2-3 Constraints on the mechanical properties of an orthotropic materials

- Compliance matrix for orthotropic mat.

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

We are interested to find the constraint on S_{ij}'s

$\Delta_1 = S_{11} > 0$ (2-3-3)

$\Delta_2 = \begin{vmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{vmatrix} = S_{11}S_{22} - S_{12}^2 > 0$ (2-3-4)

$\Delta_3 = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix} > 0$ (2-3-5)

$\Delta_4 = S_{44}\Delta_3 > 0$ (2-3-6)

$\Delta_5 = S_{44}S_{55}\Delta_3 > 0$ (2-3-7)

$\Delta_6 = S_{44}S_{55}S_{66}\Delta_3 > 0$ (2-3-8)

From (2-3-3) and (2-3-4): $S_{22} > 0$ (2-3-9)

Condition (2-3-5):

$\Delta_3 = \frac{1}{S_{11}}[(S_{11}S_{22} - S_{12}^2)(S_{11}S_{33} - S_{13}^2) - (S_{11}S_{23} - S_{12}S_{13})^2] > 0$ (2-3-10)

$\Delta_3 = \frac{1}{S_{11}}[(S_{22}S_{33} - S_{23}^2)(S_{11}S_{22} - S_{12}^2) - (S_{22}S_{13} - S_{23}S_{12})^2] > 0$ (2-3-11)

From (2-3-3) and (2-3-4), (2-3-10):

$S_{11}S_{33}-S_{13}^2>0$ (2-3-12)

From (2-3-3) and (2-3-12):

$S_{33}>0$ (2-3-13)

From (2-3-4) and (2-3-11):

$S_{22}S_{33}-S_{23}^2>0$ (2-3-14)

Fro (2-3-5) and (2-3-6) and (2-3-7):

$S_{44}>0$ & $S_{55}>0$ & $S_{66}>0$ (2-3-15)

Compliance matrix in terms of Eng. Const.

Compliance matrix in terms of Eng. Const.

$$\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \qquad (2\text{-}3\text{-}16)$$

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad i,j=1,2,3 \quad (2\text{-}3\text{-}17)$$

$E_1>0$ & $E_2>0$ & $E_3>0$ & $G_{12}>0$ & $G_{13}>0$ & $G_{23}>0$ (2-3-18)

$1-\nu_{12}\nu_{21}>0$ & $1-\nu_{13}\nu_{31}>0$ & $1-\nu_{32}\nu_{23}>0$

Also,

$\nu_{12}<(\frac{E_1}{E_2})^{1/2}$ & $\nu_{13}<(\frac{E_1}{E_3})^{1/2}$ & $\nu_{23}<(\frac{E_2}{E_3})^{1/2}$

From (2-3-5):

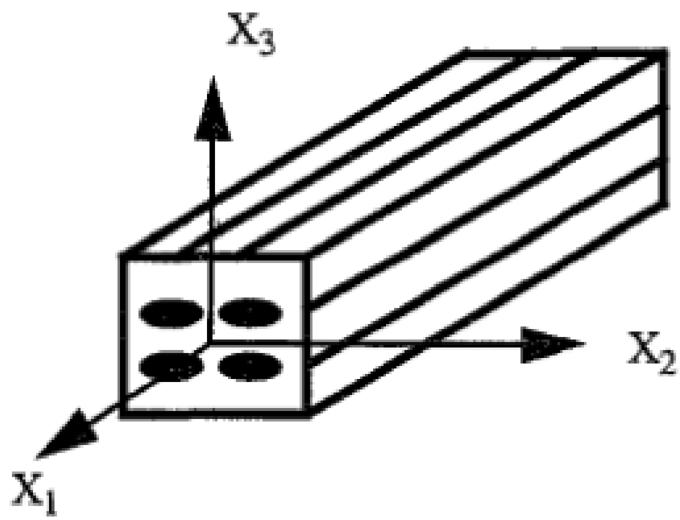
$$\Delta_3 = \frac{1}{E_1 E_2 E_3}(1-\square_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{32}\nu_{23}-2\nu_{21}\nu_{32}\nu_{13})>0 \quad (2\text{-}3\text{-}19)$$

From (2-3-18) and (2-3-19):

$$1-\square_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{32}\nu_{23}-2\nu_{21}\nu_{32}\nu_{13}>0 \quad (2\text{-}3\text{-}20)$$

2-4-On-Axis constitutive law of a orthotropic materials

Consider X_1-X_2-X_3 coordinate attached to a lamina as shown here, this coordinate is called the principle material coordinate of orthotropic Mat.



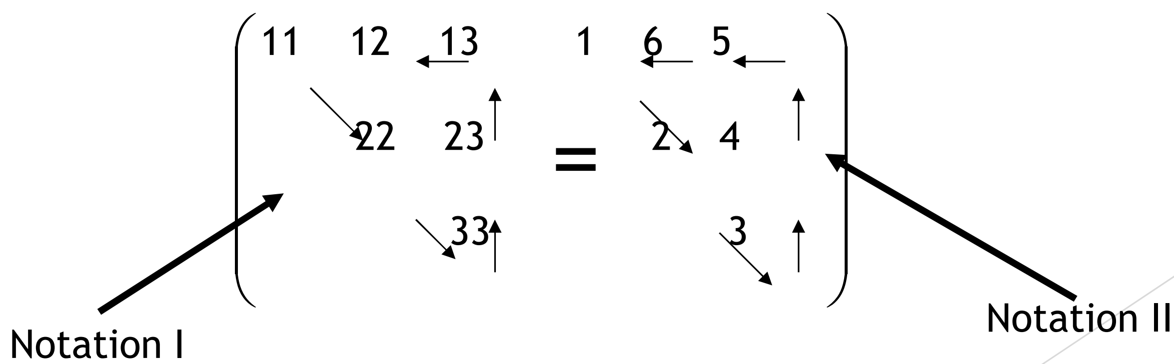
Additional simplification of the stress-strain relationship can be realized through simplifying the matrix notation for stresses and strains.

⟶ We can replace the indices as follows:

$$11 \to 1 \qquad 23 \to 4$$

$$22 \to 2 \qquad 13 \to 5$$

$$33 \to 3 \qquad 12 \to 6$$



- The foregoing transformation is easy to remember: In other to obtain notation II, one must proceed first along the diagonal ($1 \rightarrow 2 \rightarrow 3$) and then back ($4 \rightarrow 5 \rightarrow 6$).

- Notation II method makes life very easy when correlating the stresses and strains for general case, in which the elastic properties of a material are dependent on its orientations.

We now have the stress and strain, in general form, as

$$\begin{pmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{pmatrix} and \begin{pmatrix} \varepsilon_1 & \frac{\varepsilon_6}{2} & \frac{\varepsilon_5}{2} \\ \frac{\varepsilon_6}{2} & \varepsilon_2 & \frac{\varepsilon_4}{2} \\ \frac{\varepsilon_5}{2} & \frac{\varepsilon_4}{2} & \varepsilon_3 \end{pmatrix}$$

It should be noted that $\varepsilon_1 = \varepsilon_{11}$, $\varepsilon_2 = \varepsilon_{22}$, and $\varepsilon_3 = \varepsilon_{33}$, but

$$\varepsilon_4 = 2\varepsilon_{23} = \gamma_{23}$$

$$\varepsilon_5 = 2\varepsilon_{13} = \gamma_{13} \qquad (7\text{-}5)$$

$$\varepsilon_6 = 2\varepsilon_{12} = \gamma_{12}$$

The generalized Hook's law in the principal Mat. Coor.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

On-Axis constitutive law of a orthotropic Mat.

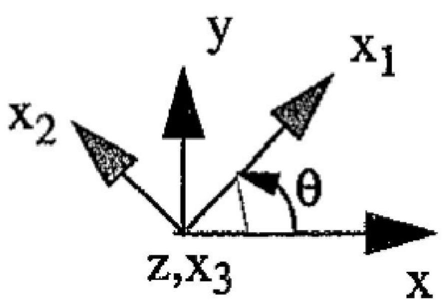
By inverting recent relationship:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

[C] is called the on-axis stiffness Matrix.

2-4-Off-Axis constitutive law of a orthotropic materials

m=cosθ and n=sinθ



$$\{\sigma\}_1 = [T_1]\{\sigma\}_x$$

$$\{\varepsilon\}_1 = [T_2]\{\varepsilon\}_x$$

where the transformation matrix $[T_1]$ is

$$[T_1] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix}$$

$$\{\sigma\}_1 = [C]\{\varepsilon\}_1$$

$$[T_1]\{\sigma\}_x = [C][T_2]\{\varepsilon\}_x$$

$$\{\sigma\}_x = [T_1]^{-1}[C][T_2]\{\varepsilon\}_x$$

$$\{\sigma\}_x = [\bar{C}]\{\varepsilon\}_x$$

$$[\bar{C}] \equiv [T_1]^{-1}[C][T_2]$$

$$\bar{C}_{11} = m^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{22}$$

$$\bar{C}_{12} = n^2 m^2 (C_{11} + C_{22} - 4C_{66}) + (n^4 + m^4) C_{12}$$

$$\bar{C}_{13} = m^2 C_{13} + n^2 C_{23}$$

$$\bar{C}_{16} = nm[m^2 (C_{11} - C_{12} - 2C_{66}) + n^2 (C_{12} - C_{22} + 2C_{66})]$$

$$\bar{C}_{22} = n^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + m^4 C_{22}$$

$$\bar{C}_{23} = n^2 C_{13} + m^2 C_{23}$$

$$\bar{C}_{26} = nm[n^2 (C_{11} - C_{12} - 2C_{66}) + m^2 (C_{12} - C_{22} + 2C_{66})]$$

$$\bar{C}_{33} = C_{33}$$

$$\bar{C}_{36} = mn(C_{13} - C_{23})$$

$$\bar{C}_{44} = m^2 C_{44} + n^2 C_{55}$$

$$\bar{C}_{45} = mn(C_{55} - C_{44})$$

$$\bar{C}_{55} = n^2 C_{44} + m^2 C_{55}$$

$$\bar{C}_{66} = n^2 m^2 (C_{11} - 2C_{12} + C_{22}) + (n^2 - m^2)^2 C_{66}$$

$$\{\varepsilon\}_x = [\bar{S}]\{\sigma\}_x$$

where the compliance $[\bar{S}]$ is the inverse of stiffness and

$$[\bar{S}] = [\bar{C}]^{-1} = ([T_1]^{-1}[C][T_2])^{-1}$$

$$[\bar{S}] = [T_2]^{-1}[C]^{-1}([T_1]^{-1})^{-1}$$

$$[\bar{S}] = [T_2]^{-1}[S][T_1]$$

The transformed compliance matrix then has the same symmetric form as the transformed stiffness matrix:

$$[\bar{S}] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & \bar{S}_{36} \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{36} & 0 & 0 & \bar{S}_{66} \end{bmatrix}$$

$$\bar{S}_{11} = m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22}$$

$$\bar{S}_{12} = n^2 m^2 (S_{11} + S_{22} - S_{66}) + (n^4 + m^4) S_{12}$$

$$\bar{S}_{13} = m^2 S_{13} + n^2 S_{23}$$

$$\bar{S}_{16} = nm[m^2(2S_{11} - 2S_{12} - S_{66}) + n^2(2S_{12} - 2S_{22} + S_{66})]$$

$$\bar{S}_{22} = n^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + m^4 S_{22}$$

$$\bar{S}_{23} = n^2 S_{13} + m^2 S_{23}$$

$$\bar{S}_{26} = nm[n^2(2S_{11} - 2S_{12} - S_{66}) + m^2(2S_{12} - 2S_{22} + S_{66})]$$

$$\bar{S}_{33} = S_{33}$$

$$\bar{S}_{36} = 2mn(S_{13} - S_{23})$$

$$\bar{S}_{44} = m^2 S_{44} + n^2 S_{55}$$

$$\bar{S}_{45} = mn(S_{55} - S_{44})$$

$$\bar{S}_{55} = n^2 S_{44} + m^2 S_{55}$$

$$\bar{S}_{66} = 4n^2 m^2 (S_{11} - 2S_{12} + S_{22}) + (n^2 - m^2)^2 S_{66}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & \bar{S}_{36} \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{36} & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$