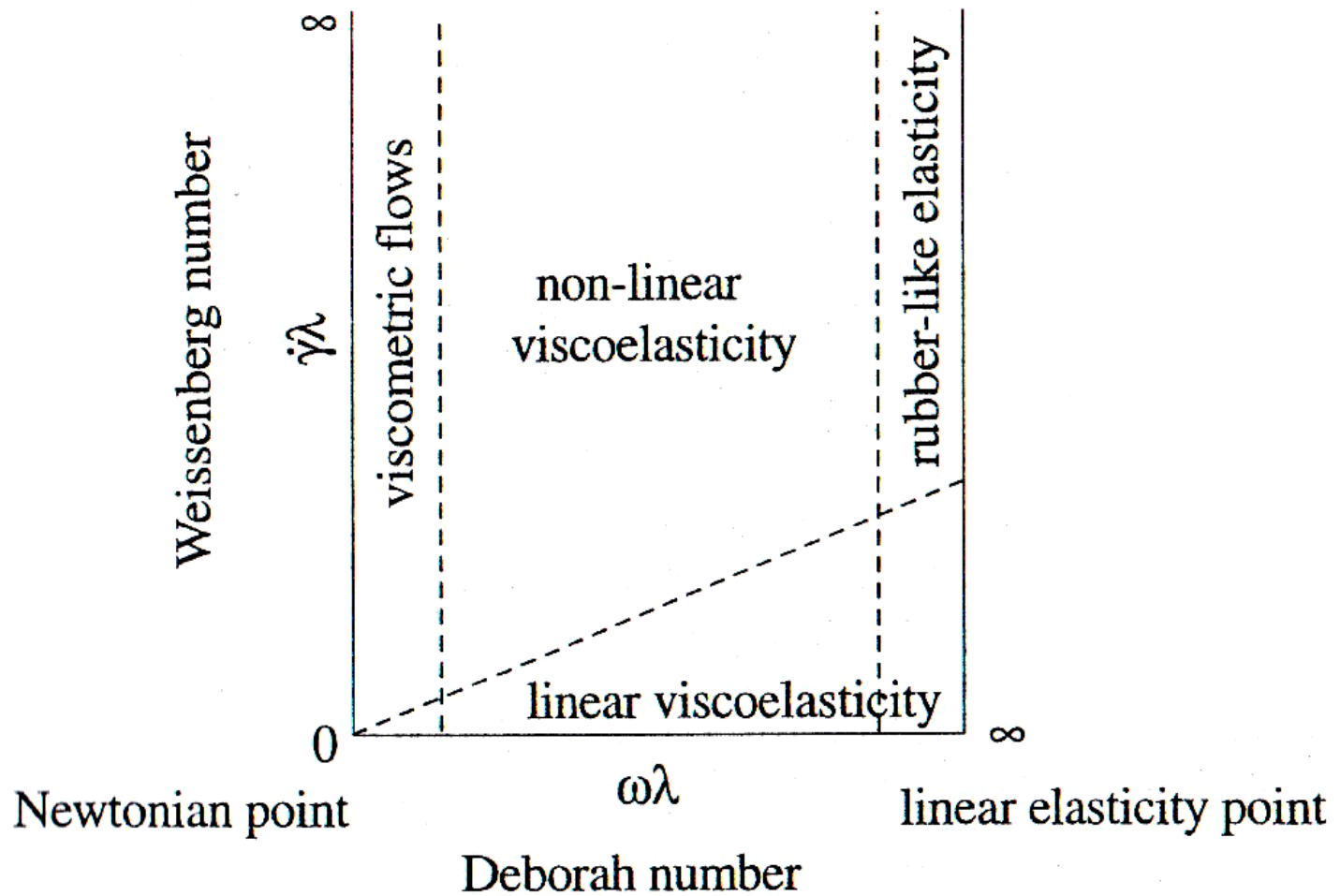


A rheological experiment is recognized in the Guinness Book of World Records as the longest-running laboratory experiment in the world. The Pitch-Drop Experiment, started by physics Professor Thomas Parnell in 1927 at the University of Queensland in Australia, is currently maintained by Professor John Mainstone (shown at right). The pitch flows slowly from an upper reservoir drop by drop. The eighth drop fell in 1990; the viscosity of the pitch is calculated to be 100 billion cp.



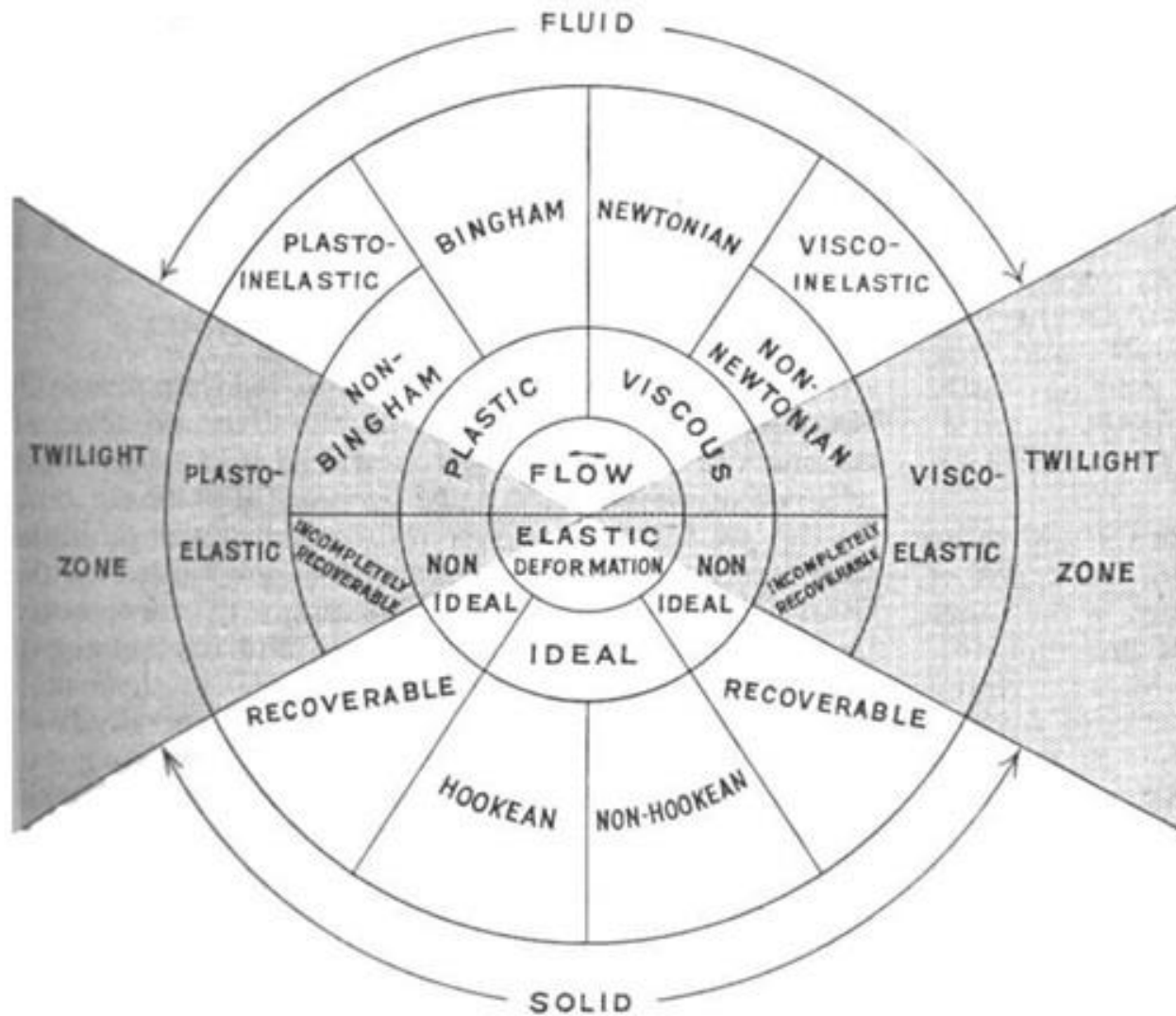


Pipkin Diagram





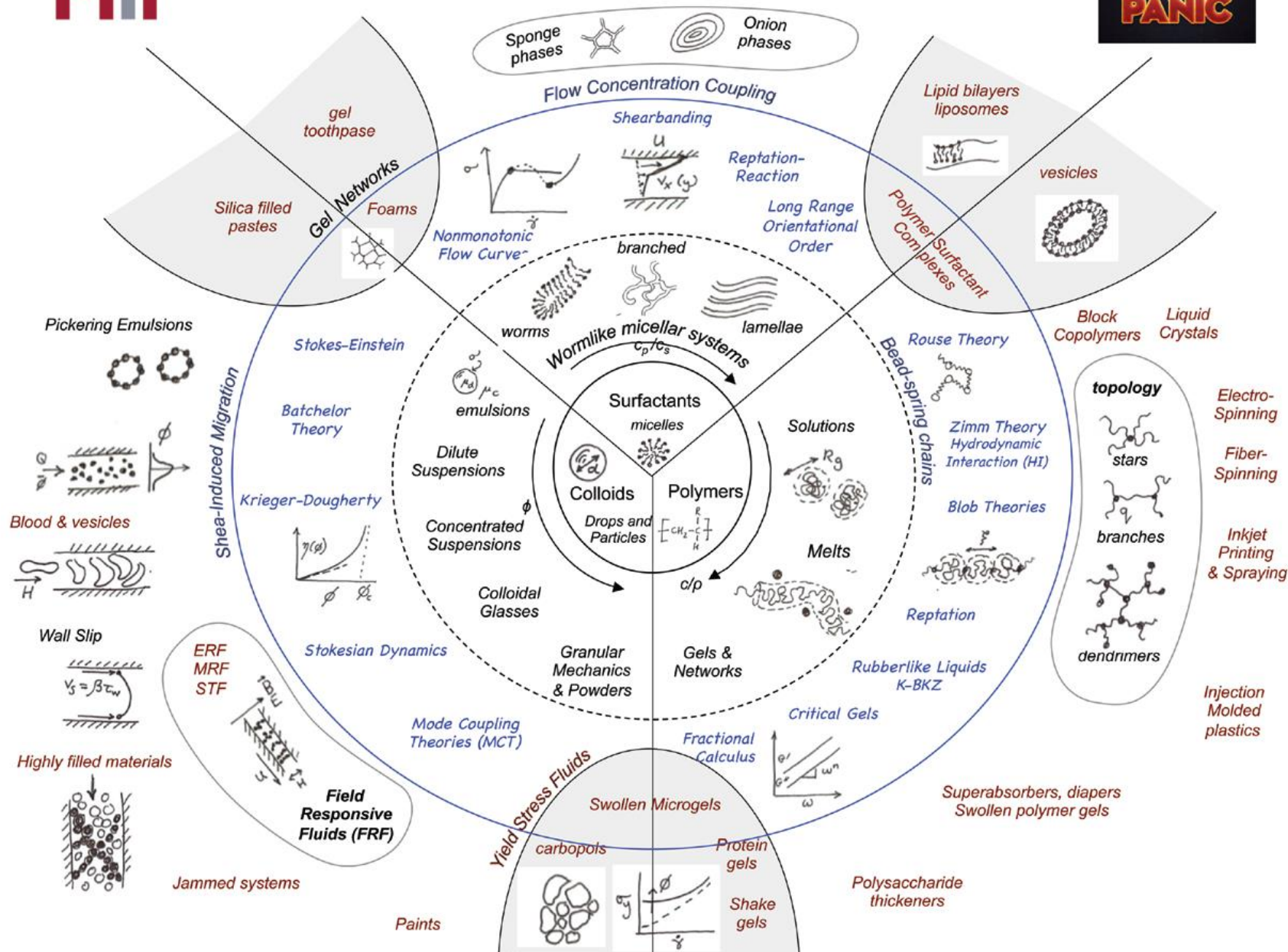
Rheological Chart





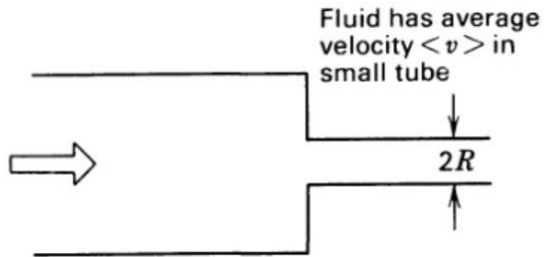
A Hitchhikers Guide to Complex Fluids

DON'T PANIC

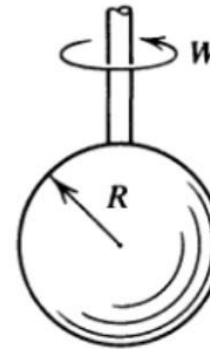




Dimensionless Groups



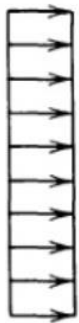
$$De = \lambda \langle v \rangle / R \quad \& \quad Wi = \frac{\lambda \langle v \rangle}{2R}$$



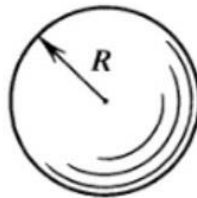
Sphere rotates with angular velocity W

$$De = \lambda W$$

$$Wi \equiv De$$



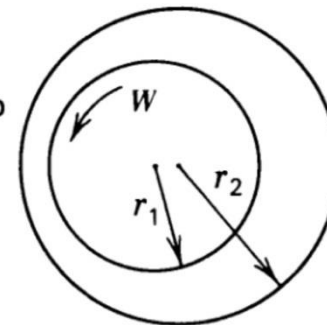
Fluid has uniform velocity v_∞ far from sphere



$$De = \lambda v_\infty / R$$

$$Wi \equiv De$$

Fluid is contained in gap with average thickness $c = r_2 - r_1$ between eccentric cylinders



Inner cylinder rotates with constant angular velocity W

$$De = \lambda W$$

$$Wi \equiv \lambda W r_1 / c$$



Dimensionless Groups



What shall we do about identical definition of dimensionless groups?

The Deborah number is intended to describe the extent to which the response of a material to a deformation is viscoelastic rather than purely viscous. It is the ratio of a characteristic time of the fluid to a time reflecting the duration of observation. The Weissenberg number, on the other hand, is a dimensionless group intended to describe the nonlinearity of the rheological response. It is the product of a characteristic time of the fluid and a characteristic rate of deformation. The problem that arises in the use of these groups is the identification of a characteristic time of the fluid and a characteristic duration of observation and/or a characteristic rate of deformation. Failure to deal with these issues correctly has led to incorrect statements in many papers and books. But even when Wi and De are used correctly, it has not been possible, except under certain conditions, to achieve the desired generalization of data. The conditions under which generalization has been possible are as follows:



1. The flow must be homogeneous;
2. If a transient deformation, its time-dependency must be describable in terms of a single variable, for example the frequency in large-amplitude oscillatory flow;
3. It must be possible to describe the rate of deformation by a single parameter;
4. The fluid must have a well-defined characteristic time that can be measured or predicted, and this time must correlate with the duration of the observation and the phenomena of interest.

These requirements are fully met for viscoelastic materials only when plotting either predictions of a theory or when correlating data for polymers that are monodisperse in terms of both molecular weight and structure.



The End

And The Beginning

