Turbulence

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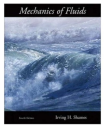
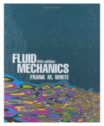
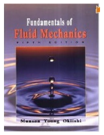
References







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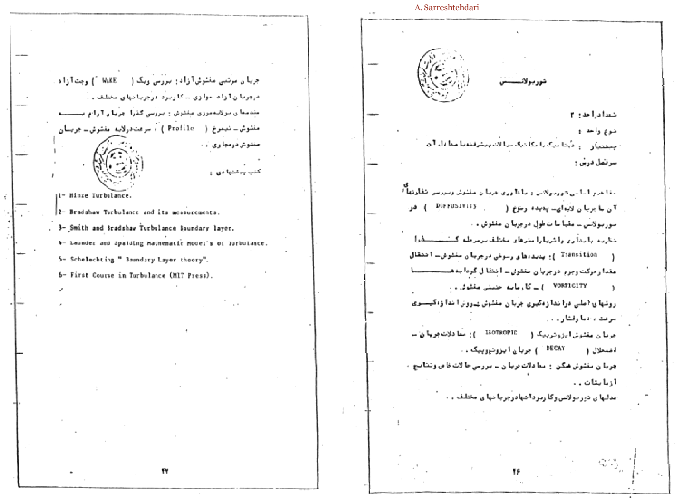


- **Course Purpose:**

  Turbulent flows, with **emphasis on engineering** methods. **Governing equations** for momentum, energy, and species transfer. Turbulence: its **production**, **dissipation**, and **scaling** laws. Reynolds averaged **equations** for momentum, energy, and species transfer. Simple closure **approaches** for free and bounded turbulent shear flows. **Applications** to jets, pipe and channel flows, boundary layers, buoyant plumes and thermals, and Taylor dispersion, etc., including heat and species transport as well as flow fields. Introduction to more complex closure schemes, including the k-epsilon, and statistical methods in turbulence.

Course Outline (tentative and not exactly sorted)

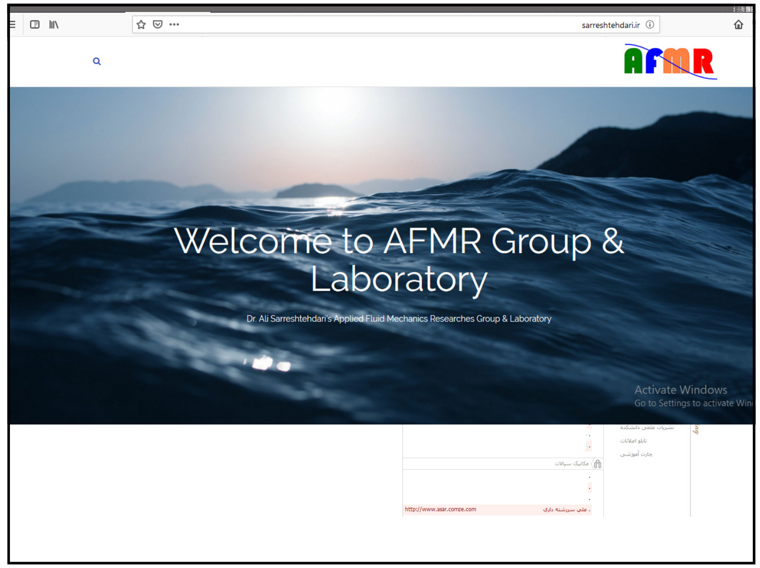
- **Review** of flow and transport equations, with particular emphasis on the energy equation and the role of viscous dissipation.
- **Instability** and transition.
- Fundamental **concepts** in turbulence; approaches to closure and turbulence modeling.
- Jets, wakes, etc. modeled via simple closure schemes. Scalar transport in free flows (temperature, concentration).
- Turbulent flow **over walls**: general near-wall scaling laws; flows in pipes, channels, etc.
- **Boundary layers**.
- Transient dispersion in laminar and turbulent shear flows (**Taylor dispersion**).
- Turbulence **models** and their application.
- Buoyant plumes, transient thermals, etc.
- Additional topics (if time).

- **Course project:**

Each student has to do a project specific title agreed with the instructor in advance. Please talk me individually four weeks before ending of the term in order to finalize it. The Project deadline is the last week of the current term. ***Take it Serious.***

- **Marking Strategy:**

Activity	Mark
Homework and Quiz	(30)%
Midterm Exam and/or Term Project	(40)%
Final Exam	(30)%



MSc/PhD course in Turbulence, 2013

Basic information

The course homepage is.www.asar.comze.com/Turbulence/2013/Turbulence91.htm The course is open to Master/PhD students at the two master programmes at Applied Mechanics (Energy Transformation, and Energy Systems), and PhD students enrolled anywhere. If you are interested in taking the course you should contact me at sarreshtehdari@shahroodut.ac.ir .so that I can maintain an e-mail list that will be used for further information until the course starts.

Prerequisites and preparations

You should make sure that you have time to take the course. You should have a background in Fluid Dynamics. You should be able to identify a suitable project that fulfils the requirements of this course, and that you are able to complete in the available time. It is beneficial if it is related to a project you are anyway doing, or planning to do (PhD project/Master project etc.), since it will be more useful to you and you will put more effort into it. We will of course discuss the project before you start doing it. It is HIGHLY recommended that you make sure that you can run Linux and OpenFOAM from your own laptop.

Syllabus

The course gives an overview on Turbulence topics such as:

- Review of flow and transport equations, with particular emphasis on the energy equation and the role of viscous dissipation.
- Instability and transition.
- Fundamental concepts in turbulence; approaches to closure and turbulence modeling.
- Jets, wakes, etc. modeled via simple closure schemes. Scalar transport in free flows (temperature, concentration).
- Buoyant plumes, transient thermals, etc.

درس توربولانس کارشناسی ارشد/دکتری، ۱۳۹۱

اطلاعات کلی

نشانی اینترنتی این درس www.asar.comze.com/Turbulence/2013/Turbulence91.htm است و قابل اخذ برای دانشجویان کارشناسی ارشد و دکتری (تبدیل و سیستم های انرژی) می باشد. در صورت نیاز به اطلاعات بیشتر می توانید با اینجانب به نشانی sarreshtehdari@shahroodut.ac.ir تماس بگیرید.

آمادگی و پیشنیاز

برای مقاطع کارشناسی ارشد و دکترای یاد شده، درس مذکور از نیم سال اول تحصیلی قابل اخذ میباشد. زمان لازم برای حضور در کلاس و انجام پروژه های درسی و پروژه نهایی آن، آشنایی با مبانی مکانیک سیالات و نرم افزارهای شبیه سازی جریان سیالات به ویژه OpenFOAM توصیه می شود.

سرفصل ها

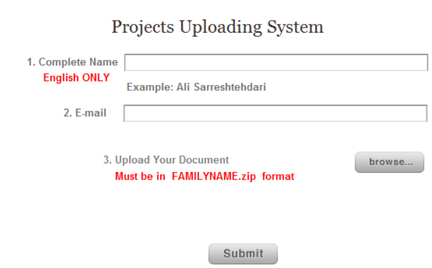
مطابق با برنامه اعلام شده توسط وزارت علوم تحقیقات و فناوری. با اختصاص اولویت های تعیین شده در کلاس.

تمرین ها و گزارش های دانشجویان

در این بخش می توانید گزارش و فایل های نهایی مربوط به پروژه دانشجویان را مشاهده و دریافت نمایید.

بارگذاری فایل

دسترسی به فایل تمرین های دانشجویان



"Congratulation & Thank you Page"

9

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- **ارسطو مبادی علوم را به ۳ دسته تقسیم می‌کند و معتقد است که هر کس به آموختن دانشی می‌پردازد، باید تا جایی که به آن دانش ارتباط می‌یابد، شناختهایی در آن‌ها حاصل کند.**
- **این مبادی عبارتند از:**
  - **حدود (یا تعاریف)،**
  - **اصول متعارفه و**
  - **اصول موضوعه.**

- **حد بیانگر ماهیت چیزهاست و معانی الفاظی را بیان می‌کند که در گزاره‌های دانشی خاص، موضوع و محمول واقع می‌شوند.** Definitions

بنابراین، هرچند وجود یا عدم آن معانی در جای دیگری از آن علم مورد بحث قرار گیرد، تعریف آنها با آنکه در قالب گزاره بیان می‌شود، حکمی نیست که خود نیازمند اثبات باشد، بلکه از مبادیِ «تصوری» علم به شمار می‌رود. مثلاً «خط راست» در علم هندسه.

- **اصول متعارفه قضایایی هستند که درستی آنها به خودی خود یقینی و ثابت است و عقل سلیم آنها را بدون واسطه تصدیق می‌کند.** Axioms

اینگونه قضایا قواعد حاکم بر هرگونه شناخت علمی را تشکیل می‌دهند، بدین معنا که روابط معلومات جدید، از هر موضوعی که باشند، ضرورتاً تابع این اصول عام است که عقل به نحو ذاتی به آنها گواهی می‌دهد. بنیادی‌ترین اصل متعارف، اصل امتناع تناقض است که بدون پذیرش ضمنی آن هیچ شناختی امکان‌پذیر نیست. مثلا بزرگتر بودن کل از جز.

- **اصل موضوع قضیه‌ای است کلی و غیر بدیهی که ذاتاً بی‌نیاز از اثبات نیست، اما در دانشی خاص بدون استدلال پذیرفته می‌شود و استنتاج احکام دیگر بر پایهٔ آن صورت می‌گیرد.** Postulates

چنین قضیه‌ای در علمی که آن را اصل می‌گیرد، قابل اثبات نیست، زیرااثبات آن مبتنی بر مقدماتی بیرون از حوزهٔ آن علم است، و اگر فرض شود که چنین مقدماتی در آن علم وجود دارد، همانها را باید اصل تلقی کرد. مثل اصول موضوعه هندسه اقلیدسی.

10

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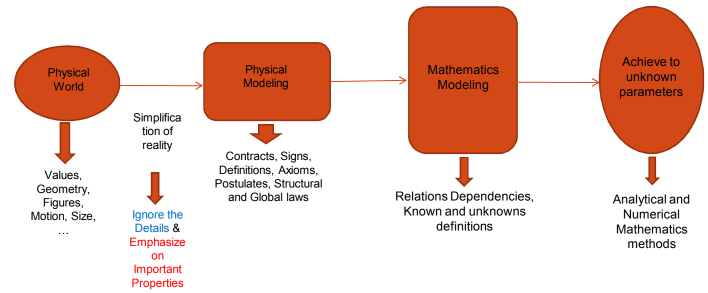
- **استدلال ریاضی: آنچه از مبادی فوق نتایج قطعی و یقینی جدید به دست می دهد (برهان).**
- **مدل ریاضی**



11

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Analysis Techniques of Turbulence Problems

Methods	Status: Now	Status: Past
Numerical Methods	*****	**
Experimental	****	***
Analytical Methods	***	****

12

Deterministic Chaos

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- A **chaotic system** is defined as one in which the **solution** is **extremely sensitive** to **initial** conditions (or solutions are **aperiodic**).
- Consider following PDE: $\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = -\beta z + xy, \end{cases}$ (Lorenz equation)

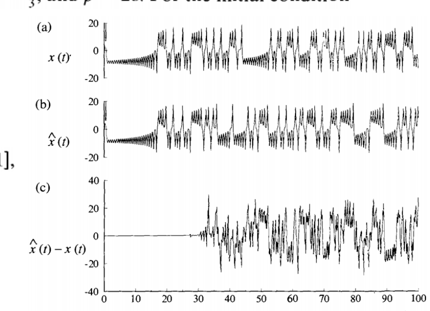
where the coefficients are $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. For the initial condition

- a)

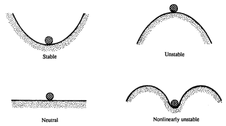
$[x(0), y(0), z(0)] = [0.1, 0.1, 0.1]$,

- b)

different initial condition $[x(0), y(0), z(0)] = [0.100001, 0.1, 0.1]$,



Instability



- Smooth **laminar flows** are **stable** to **small disturbances** only when certain conditions are satisfied.
- when the flow **becomes** a **superposition** of various **large disturbances** of random phases, and reaches a **chaotic** condition that is commonly described as "**turbulent**."
- A real flow may be stable to **infinitesimal disturbances** (**linearly stable**), but still can be unstable to sufficiently **large disturbances** (**nonlinearly** unstable)
- The method of linear **stability analysis** consists of **introducing sinusoidal disturbances** on a **basic state** (background or initial state), which is the flow whose stability is being investigated.

Orr–Sommerfeld Equation

- **flow** along **x** direction, vary in the **y**: U =[U(y), 0, 0]

Decompose as: **basic** flow **plus** the **perturbation**: $\tilde{\mathbf{u}} = [U+u, v, w]$, $\tilde{p} = P + p$.

background and the **perturbed** flows **satisfy** the **Navier–Stokes** equations.
The **perturbed** flow **satisfies** the **x-momentum** equation:

$$\frac{\partial u}{\partial t} + (U+u)\frac{\partial}{\partial x}(U+u) + v\frac{\partial}{\partial y}(U+u) = -\frac{\partial}{\partial x}(P+p) + \frac{1}{\mathrm{Re}}\nabla^2(U+u),$$

<u>NOTE!</u> **Nondimensionalized** by:
L (the width of flow), **Uo** (the maximum velocity of the basic flow);
time is scaled **by L/Uo** and the pressure is scaled by ρUo^2 . The Reynolds number is defined as **Re = UoL/v**.

- The background flow satisfies $0 = -\frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 U$
- **Subtracting** from last equation and **neglecting** terms **nonlinear** in the perturbations, the **x-momentum** equation **for** the **perturbations**:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u$$

- Similarly the **y-momentum**, **z-momentum**, and **continuity** equations:

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v,$$
$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w,$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

- **Squire's Theorem**, 1933, showing that to each unstable three-dimensional disturbance there corresponds a more unstable two-dimensional one.
- The **critical Reynolds** number at which the instability starts is lower for two-dimensional disturbances.

$\hat{u} = \phi_y, \quad \hat{v} = -ik\phi.$

$$(U-c)(\phi_{yy} - k^2\phi) - U_{yy}\phi = \frac{1}{ik\,\mathrm{Re}}[\phi_{yyyy} - 2k^2\phi_{yy} + k^4\phi],$$

Is the well-known Orr–Sommerfeld equation, which governs the stability of nearly parallel viscous flows such as those in a straight channel or in a boundary layer.

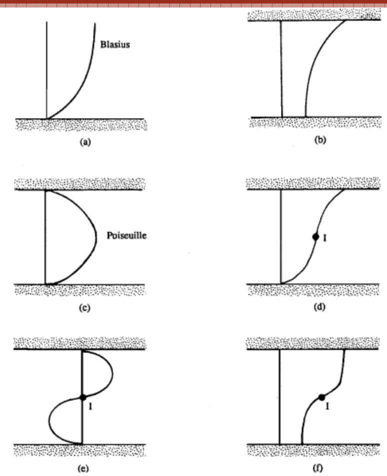
- <u>**Rayleigh's Inflection Point Criterion**</u>

A **necessary** (but not sufficient) **criterion** for instability of an inviscid parallel flow is that the basic velocity **profile** U(y) has a point of **inflection**.

- <u>**Fjortoft's Theorem**</u>

A **necessary** condition for instability of inviscid parallel flows is that
Uyy (U – U1)< 0 somewhere in the flow, where U1 is the value of U at the point of **inflection**.

Note! an alternate way of stating Fjortoft's theorem is that the magnitude of vorticity of the basic flow must have a maximum within the region of flow, not at the boundary.



- Examples of parallel flows.

Points of inflection are denoted by I.

Only (e) and (f) satisfy Fjortoft's criterion of inviscid instability.

Some Results of Parallel Viscous Flows

- **Plane Poiseuille Flow**
  - Channel flow with parabolic velocity distribution (**no** point of **inflection** and is inviscidly stable) -> **complicated** solution of **Sommerfeld** eq.
  - ❑ **linear viscous calculations:** flow becomes unstable at a critical Reynolds number of **5780**
  - ❑ **Nonlinear calculations:** give a critical number of **2510** (which agrees better with the observed transition)

Note! In flows with **inflection** points, **viscosity acts** as a **singular** perturbation. **Instability** caused waves in these flows are called **Tollmien–Schlichting waves**.

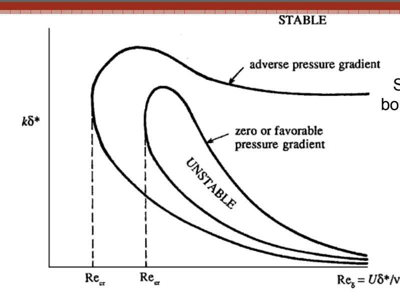
- **pipe Flow**
  - Absence of an **inflection** point -> Inviscidly stable.
  - **Stability calculations** of the viscous problem have also shown stable **flow** to small disturbances.
  - In contrast, most experiments show that the **transition** to turbulence takes place at about **Re ~ 3000** .
  - Careful **experiments**, have been able to **maintain laminar** flow until **Re = 50,000**.

- Transition attributed to:

(1) It could be a finite amplitude effect;
(2) the turbulence may be initiated at the entrance of the tube by boundary layer instability
(3) the instability could be caused by a slow rotation of the inlet flow shown to result in instability

Note! New **insights into** the **instability** and transition of **pipe flow** were described by Eckhardt et al. (**2007**) **by analysis** via dynamical systems **theory** and comparison **with** recent very carefully crafted **experiments** by them and others.

- **Boundary Layers with Pressure Gradients**
  - pressure falling in the flow direction: "**favorable**" gradient,
  - pressure rising in the flow direction:"**adverse**" gradient.
  - **Adverse** pressure gradient provide a **inflection** point in the velocity profile.



Sketch of marginal stability curves for a boundary layer with favorable and adverse pressure gradients.

Note! critical Reynolds number is lower for flows with adverse pressure gradients.

- **stable** flow for **low Reynolds** and **unstable** at **higher Reynolds** numbers.
- Increasing **viscosity effects** on **stabilizing** in this range.
- For **zero pressure gradient** boundary layers (Blasius flow) **or** a **favorable** pressure gradient, the instability **loop shrinks** to **zero** as Reδ→∞ (these flows do not have a point of inflection in the velocity profile and are therefore inviscidly stable).
- For an **adverse pressure** gradient boundary layers, the instability **loop** does **not shrink** to zero.
- Upper **branch** of the curve **becomes flat** (with a limiting value of k∞ as Reδ→∞).

Linear Stability Results of Common Viscous Parallel Flows

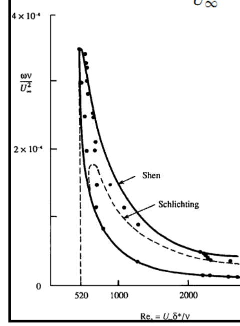
Flow	$U(y)/U_0$	Re_{cr}	Remarks
Jet	$\operatorname{sech}^2(y/L)$	4	
Shear layer	$\tanh(y/L)$	0	Always unstable
Blasius		520	Re based on δ^*
Plane Poiseuille	$1-(y/L)^2$	5780	L = half-width
Pipe flow	$1-(r/R)^2$	∞	Always stable
Plane Couette	y/L	∞	Always stable

- The **first two flows** have points of **inflection** and are **inviscidly unstable**; (viscous **solution** shows **zero or** a **small** critical **Reynolds** number).
- The **remaining** flows are **stable in** the **inviscid** limit.
- **Blasius** boundary layer **and** the plane **Poiseuille** flow are **unstable** in the presence of viscosity, but **have high** critical **Reynolds** numbers.

Experimental Verification of Boundary Layer Instability

- The **first** calculations of the **Blasius flow** based on an analysis of the **Orr–Sommerfeld** equation were performed by **Tollmien** in 1929 and **Schlichting** in 1933 using the profile:

$$\frac{U}{U_\infty} = \begin{cases} 1.7(y/\delta) & 0 \leqslant y/\delta \leqslant 0.1724, \\ 1 - 1.03\,[1-(y/\delta)^2] & 0.1724 \leqslant y/\delta \leqslant 1, \\ 1 & y/\delta \geqslant 1, \end{cases}$$



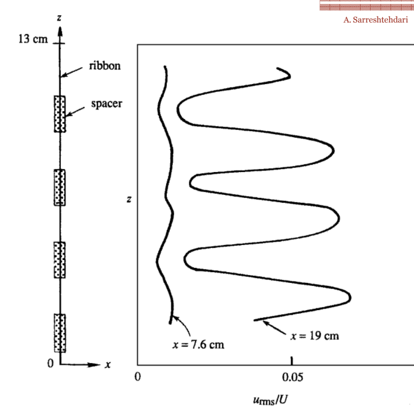
Unstable (**Tollmien–Schlichting**) **waves appear** when the **Reynolds** number is **high enough.**

Marginal stability curve for a Blasius boundary layer. Theoretical solutions of Shen and Schlichting are compared with experimental data of Schubauer and Skramstad.

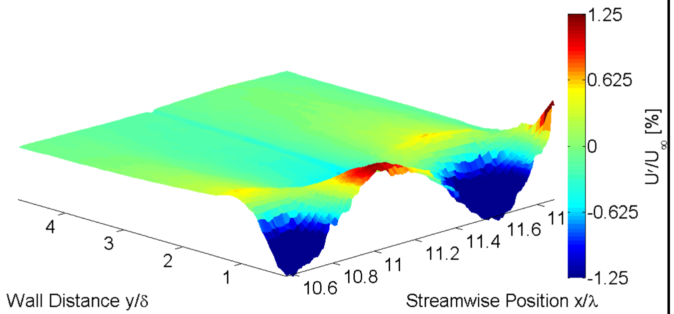
Transition

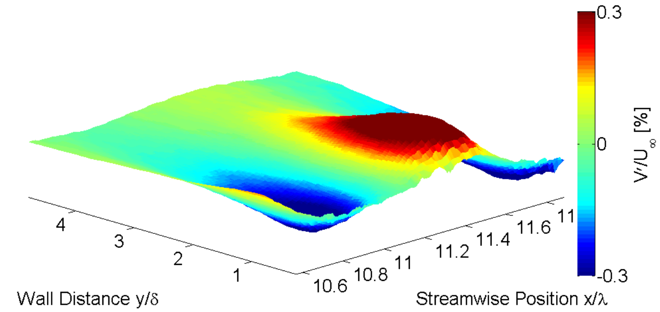
- The **process** by which a **laminar** flow changes **to** a **turbulent** one is called transition.
- The process of **transition** is greatly **affected** by such **experimental conditions** as intensity of **fluctuations** of the free stream, **roughness** of the walls, and **shape** of the inlet.
- The **basic state** of wall-bounded parallel shear flows becomes unstable to **two-dimensional TS** waves, which **grow** and eventually reach equilibrium at some finite amplitude. This steady state can be considered a new background state, and calculations show that it is generally unstable to **three-dimensional waves** of short wavelength, which vary in the "spanwise" direction (**secondary instability**).

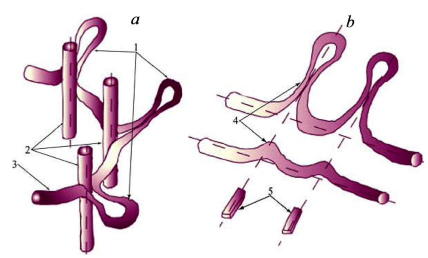
Note! If x is the direction of flow and y is the directed normal to the boundary, then the z-axis is spanwise.



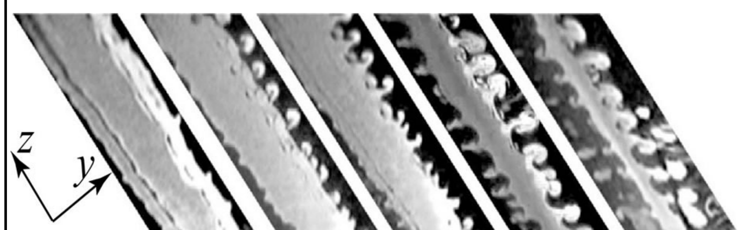
Three-dimensional unstable waves initiated by vibrating ribbon. Measured distributions of intensity of the u-fluctuation at two distances from the ribbon are shown.



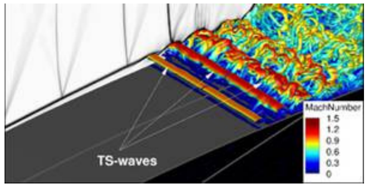




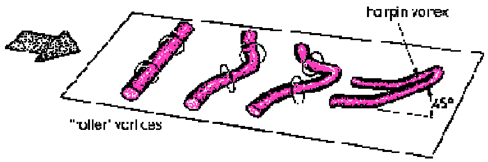
Three-dimensional deformation of the **ring vortex (a)** as compared to that of the **Tollmien-Schlichting wave in a boundary layer (b)**: 3D bursts (1), longitudinal disturbances (2), ring vortex (3), 3D distortion of the 2D instability wave (4), roughness elements (5).

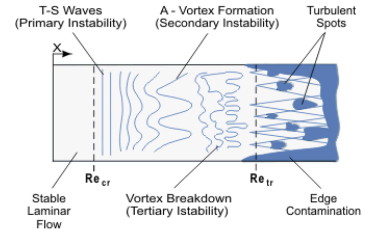


- Smoke visualization of the jet cross sections at x/h = 0.5, 2.5, 4.5, 6.5 and 8.5 from the nozzle exit (left to right); nozzle width h = 10 mm, jet core velocity U_o = 3 m/s, Re_h = 2000.

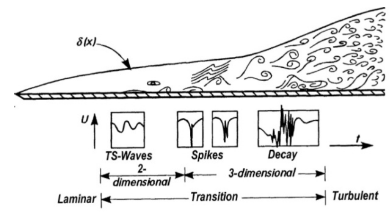


airfoil flow; M=0.7, Re=1 million





Idealized sketch of the transition process from White (1991)

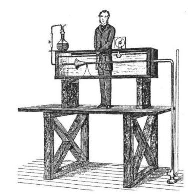


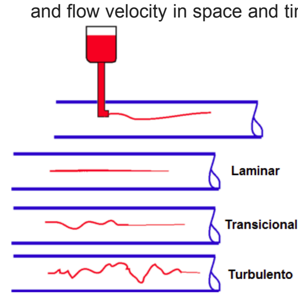
Turbulence

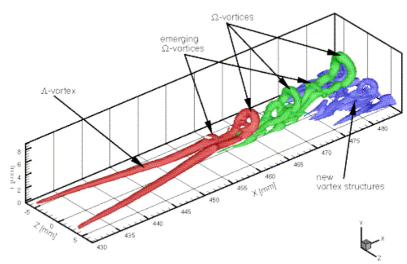
- **Most flows** encountered in **engineering** practice and in **nature** are turbulent.
- Turbulence is **not easy** to define precisely.
- Lesieur (1987) :"**turbulence** is a **dangerous topic** which is at the origin of serious fights in scientific meetings since it represents extremely **different** points of **view**, all of which have in **common** their **complexity**, as well as an **inability** to solve the problem. It is even difficult to agree on what exactly is the problem to be solved."

Turbulence

In fluid dynamics, **turbulence** or **turbulent** flow is a flow regime characterized by chaotic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and flow velocity in space and time.

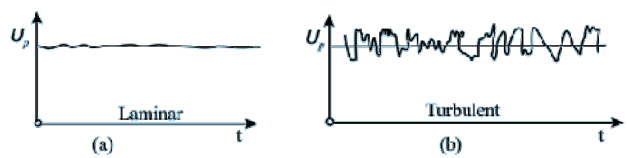






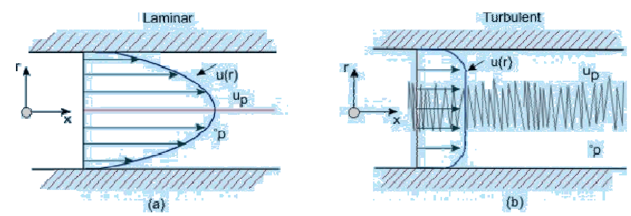
Turbulence

- Definition of Turbulence. **Hinze (1959):**
  "Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned."



Some of turbulence's characteristics

- (1) **Randomness**: Turbulent flows seem **irregular**, **chaotic**, and **unpredictable**.
- (2) **Nonlinearity**: Turbulent flows are **highly** nonlinear.
  - Note! Two purposes of nonlinearity :
    - it causes the relevant nonlinearity parameter, (e.g. Re)
    - nonlinearity of a turbulent flow results in vortex stretching, a key process by which three-dimensional turbulent flows maintain their vorticity.

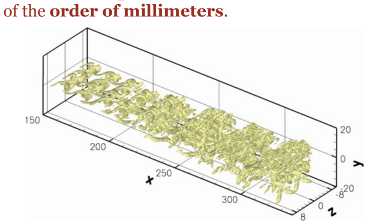


Some of turbulence's characteristics

- (3) **Diffusivity**: Due to the macroscopic mixing of fluid particles, turbulent flows are characterized by a **rapid rate** of **diffusion** of **momentum** and **heat**.

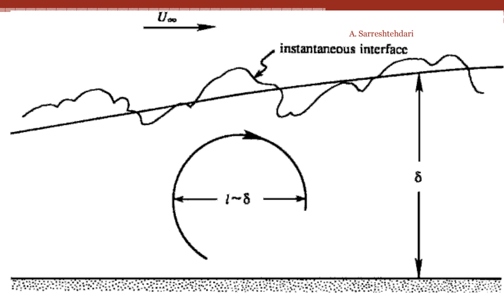
Some of turbulence's characteristics

- (4) **Vorticity**: Turbulence is characterized by **high levels** of **fluctuating vorticity**.
  - Note! The identifiable structures in a turbulent flow are vaguely called eddies. A **characteristic** feature **of turbulence** is the **existence** of an **enormous range** of **eddy sizes**.
  - The **large eddies** have a size of **order** of the **width of** the **region** of turbulent flow.
  - The **large eddies contain most** of the **energy.**
  - The **energy** is **handed down** from large **to small** eddies **by nonlinear** interactions, **until** it is **dissipated by** viscous **diffusion in** the **smallest** eddies, whose size is of the **order of millimeters**.



37

A. Sarreshtehdari



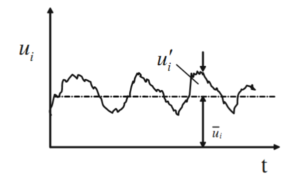
Turbulent flow in a boundary layer, showing that a large eddy has a size of the order of boundary layer thickness.

- 5. **Dissipation:** The **vortex transfers** energy and vorticity stretching mechanism to increasingly smaller scales, until the gradients become so large that they are smeared out (i.e., dissipated) by viscosity.
  - **Note! Turbulent** flows therefore **require** a continuous **supply** of **energy** to make up for the viscous losses.

38

A. Sarreshtehdari

- many **flows** that **seem "random,"** such as **gravity waves** in the ocean or the atmosphere, are **not** turbulent because they are **not dissipative**, **vortical**, and **nonlinear**.
- The **turbulent** flow **variables** are **not deterministic** in details and have to be **treated** as **stochastic** or random variables.
- **Averaging**



Mean and fluctuating components of a flow property in a turbulent flow

$$\bar{u} \equiv \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} u(t)\, dt .$$

39

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- **Averaging**

$$\bar{u}(x,y,z) = \lim_{T\to\infty} \frac{1}{T} \int_{to}^{to+T} u(x,y,z,t)\mathrm{d}t$$

- Time-Average
  - **appropriate** for **stationary** turbulence which is the one whose time average does not vary with time.
  - Characteristics of time-average:

$$\overline{g'} = 0, \quad \overline{g\bar{h}} = \bar{g}\bar{h}, \quad \bar{\bar{g}} = \bar{g}$$

$$\overline{g+h} = \bar{g} + \bar{h}, \quad \overline{gh} = \bar{g}\bar{h} + \overline{g'h'}, \quad \overline{g'\bar{h}} = 0$$

$$\overline{a_1 g + b_1 h} = a_1 \bar{g} + b_1 \bar{h}$$

$$\overline{\frac{\partial u}{\partial t}} = \frac{\partial \bar{u}}{\partial t}, \qquad \overline{\int_a^b u\, dt} = \int_a^b \bar{u}\, dt .$$

- Spatial-Average
  - **appropriate** for **homogeneous** turbulence, i.e., the one whose space average is uniform in all flow directions.

$$\bar{u}(t) = \lim_{V\to\infty} \frac{1}{V} \iiint u(x,y,z,t)\mathrm{d}V$$

40

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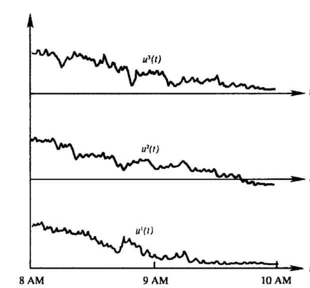
- Ensemble-Average
  - is the **most** general average and is also valid for time dependent mean flows.

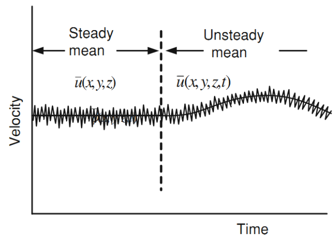
$$\bar{u}(x,y,z,t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u(x,y,z,t)$$

  - **N** denotes large number of identical experiments.

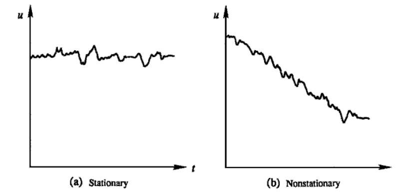


An ensemble of functions $u(t)$

- For **stationary** and **homogeneous** turbulent flow, the three **averages** are **equal**.



Steady and unsteady mean turbulent flows



Stationary and nonstationary time series

Note! For a **stationary** process the **time average** can be shown to **equal** the **ensemble** average.

- The **mean square** value of a variable is called the **variance**.
- The **square root** of **variance** is called the root-mean-square (**rms**).

$$\text{variance} \equiv \overline{u^2},$$

$$u_{\text{rms}} \equiv (\overline{u^2})^{1/2}$$

- The **rms** value **of** the **fluctuation** is called the **standard deviation.**

$$u_{\text{SD}} \equiv [\overline{(u-\bar{u})^2}]^{1/2}$$

Correlations and Spectra

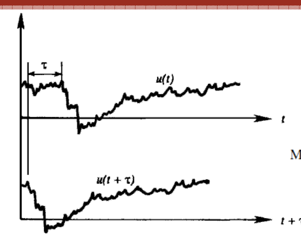
- The **autocorrelation** of a single **variable** u(t) **at two times** t1 and t2 is defined as: $R(t_1, t_2) \equiv \overline{u(t_1)u(t_2)}$
- For a **stationary** process the **statistics** (i.e., the various kinds of averages) are **independent of** the origin of **time**,

$$R(\tau) = \overline{u(t)u(t+\tau)}$$

- **normalized** autocorrelation function:

$$r(\tau) \equiv \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}}$$

$$r \leqslant 1$$



Method of calculating autocorrelation $\overline{u(t)u(t+\tau)}$

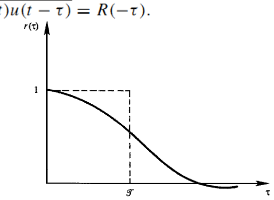
Obviously, $r(0) = 1$. For a stationary process the autocorrelation is a symmetric function, because then

$$R(\tau) = \overline{u(t)u(t+\tau)} = \overline{u(t-\tau)u(t)} = \overline{u(t)u(t-\tau)} = R(-\tau).$$

Note! Under **normal conditions** r goes to 0 as $\tau \to \infty$, because a process becomes **uncorrelated** with **itself** after a **long time**.

integral time scale $\mathcal{T} \equiv \int_0^\infty r(\tau)\, d\tau$

Is measure of the **time** over which u(t) is **highly correlated** with **itself** or measure of the "**memory**" **of the process**.



Autocorrelation function and the integral time scale

- If S(ω) : Fourier transform of R(τ)

$$S(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau)\, d\tau \;\Rightarrow\; R(\tau) \equiv \int_{-\infty}^{\infty} e^{i\omega\tau} S(\omega)\, d\omega \;\Rightarrow\; \overline{u^2} = \int_{-\infty}^{\infty} S(\omega)\, d\omega$$

"Fourier transform pair."

⇒
- S(ω)dω: energy (variance) in a **frequency band** dω centered at ω.
- S(ω): represents the way **energy** is **distributed** as a function of frequency ω.
- S(ω) is the **energy spectrum**

$$S(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\, d\tau = \frac{\overline{u^2}}{\pi}\int_0^{\infty} r(\tau)\, d\tau = \frac{\overline{u^2}\mathcal{T}}{\pi}$$

⇒ **Spectrum** value **at zero** frequency is **proportional** to the integral **time scale**.

Taylor 's hypothesis

- The **assumption** that the turbulent **fluctuations** at a **point** are caused by the **advection** of a **frozen** field past the point.
- If the turbulence field is "**frozen**" and does not change during the measurement it **is possible** to transfer **time** series u(t), **to** a **spatial** series u(x) by replacing t by x/Uo .

- **cross-correlation** function between **two** stationary variables **u(t)** and **v(t)**: $C(\tau) \equiv \overline{u(t)v(t+\tau)}$
- is **not** a **symmetric** function of the time lag τ,

$$C(-\tau) = \overline{u(t)v(t-\tau)} \neq C(\tau)$$

Note! cross-correlation at **zero** lag: $\overline{u(t)v(t)}$, **simply** written: $\overline{uv}$ ("**correlation**" of u and v).

Averaged Equations of Motion

Reynolds decomposition
$$\tilde{u}_i = U_i + u_i,\quad \tilde{p} = P + p,\quad \tilde{T} = \bar{T} + T'.$$
⇒ $\bar{u}_i = \bar{p} = \overline{T'} = 0,$

Averaging the continuity equation

$$\overline{\frac{\partial}{\partial x_i}(U_i + u_i)} = \frac{\partial U_i}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_i} = 0 \xrightarrow{\text{Using } \bar{u}_i = 0,} \boxed{\frac{\partial U_i}{\partial x_i} = 0,}$$

Subtracting ⇒ $\boxed{\frac{\partial u_i}{\partial x_i} = 0,}$

Mean Momentum Equation

The momentum equation

$$\frac{\partial}{\partial t}(U_i + u_i) + (U_j + u_j)\frac{\partial}{\partial x_j}(U_i + u_i)$$
$$= -\frac{1}{\rho_0}\frac{\partial}{\partial x_i}(P + p) - g[1 - \alpha(\bar{T} + T' - T_0)]\,\delta_{i3} + \nu\frac{\partial^2}{\partial x_j^2}(U_i + u_i)$$

$$\overline{\frac{\partial}{\partial t}(U_i + u_i)} = \frac{\partial U_i}{\partial t} + \overline{\frac{\partial u_i}{\partial t}} = \frac{\partial U_i}{\partial t} + \frac{\partial \bar{u}_i}{\partial t} = \frac{\partial U_i}{\partial t},$$ The average of the time derivative term

$$\overline{(U_j + u_j)\frac{\partial}{\partial x_j}(U_i + u_i)} = U_j\frac{\partial U_i}{\partial x_j} + U_j\frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_j\frac{\partial U_i}{\partial x_j} + \overline{u_j\frac{\partial u_i}{\partial x_j}}$$
$$= U_j\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}),$$ The average of the advective term

$$\overline{\frac{\partial}{\partial x_i}(P + p)} = \frac{\partial P}{\partial x_i} + \frac{\partial \bar{p}}{\partial x_i} = \frac{\partial P}{\partial x_i}$$ The average of the pressure gradient term

$$\overline{g[1 - \alpha(\bar{T} + T' - T_0)]} = g[1 - \alpha(\bar{T} - T_0)]$$ The average of the gravity term

$$\overline{\nu\frac{\partial^2}{\partial x_j \partial x_j}(U_i + u_i)} = \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}$$ The average of the viscous term

⇒
$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - g[1 - \alpha(\bar{T} - T_0)]\,\delta_{i3} + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

Reynolds Stress

Writing the term $\overline{u_i u_j}$ on the right-hand side, the mean momentum equation becomes

$$\frac{DU_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - g[1 - \alpha(\bar{T} - T_0)]\,\delta_{i3} + \frac{\partial}{\partial x_j}\left[\nu\frac{\partial U_i}{\partial x_j} - \overline{u_i u_j}\right],$$

which can be written as

$$\boxed{\frac{DU_i}{Dt} = \frac{1}{\rho_0}\frac{\partial \bar{\tau}_{ij}}{\partial x_j} - g[1 - \alpha(\bar{T} - T_0)]\,\delta_{i3},}$$

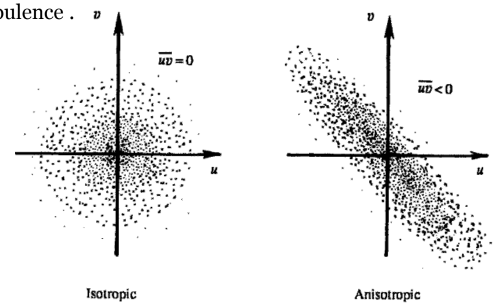
where

$$\boxed{\bar{\tau}_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho_0\overline{u_i u_j}.}$$ ⟶ *additional* stress $-\rho_0\overline{u_i u_j}$

The tensor $-\rho_0\overline{u_i u_j}$ is called the *Reynolds stress tensor* and has the nine Cartesian components

$$\begin{bmatrix} -\rho_0\overline{u^2} & -\rho_0\overline{uv} & -\rho_0\overline{uw} \\ -\rho_0\overline{uv} & -\rho_0\overline{v^2} & -\rho_0\overline{vw} \\ -\rho_0\overline{uw} & -\rho_0\overline{vw} & -\rho_0\overline{w^2} \end{bmatrix}.$$ (symmetric tensor)

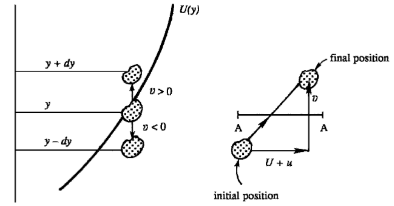
- In **isotropic** turbulent, the **off-diagonal** components of **vanish**, and $\overline{u^2} = \overline{v^2} = \overline{w^2}$
- **Isotropic**: there is not **any directional preference.**
- The **average** value of the product uv **is zero** in **isotropic** turbulence .



Isotropic and anisotropic turbulent fields. Each dot represents a uv-pair at a certain time.

- **Exercise:**

  regard to following fig. describe: why the average product of the velocity fluctuations in a turbulent flow is not expected to be zero.



- **Reynolds stress** is that it is the rate of **mean momentum** transfer **by** turbulent **fluctuations.**

$$\rho_0\overline{(U+u)v} = \rho_0 U\bar{v} + \rho_0\overline{uv} = \rho_0\overline{uv}.$$

Generalizing, $\rho_0\overline{u_iu_j}$ *is the average flux of j-momentum along the i-direction, which also equals the average flux of i-momentum along the j-direction.*

Kinetic Energy Budget of Mean Flow

- A kinetic energy equation can be obtained by multiplying the equation for DU/Dt by U .

The equation of motion for the mean flow,

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = \frac{1}{\rho_0}\frac{\partial \bar{\tau}_{ij}}{\partial x_j} - \frac{g}{\rho_0}\bar{\rho}\delta_{i3},$$

where the stress is given by

$$\bar{\tau}_{ij} = -P\delta_{ij} + 2\mu E_{ij} - \rho_0\overline{u_iu_j}.$$

Here we have introduced the mean strain rate

$$E_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

Multiplying by Ui,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}U_i^2\right) + U_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}U_i^2\right) = \frac{1}{\rho_0}\frac{\partial}{\partial x_j}(U_i\bar{\tau}_{ij}) - \frac{1}{\rho_0}\bar{\tau}_{ij}\frac{\partial U_i}{\partial x_j} - \frac{g}{\rho_0}\bar{\rho}U_i\delta_{i3}.$$

$$\frac{D}{Dt}\left(\frac{1}{2}U_i^2\right) = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho_0}U_iP\delta_{ij} + 2\nu U_iE_{ij} - \overline{u_iu_j}U_i\right) + \frac{1}{\rho_0}P\delta_{ij}\frac{\partial U_i}{\partial x_j} - 2\nu E_{ij}\frac{\partial U_i}{\partial x_j} + \overline{u_iu_j}\frac{\partial U_i}{\partial x_j} - \frac{g}{\rho_0}\bar{\rho}U_3.$$

proportional to $\delta_{ij}(\partial U_i/\partial x_j) = \partial U_i/\partial x_i$ → 0 (Continuity)

Exercise: describe the physical meaning of this Zero term.

The mean kinetic energy balance then becomes

$$\frac{D}{Dt}\left(\frac{1}{2}U_i^2\right) = \frac{\partial}{\partial x_j}\left(-\frac{PU_j}{\rho_0} + 2\nu U_iE_{ij} - \overline{u_iu_j}U_i\right) - 2\nu E_{ij}E_{ij} + \overline{u_iu_j}\frac{\partial U_i}{\partial x_j} - \frac{g}{\rho_0}\bar{\rho}U_3.$$

transport; viscous dissipation; loss to turbulence; loss to potential energy

- The **first** term: **transport** of **mean** kinetic energy **by** the mean **pressure**, **the second** by the mean **viscous** stresses , and the **third** by **Reynolds** stresses.
- product of the **mean strain rate** and the mean **viscous stress**. It is a loss at every point, (**direct viscous dissipation**). The energy is lost to **heat**.
- **loss** of **mean** kinetic energy and a **gain** of **turbulent** kinetic energy (the **shear production of turbulence** by the **interaction** of **Reynolds** stresses and the **mean shear**).
- the **work** done **by gravity** on the mean vertical motion.

- The **two viscous** terms, namely, the viscous **transport** and the viscous **dissipation**, are **small** in a fully turbulent flow at high Reynolds numbers.(e.g.) Compare, the viscous dissipation and the shear production terms:

$$\frac{2\nu E_{ij}^2}{\overline{u_i u_j}(\partial U_i/\partial x_j)} \sim \frac{\nu (U/L)^2}{u_{\mathrm{rms}}^2 U/L} \sim \frac{\nu}{UL} \ll 1,$$

- The **mean flow loses** energy **to** the **turbulent** field by means of the **shear** production; the **turbulent** kinetic energy so **generated** is **then dissipated** by viscosity.

Kinetic Energy Budget of Turbulent Flow

Equations of motion for the total and mean flows are, respectively,

$$\frac{\partial}{\partial t}(U_i + u_i) + (U_j + u_j)(U_i + u_i)_{,j}$$
$$= -\frac{1}{\rho_0}(P+p)_{,i} - g[1 - \alpha(\bar{T} + T' - T_0)]\delta_{i3} + \nu(U_i + u_i)_{,jj},$$
$$\frac{\partial U_i}{\partial t} + U_j U_{i,j} = -\frac{1}{\rho_0}P_{,i} - g[1 - \alpha(\bar{T} - T_0)]\delta_{i3} + \nu U_{i,jj} - (\overline{u_i u_j})_{,j}.$$

Subtracting, we obtain the equation of motion for the turbulent velocity u_i:

$$\frac{\partial u_i}{\partial t} + U_j u_{i,j} + u_j U_{i,j} + u_j u_{i,j} - (\overline{u_i u_j})_{,j} = -\frac{1}{\rho_0} p_{,i} + g\alpha T'\delta_{i3} + \nu u_{i,jj}.$$

multiplying this equation by ui and averaging.

The first two terms on the left-hand side of equation

$$\overline{u_i \frac{\partial u_i}{\partial t}} = \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i^2}\right),$$
$$\overline{u_i U_j u_{i,j}} = U_j \left(\frac{1}{2}\overline{u_i^2}\right)_{,j}.$$

The third, fourth and fifth terms on the left-hand side of equation (

$$\overline{u_i u_j U_{i,j}} = \overline{u_i u_j} U_{i,j},$$
$$\overline{u_i u_j u_{i,j}} = (\tfrac{1}{2}\overline{u_i^2 u_j})_{,j} - \tfrac{1}{2}\overline{u_i^2 u_{j,j}} = \tfrac{1}{2}(\overline{u_i^2 u_j})_{,j}$$
$$-\overline{u_i (\overline{u_i u_j})_{,j}} = -\bar{u}_i (\overline{u_i u_j})_{,j} = 0,$$

the continuity equation $u_{i,i} = 0$ and $\bar{u}_i = 0$.

The first and second terms on the right-hand side of equation

$$\overline{-u_i \frac{1}{\rho_0} p_{,i}} = -\frac{1}{\rho_0}(\overline{u_i p})_{,i},$$
$$\overline{u_i g\alpha T'\delta_{i3}} = g\alpha \overline{wT'}.$$

The last term on the right-hand side of equation

$$\nu \overline{u_i u_{i,jj}} = \nu\{\overline{u_i u_{i,j}}_{,j} + \tfrac{1}{2}\overline{(u_{i,j} + u_{j,i})(u_{i,j} - u_{j,i})}\},$$

Defining the fluctuating strain rate by

$$e_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

we finally obtain

$$\nu \overline{u_i u_{i,jj}} = 2\nu[\overline{u_i e_{ij}}]_{,j} - 2\nu \overline{e_{ij} e_{ij}}.$$

Collecting terms, the turbulent energy equation becomes

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{u_i^2}\right) = -\frac{\partial}{\partial x_j}\underbrace{\left(\frac{1}{\rho_0}\overline{p u_j} + \frac{1}{2}\overline{u_i^2 u_j} - 2\nu \overline{u_i e_{ij}}\right)}_{\text{transport}}$$
$$\underbrace{-\overline{u_i u_j} U_{i,j}}_{\text{shear prod}} + \underbrace{g\alpha \overline{wT'}}_{\text{buoyant prod}} - \underbrace{2\nu \overline{e_{ij} e_{ij}}}_{\text{viscous diss}}.$$

$$\text{Shear production} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}.$$

$$\text{Buoyant production} = g\alpha \overline{wT'}.$$

$$\varepsilon = \text{Viscous dissipation} = 2\nu \overline{e_{ij} e_{ij}}.$$

- the viscous **dissipation** ε is of the **order** of the **turbulence** production terms $(\overline{u_i u_j} U_{i,j}$ or $g\alpha \overline{wT'})$ in most locations.

Turbulence Production and Cascade

•**Order** of **largest eddies**: **width** of the **shear** flow.
•These eddies **extract** kinetic **energy from** the **mean** field.
•The **smaller** eddies are **strained** by the **velocity** field of the largest eddies, and **extract** energy **from** the **larger eddies by** the same mechanism of vortex **stretching**.
•The **much smaller** eddies are essentially **advected** in the velocity field of the large eddies.
•The **small** eddies **do not interact** with either the **large** eddies **or** the **mean** field.

- *The **kinetic** energy is **cascaded down** from large **to small** eddies in **a series** of small **steps**. This process of energy cascade is **essentially inviscid**, as the vortex **stretching mechanism** arises **from the nonlinear** terms of the equations of motion .*

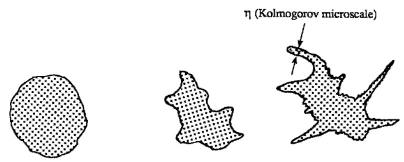
- In a **completely isotropic** field the **off-diagonal** components **of** the **Reynolds stress are zero** and **no turbulent** energy **can** be **extracted** from the **mean** field. Therefore,

***turbulence must develop anisotropy** if it has to **sustain** itself against **viscous** dissipation.*

- **viscosity** does **not affect** the **shear** production, however, **determine** the **scales** at which turbulent energy is **dissipated** into **heat**.

$\varepsilon = 2\nu \overline{e_{ij} e_{ij}}$

- The continuous **stretching** and cascade **generate long** and **thin filaments**, "**spaghetti**." When they become **thin enough**, **molecular** diffusive effects are able to **smear out** their velocity gradients.
- ε is **determined** by the inviscid **properties of** the **large eddies.**



Successive deformations of a marked fluid element.

if **l** is a typical **length scale** of the large eddies, and **u** is a typical **scale of** the **fluctuating** velocity, the **dissipation** rate must then be of **order**

$$\varepsilon \sim \frac{u'^3}{l},$$

Kolmogorov **suggested** in 1941 that the **size** of the **dissipating eddies depends** On **smallest eddies parameters**. i.e. ε and the diffusivity **ν**.
Dimensional **reasoning** shows that the **length scale** formed **from** ε and **ν** is:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4},$$

Kolmogorov microscale

Landahl and Mollo-Christensen (1986)

Suppose we are using a 100-W household mixer in 1 kg of water. As all the power is used to generate the turbulence, the rate of dissipation is $\varepsilon = 100\ \text{W/kg} = 100\ \text{m}^2/\text{s}^3$. Using $\nu = 10^{-6}\ \text{m}^2/\text{s}$ for water, we obtain $\eta = 10^{-2}$ mm.

Spectrum of Turbulence in Inertial Subrange

Wavenumber spectrum S(K), representing turbulent **kinetic energy** as a **function of** the **wavenumber** vector K . In **isotropic** turbulence, it is **independent** of the **orientation** of the **wavenumber** vector and **depends on** its **magnitude** K **only,**

$$\overline{u^2} = \int_0^\infty S(K)\, dK$$

- Somewhat vaguely, a **wavenumber** K associate with an eddy **of size** K^−1.

$$S = S(K, \varepsilon, \nu) \qquad K \gg l^{-1}.$$

- In **small scales** there is **no direct** interaction **between** the **turbulence** and the **motion** of the large, energy-containing eddies.
- The **spectrum** here does **not depend** on how much energy is present at large scales, it **depends only** on the small-scale flow nature.
- The **spectrum** in the range of large wave numbers is **nearly isotropic** and usually **called** the **equilibrium** range **.**

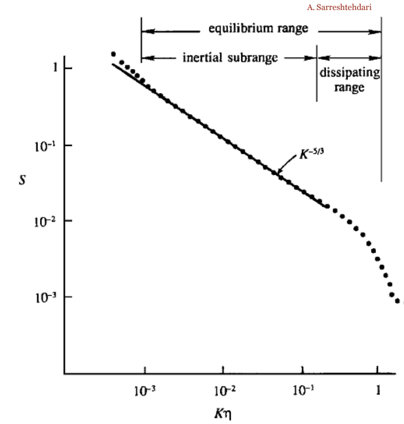
Kolmogorov argued that, in the inertial subrange part of the equilibrium range, S is independent of ν also, so that

$$S = S(K, \varepsilon) \qquad l^{-1} \ll K \ll \eta^{-1}.$$

dimensional reasoning gives

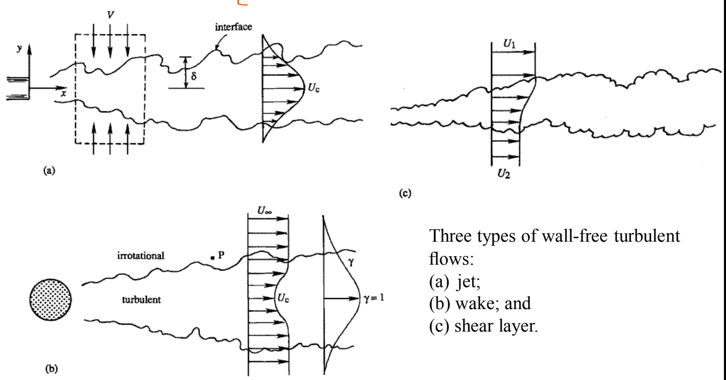
Kolmogorov's $K^{-5/3}$ law

$$S = A\varepsilon^{2/3} K^{-5/3} \qquad l^{-1} \ll K \ll \eta^{-1}, \qquad A \simeq 1.5$$



A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale.

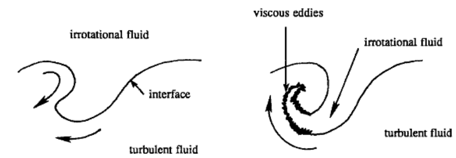
Nearly parallel shear flows
- **wall-free shear flows** (such as jets, wakes, and shear layers)
- wall-bounded shear flows



Three types of wall-free turbulent flows:
(a) jet;
(b) wake; and
(c) shear layer.

Intermittency γ is defined as the fraction of time the flow at a point is turbulent.

- A **flow** can slowly **pull** the **surrounding** irrotational fluid **inward** by "frictional" effects; the process is called **entrainment**.
- The source of this "friction" is **viscous** in **laminar** flow and **inertial** in **turbulent** flow.



- Experiments: Far **downstream**, the **mean** field in a wall-free shear flow **becomes** approximately **self-similar** at **various** downstream distances("**moving equilibrium**").
- When **both** the **mean** and the **turbulent** fields are **determined** solely **by** the **local scales** of length and velocity is called **(self-preservation)**.

- In the **self-similar** state, the **mean** velocity at various downstream distances:

$$\frac{U}{U_c} = f\left(\frac{y}{\delta}\right) \quad \text{(jet)},$$

$$\frac{U_\infty - U}{U_\infty - U_c} = f\left(\frac{y}{\delta}\right) \quad \text{(wake)},$$

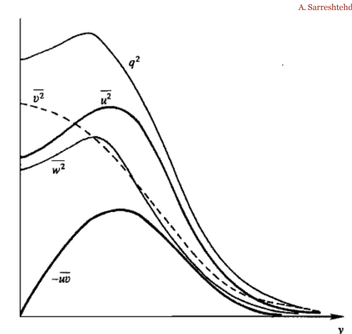
$$\frac{U - U_1}{U_2 - U_1} = f\left(\frac{y}{\delta}\right) \quad \text{(shear layer)}.$$

Here $\delta(x)$ is the width of flow, $U_c(x)$ is the centerline velocity for the jet and the wake, and U_1 and U_2 are the velocities of the two streams in a shear layer

For two-dimensional wakes and shear layers, it can be shown (Townsend, 1976; Tennekes and Lumley, 1972) that the assumption of self similarity requires

$$U_\infty - U_c \propto x^{-1/2}, \quad \delta \propto \sqrt{x} \quad \text{(wake)},$$

$$U_1 - U_2 = \text{const.}, \quad \delta \propto x \quad \text{(shear layer)}.$$

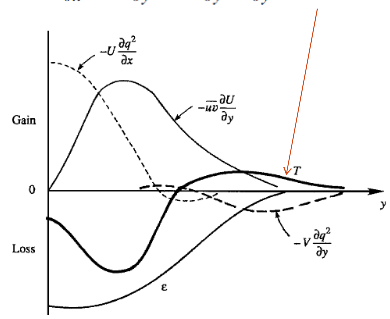


Sketch of observed variation of turbulent intensity and Reynolds stress across a jet.

$\overline{u^2}$ is the intensity of fluctuation in the downstream direction x, $\overline{v^2}$ is the intensity along the cross-stream direction y, and $\overline{w^2}$ is the intensity in the z-direction; $q^2 \equiv (\overline{u^2} + \overline{v^2} + \overline{w^2})/2$ is the turbulent kinetic energy per unit mass.

The **kinetic energy** budget For a **two-dimensional** jet under the boundary layer **assumption** $\partial/\partial x \ll \partial/\partial y$:

$$0 = -U\frac{\partial q^2}{\partial x} - V\frac{\partial q^2}{\partial y} - \overline{uv}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\left[\overline{q^2 v} + \overline{pv}/\rho\right] - \varepsilon$$



Sketch of observed kinetic energy budget in a turbulent jet. Turbulent transport is indicated by T .

•The shear production is zero at the center where both ∂U/∂y and uv are zero, and reaches a maximum close to the position of the maximum Reynolds stress.

•Near the center, the dissipation is primarily balanced by the downstream advection, which is positive because the turbulent intensity q^2 decays downstream.

•Away from the center, but not too close to the outer edge of the jet, the production and dissipation terms balance.

•In the outer parts of the jet, the transport term balances the cross-stream advection where V is negative (i.e., toward the center) due to entrainment of the surrounding fluid, (q^2 decreases with y).

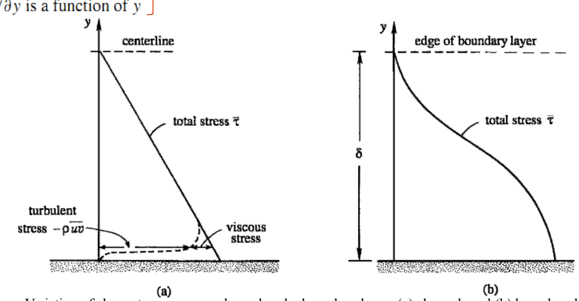
Conclusion:

The gross characteristics of free shear flows, are independent of viscosity.

Nearly parallel shear flows
- wall-free shear flows
- **wall-bounded shear flows** (e.g. channel flows)

The mean equation of motion: $0 = -\frac{\partial P}{\partial x} + \frac{\partial \bar{\tau}}{\partial y}$ where $\bar{\tau} = \mu(dU/dy) - \rho\overline{uv}$ is the total stress.

$\partial P/\partial x$ is a function of x
$\partial \bar{\tau}/\partial y$ is a function of y
→ both of them must be constants → The stress distribution is linear



Variation of shear stress across a channel and a boundary layer: (a) channel; and (b) boundary layer.

In a boundary layer on a flat plate there is no pressure gradient and the mean flow equation is

$$\rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} = \frac{\partial \bar{\tau}}{\partial y}, \quad \text{where } \bar{\tau} \text{ is a function of } x \text{ and } y.$$

Inner Layer: Law of the Wall

Consider : the wall bounded flow near the wall

- U∞: the free-stream velocity (or the centerline velocity)
- δ : the width of flow
- wall : is smooth

The near wall **velocity** profile depends **only** on **near wall parameters** (not on U∞ or δ)

$$U = U(\rho, \tau_0, \nu, y)$$

only ρ and τ_0 involve the dimension of mass, so occur together in any nondimensional group.

$u_* \equiv \sqrt{\frac{\tau_0}{\rho}}$, friction velocity → $U = U(u_*, \nu, y)$

4 variables involving (**length** and **time**) pi theorem → only 4 − 2 = 2 nondimensional groups U/u∗ and yu∗/v

$$\frac{U}{u_*} = f\left(\frac{yu_*}{\nu}\right) = f(y_+) \qquad \text{(law of the wall)}$$

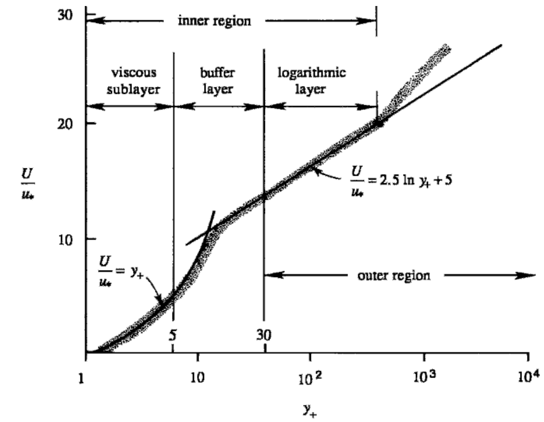
where $y_+ \equiv yu_*/\nu$ is the distance nondimensionalized by the *viscous scale* ν/u_*.

The **inner** part of the wall layer is **dominated** by **viscous** effects (**viscous sublayer** or "laminar sublayer," until experiments revealed the presence of considerable fluctuations within the layer).

$$\mu\frac{dU}{dy} = \tau_0$$

No-slip boundary condition, $U = \frac{y\tau_0}{\mu}$ $\qquad \frac{U}{u_*} = y_+ \qquad$ (viscous sublayer)

Experiments show that the linear distribution holds up to $yu_*/\nu \sim 5$, which may be taken to be the limit of the viscous sublayer.



Law of the wall. A typical data cloud is shaded

Outer Layer: Velocity Defect Law

Characteristics:
•**Inviscid**
•**wall-free**
•**Reynolds stresses** generates a **velocity defect** (U∞ − U), **proportional** to the wall friction (u∗).

In the outer region:

$$\frac{U - U_\infty}{u_*} = F\left(\frac{y}{\delta}\right) = F(\xi) \qquad \text{(velocity defect law)}$$

where $\xi \equiv y/\delta$. This is called the *velocity defect law*.

Overlap Layer: Logarithmic Law

•**Distances** in the **outer** part are **scaled** by **δ**
• in the **inner** part are measured **by** the much smaller viscous scale **v/u∗**.
• The **small distances** in the inner layer are **magnified by** expressing them as **yu∗/v**.
•The **inner** and **outer solutions** are **matched** together **in** a region of **over-lap** by taking the limits y+ →∞ and ξ → 0 simultaneously .

the velocity gradients in the inner and outer regions are given

$$\frac{dU}{dy} = \frac{u_*^2}{\nu}\frac{df}{dy_+}, \qquad \frac{dU}{dy} = \frac{u_*}{\delta}\frac{dF}{d\xi}.$$

→ and multiplying by y/u_*, we obtain

$$\xi\frac{dF}{d\xi} = y_+\frac{df}{dy_+} = \frac{1}{k},$$

valid for large y_+ and small ξ. As the left-hand side can only be a function of ξ and the right-hand side can only be a function of y_+, both sides must be equal to the same universal constant, say $1/k$, where k is called the *von Karman constant*. Experiments show that $k \simeq 0.41$. Integration of equation $f(y_+) = \frac{1}{k}\ln y_+ + A$,

$$F(\xi) = \frac{1}{k}\ln\xi + B.$$

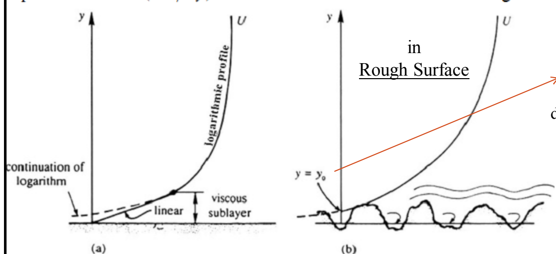
Experiments show that $A = 5.0$ and $B = -1.0$ for a smooth flat plate,

$$\frac{U}{u_*} = \frac{1}{k}\ln\frac{yu_*}{\nu} + 5.0,$$

$$\frac{U - U_\infty}{u_*} = \frac{1}{k}\ln\frac{y}{\delta} - 1.0.$$

These are the velocity distributions in the *overlap layer*, also called the *inertial sub-layer* or simply the *logarithmic layer*. As the derivation shows, these laws are only valid for large y_+ and small y/δ.

The region $5 < y_+ < 30$, where the velocity distribution is neither linear nor logarithmic, is called the *buffer layer*. Neither the viscous stress nor the Reynolds stress is negligible here. This layer is dynamically very important, as the turbulence production $-\overline{uv}(dU/dy)$ reaches a maximum here due to the large velocity gradients.

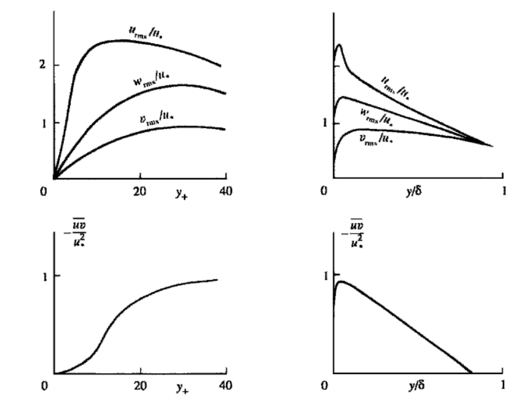


y0 is a measure of the roughness heights and is defined as the value of y at which the logarithmic distribution gives U = 0 .

Logarithmic velocity distributions near smooth and rough surfaces: (a) smooth wall; and (b) rough wall.

$$\frac{U}{u_*} = \frac{1}{k} \ln \frac{y}{y_0}.$$

Variation of Turbulent Intensity



Sketch of observed variation of turbulent intensity and Reynolds stress across a channel of half-width δ. The left panels are plots as functions of the inner variable y+, while the right panels are plots as functions of the outer variable y/δ.

- The turbulent velocity **fluctuations** are of **order u∗.**
- The longitudinal fluctuations are the largest because:
  - the **shear** production **initially feeds** the energy **into** the **u-component**;
  - **subsequently** distributed **into** the lateral components **v and w**.
- The turbulent **intensity** initially **rises** as the **wall** is **approached**,
- It **goes** to **zero** right **at** the **wall in** a very **thin** wall **layer**.
- The **normal** component v_{rms} **starts** to **feel** the **wall** effect **earlier**.
- The **distribution**, close to the wall, **becomes clear** only **when** the **distances** are **magnified** by the viscous scaling **v/u∗**.
- The **Reynolds** stress **profile** shows: the **stresses** are **negligible within** the viscous sublayer **(y+ < 5),**
- The **Reynolds stress** is nearly **constant** throughout the **wall layer**. (the constant stress layer).

Eddy Viscosity and Mixing Length

- The **mean motion** equations, **cannot** be **solved** for Ui (x) **unless** to have an expression **relating** the **Reynolds** stresses **in terms** of the **mean** velocity **field**.
- **Semiempirical theories** (e.g. Prandtl and von Karman) **based** on an analogy between the **momentum exchanges** both **in turbulent** and **in laminar** flows.
- in a unidirectional laminar flow U(y), the shear stress is $\frac{\tau_{lam}}{\rho} = \nu \frac{dU}{dy}$
- where **v** is a **property** of the **fluid**.
- The diffusive properties of a gas are due to the molecular motions, It can be **shown** that the **viscosity of** a **gas** is **of order**

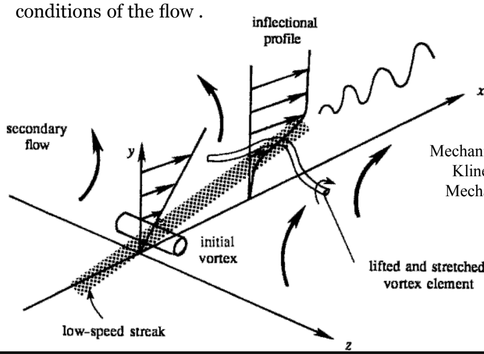
$$\nu \sim a\lambda$$

- where **a** is the **rms speed** of **molecular motion**, and **λ** is the **mean free path** defined as:
  - the average distance traveled by a molecule between collisions.
    - The proportionalityconstant in equation is of order 1.

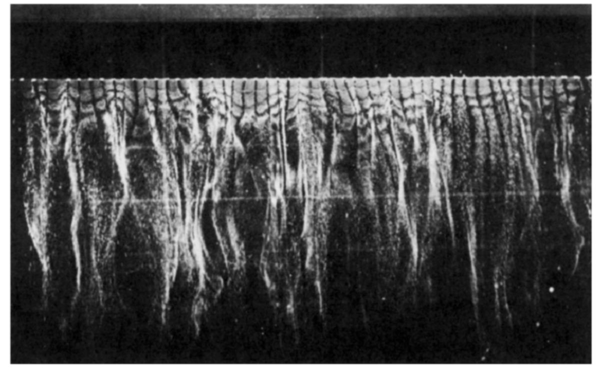
- **Boussinesq** speculated that the **diffusive** behavior of a turbulent flow may be **qualitatively similar** to that of a **laminar** flow and may simply be **represented by** a much **larger diffusivity:**

$$-\overline{uv} = \nu_e \frac{dU}{dy}$$ where ν_e is the eddy viscosity.

- **Note:** whereas ν is a known property of the fluid, but ν_e depends on the conditions of the flow .



Mechanics of streak break up. S. J. Kline et al., Journal of Fluid Mechanics 30: 741–773, 1967



Top view of near-wall structure (at y+ = 2 .7) in a turbulent boundary layer on a horizontal flat plate. The flow is visualized by hydrogen bubbles. S. J. Kline et al., Journal of Fluid Mechanics 30 :741–773, 1967

Boussinesq Approximation Based Models

- **Boussinesq approximation:**
  - the Reynolds stresses can be expressed in terms of the mean strain rate
  - or turbulent momentum transport is assumed proportional to the mean strain rate).

$$\tau_{ij} = \mu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

- **key issue:** the computation of the eddy viscosity, by using a suitable prescription.
- **Note!** The **eddy viscosity** is a **flow** property and **not** a **fluid property** (unlike molecular viscosity) and therefore depends on flow characteristics.
- The **turbulent transport** of **heat**, **mass** or **other** scalar quantities $(\overline{u_i'\varphi'})$ is **modeled similar** to that for **momentum** (**proportional to** the gradient of **mean** value of the transported quantity):

$$-\rho\overline{u_i'\varphi'} = \Gamma_t \left(\frac{\partial \bar{\phi}}{\partial x_i} \right)$$

Proportionality constant: turbulent diffusivity of a scalar variable.

- Turbulent **transport** of **momentum** or **heat** or mass is due to the **same mechanism**, (i.e. **eddy mixing**).
  - turbulent Prandtl number (Pr_t) for heat transfer
  - turbulent Schmidt number (Sct) for mass transfer

$$Pr_t = \frac{\mu_t}{\Gamma_t} \text{ and } Sc_t = \frac{\mu_t}{m_t}.$$

Here m_t denotes the turbulent diffusivity of mass. A typical value of turbulent Prandtl number or turbulent Schmidt number used in engineering computations used is approximately 1.0.

Models Based on Boussinesq Approximation

- Turbulence simplified by the Boussinesq approximation:
  - Reynolds stresses and turbulent transport quantities are related to the mean flow And scalar fields, respectively.
- Boussinesq models based on transport equations used to compute the eddy viscosity.
  - Models types :
    - (a) zero equation (or mixing length) models;
    - (b) one equation models;
    - (c) two equation models.

- The **first two require** the specification of flow **at least** one flow **variable** for a particular flow configuration and **therefore** are **incomplete** models.
- **Two equation** models are the **simplest complete models** and therefore are widely used.

Mixing Length Models

- It **assumed** that the **eddy viscosity** can be **expressed** as a **product** of a turbulent **velocity** scale (related to the **mean flow** properties) **and** a **length** scale (related to some **typical width** of the flow).
- The **Prandtl's** (1925) **mixing length** model (for a thin-shear layer (such as, boundary-layer, jet and plume):

$$\nu_t = l_{mix}^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$ where l_{mix} denotes the mixing length.

- The **hypothesis idea**: Turbulent moving eddies, typically retain their momentum in x-direction over a distance in the y-direction equal to the mixing length (l_{mix}).

$$\nu_t = \frac{\mu_t}{\rho}.$$

- **Mixing length models**
  - **need modifications** to damp the eddy viscosity **near wall** and the **outer** intermittent region.
  - Can be **used to** predict **turbulent diffusivity** for the transport of a scalar variable by using **turbulent Prandtl** number **or turbulent Schmidt** number approximately **equal** to **one**.

$$l_{mix} = cl,$$

Where, c: coefficient of proportionality
l: width of the flow

Constant of proportionality for different turbulent free-shear flows

Flow	Constant of proportionality
Plane jet	0.09
Round jet	0.075
Far wake	0.16
Mixing layer	0.07

- For the wall bounded flows in the inner region, the mixing length:

l_{mix} = ky,
k: the von Karman constant
y: the distance from the wall.

- **Damping** function is included to damp the **near wall** eddy **viscosity.**
- Since **no additional transport** equation is **used**, such models are called the **zero** equation **or algebraic** model.
- **Baldwin** and **Lomax** (1978) and **Cebeci** and **Smith** (1974) Mixing length models are the **most widely used** turbulent models for the **aerodynamics.**
- Cebeci and Smith (1974) **two-layer model** :

$$\mu_t = \begin{cases} \mu_{ti} & \text{for } y \le y_m \\ \mu_{to} & \text{for } y > y_m \end{cases}$$ at a particular location y_m

outer eddy **viscosity** is more than the **inner viscosity**.

$$\mu_{ti} = \rho l_{mix}^2 \left[\left(\frac{\partial \bar{u}}{\partial y}\right)^2 + \left(\frac{\partial \bar{v}}{\partial x}\right)^2\right]^{1/2} \qquad l_{mix} = ky\left[1 - e^{-y^+/A_o^+}\right]$$

$$\mu_{to} = \alpha \rho U_e \delta_v^* F_{Kleb}$$

U_e denotes the free stream velocity, F_{Kleb} is a curve fit by Klebanoff (1955) to the outer intermittency of turbulent flows. δ_v* denotes the velocity thickness and for incompressible turbulent flows it reduces to the displacement thickness The model constants are $k = 0.41$, $\alpha = 0.0168$ and the variable A_0^+ is given as

$$A_0^+ = 26\left[1 + y\frac{dP/dx}{\rho u_\tau^2}\right]^{-1/2}.$$

A_0^+ takes care of the dependence of the mixing length on the pressure gradient.

Deficiencies of the zero equations models:

- do **not directly account** for the flow **history effects**, as the eddy viscosity is related to local mean flow properties.
- **Eddy viscosity** reduces to **zero when** the mean **strain** rate equals **zero**, but this condition may not be valid in all cases.
- Cannot be **directly applied** to **three-dimensional** flows without any modification.
- **Incomplete** models **because** the mixing **length needs** to be specified.
- The mixing **length prescription** is **not unique** and depends on a particular flow configuration being studied.
- The **formulation** of the model **becomes difficult if** there is a **sudden change** in the flow conditions (e.g. Mixing length for separation flow on airfoil).

One Equation Model

- **One transport equation** in **addition** to the **continuity** and **momentum** equations is **solved**.
- The **extra** transport **equation** used can be **for any** turbulence **variable**.
- The **most** widely **used** one equation models are **based on** the **transport** equation **for** turbulence **kinetic energy**.

$$K + k = \left[\frac{1}{2}\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right)\right] + \left[\frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right]$$

from the trace of transport equation for the Reynolds stress tensor

$$\frac{\partial(\rho\overline{u_i'u_j'})}{\partial t} + \underbrace{\frac{\partial}{\partial x_k}\left(\rho\bar{u}_k\overline{u_i'u_j'}\right)}_{C_{ij} = \text{Convection}} = \underbrace{-\frac{\partial}{\partial x_k}\left[\rho\overline{u_i'u_j'u_k'} + \overline{p'(\delta_{kj}u_i' + \delta_{ik}'u_j')}\right]}_{D_{ij}^T = \text{Turbulent Diffusion}}$$

$$+ \underbrace{\frac{\partial}{\partial x_k}\left[\mu\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'}\right)\right]}_{D_{ij}^L = \text{Molecular Diffusion}} \underbrace{-\rho\left(\overline{u_i'u_k'}\frac{\partial\bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial\bar{u}_i}{k}\right)}_{P_{ij} = \text{Stress Production}}$$

$$+ \underbrace{\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\phi_{ij} = \text{Pressure Strain}} - \underbrace{2\mu\overline{\frac{\partial u_i'}{\partial x_k}}}_{\varepsilon_{ij} = \text{Dissipation}} \overline{\frac{\partial u_j'}{\partial x_k}}$$

Is a symmetric in tensor i and j, by substituting i = j :

$$\frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{k}}{\partial x_j}\right) - \nu\overline{\frac{\partial u_i'\partial u_i'}{\partial x_j\partial x_j}} - \overline{u_i'u_j'}\frac{\partial\overline{u_i}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{\overline{u_j'p'}}{\rho}\right)$$

From left to right:
(a) time rate of change of turbulence kinetic energy;
(b) convection of turbulence kinetic energy by the mean flow;
(c) molecular diffusion of turbulence kinetic energy;
(d) dissipation of turbulence kinetic energy which denotes its conversion to thermal energy due to viscous effects;
(e) production of turbulence kinetic energy by Reynolds stress acting on mean velocity gradient;
(f) transport of turbulence kinetic energy by fluctuating velocity and by pressure-velocity correlation.

The **second** and **last terms** on the **RHS require modeling** and,

$$\tau_{ij} = -\rho\overline{u_i'u_j'} = \mu_t\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij}$$

For $i = j$

$$-\rho\overline{u_i'u_i'} = -\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \mu_t\left(2\frac{\partial\overline{u_i}}{\partial x_i}\right) - \frac{2}{3}\rho k$$

RHS is zero due to continuity equation

The **rate of dissipation** of turbulence kinetic energy is **modeled** as:

$$\varepsilon = C_D k^{3/2}/l$$

C_D: is a model constant
l: a turbulent length scale that needs to be specified.

The **sum of** the turbulent **transport and pressure fluctuation** term is modeled based on the gradient diffusion hypothesis as:

$$\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{\overline{u_j'p'}}{\rho}\right) = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

The turbulent Prandtl number for turbulence kinetic energy (a flow property and not a fluid property).

- **Modeled** transport **equation** for turbulence kinetic energy:

$$\rho\frac{\partial k}{\partial t} + \rho\bar{u}_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \tau_{ij}\frac{\partial\overline{u_i}}{\partial x_j} - \varepsilon\rho$$

- Major **disadvantage**: this model is is an **incomplete model**, since we **need** to specify the **length** scale **l**. Further **no unique** prescription **of** the **length scale** can be specified.

Based on Equation for Eddy Viscosity

The Spalart and Allmaras model (1992) is a simple one equation model. It directly solves a modeled transport equation for the kinematic eddy (turbulent) viscosity itself. In this model, a length scale related to the local shear layer thickness need not be calculated. The relevant transport equation for Spalart–Allmaras variable is

$$\frac{\partial \tilde{v}}{\partial t} + \bar{u}_j \frac{\partial \tilde{v}}{\partial x_j} = c_{b1}(1 - f_{t2})\tilde{S}\tilde{v} - \left[c_{w1} f_w - \frac{c_{b1}}{k^2} f_{t2}\right]\left(\frac{\tilde{v}}{d}\right)^2 + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}(v + \tilde{v})\frac{\partial \tilde{v}}{\partial x_j}\right] + c_{b2}\frac{\partial \tilde{v}}{\partial x_i}\frac{\partial \tilde{v}}{\partial x_i}$$

The kinematic turbulent viscosity is given as product of Spalart–Allmaras variable and f_{v1}. Model constants: $c_{b1} = 0.1355$, $c_{b2} = 0.662$, $c_{v1} = 7.1$, $\sigma = 2/3$, $c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{1+c_{b2}}{\sigma}$, $c_{w2} = 0.3$, $c_{w3} = 2.0$, $\kappa = 0.41$ and the model relations:

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}, f_w = g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6},$$

$$\chi = \frac{\tilde{v}}{v}, g = r + c_{w2}(r^6 - r), r = \min\left[\frac{\tilde{v}}{\tilde{S}\kappa^2 d^2}, 10\right], v_t = \tilde{v} f_{v1},$$

$$\tilde{S} = S + \frac{\tilde{v}}{\kappa^2 d^2} f_{v2}, S = \sqrt{2\Omega_{ij}\Omega_{ij}}, f_{t2} = c_{t3}\exp(-c_{t4}\chi^2), c_{t3} = 1.2, c_{t4} = 0.5$$

Ω_{ij} denotes the rotation tensor $\Omega_{ij} = 1/2\left(\partial \overline{u_i}/\partial x_j - \partial \overline{u_j}/\partial x_i\right)$ and d the distance from the surface. The Spalart and Allmaras model gives good results for boundary layers subjected to adverse pressure gradients. This model was designed specifically for aerospace applications involving wall-bounded flows. It can also be used for turbo-machinery applications. The model of Baldwin and Barth (1990) is another one equation based on the transport equation for the eddy viscosity.

Two Equation Models

- **At least two variables** (for example, velocity and length scales) are **needed** to characterize turbulent flows completely.
- Therefore, **two equation models** are the **simplest complete** models
- The standard **k-e model** is one of the **most widely used** turbulence models.

Limitations of Boussinesq Approximation

- The key assumption that the turbulent stresses are proportional to the mean strain rate may not hold true in many situations.
- assume an isotropic eddy viscosity (i.e. which is same in all the directions) and this assumptionmay also fail in some situations.

Proper cases

Zero equation models **work well** in **simple flows** (which do not separate and where thin shear layer assumption is valid) such as

- jets, mixing layers, wakes,
- boundary layer flow, flow through pipe,
- flow between parallel plates, etc.

k–ε and Other Two Equations Models

- **Standard k–ε Model** (Jones and Launder (1972))
  - The **exact** transport **equation** for the rate of **dissipation** of turbulence kinetic energy (**ε**), **obtained** from mathematical **operation**:

$$\overline{2v\frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}[N(u_i)]} = 0$$

where $N(u_i)$ denotes the Navier–Stokes operator and is defined as

$$N(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho\, u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j\frac{\partial \varepsilon}{\partial x_j} = -2v\left[\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}\right]\frac{\partial \bar{u}_i}{\partial x_j} - 2v\overline{u'_k u'_{i,j}}\frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_j}$$

$$-2v\overline{u'_{i,k}u'_{i,m}u'_{k,m}} - 2v^2\overline{u'_{i,km}u'_{i,km}} + \frac{\partial}{\partial x_j}\left[v\frac{\partial \varepsilon}{\partial x_j} - v\overline{u'_j u'_{i,m}u'_{i,m}} - 2\frac{v}{\rho}\overline{p'_{,m}u'_{j,m}}\right]$$

- **More complex compared** to the exact equation for turbulent **kinetic** energy.
- On LHS: the standard **unsteady** and **convection** terms.
- On the RHS:
  - the production of dissipation,
  - dissipation of dissipation,
  - turbulent transport
  - molecular diffusion of dissipation.
- The **unsteady**, **convection** and **molecular diffusion** terms **do not require** any **modeling**.
- The remaining seven terms require modeling.
- The **modeled** form of the dissipation **equation used** in the literature is a **major weakness of** the k–e **model**.

Modelled Transport Equations for k and e

The modeled transport equation for turbulent kinetic energy (k):

$$\frac{\partial k}{\partial t}+\bar{u}\frac{\partial k}{\partial x}+\bar{v}\frac{\partial k}{\partial y}+\bar{w}\frac{\partial k}{\partial z}=\frac{\partial}{\partial x}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right]+\frac{\partial}{\partial z}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial z}\right]+P_k-\varepsilon$$

- The **modeled** transport equation for the **dissipation**:

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}_j\frac{\partial \varepsilon}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]+C_{\varepsilon 1}\frac{\varepsilon}{k}\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}-C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

- in the expanded form:

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}\frac{\partial \varepsilon}{\partial x}+\bar{v}\frac{\partial \varepsilon}{\partial y}+\bar{w}\frac{\partial \varepsilon}{\partial z}=\frac{\partial}{\partial x}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right]+\frac{\partial}{\partial z}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial \varepsilon}{\partial z}\right]+C_{\varepsilon 1}\frac{\varepsilon}{k}P_k-C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

- where P_k denotes the rate of shear production of k and is:

$$P_k=\nu_t\left[\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right)^2+\left(\frac{\partial \bar{v}}{\partial z}+\frac{\partial \bar{w}}{\partial y}\right)^2+\left(\frac{\partial \bar{u}}{\partial z}+\frac{\partial \bar{w}}{\partial x}\right)^2\right]+\nu_t\left[2\left(\frac{\partial \bar{u}}{\partial x}\right)^2+2\left(\frac{\partial \bar{v}}{\partial y}\right)^2+2\left(\frac{\partial \bar{w}}{\partial z}\right)^2\right]$$

The eddy viscosity (kinematic) is given as $\nu_t = C_\mu\frac{k^2}{\varepsilon}$ and the following standard values of the model constants are used (Launder and Spalding 1974) $C_\mu = 0.09$, $\sigma_k = 1.00$, $\sigma_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$.

Features of the k–e Model

- It is a **high turbulence** Reynolds number **model**.
- It **cannot** be applied **without** suitable **modifications** in the **low Re regions.**
- The solution of two **separate** transport **equations** for **k** and **e allows** the turbulent **velocity** and **length scales** to be **independently determined**.
- Each term of the **modeled** transport **equation for k** almost **accurately represents** the corresponding term in the **exact equation**.
- **However**, the gross effect of **several terms** in the exact dissipation equation **is modeled by few** terms.
- Or there is no **one to one correspondence** between different terms **in** the **modeled and exact** transport equations **for dissipation**.

Boundary Conditions

1. Inlet: It is **difficult** to obtain values of **k** and **e** at the inlet, **based** on an **approximation** from the turbulent **intensity** T_i and a **characteristic length** L of the flow configuration:

$$k=\frac{3}{2}(U_{ref}T_i)^2,\quad \varepsilon=C_\mu^{3/4}\frac{k^{3/2}}{l},\quad l=0.07L$$

l denotes a turbulent length scale.

2. Outlet: At the outlet usually turbulence (k and ε) is taken equal to zero, the mean temperature ($\overline{T_m}$) equal to the ambient temperature (T_∞) and pressure (p) equal to the atmospheric pressure p_∞.
3. Symmetry plane: Gradients of all flow properties normal to the plane or line of symmetry are taken equal to zero, i.e., $\partial\bar{u}_i/\partial n = 0, \partial\bar{T}/\partial n = 0, \partial k/\partial n = 0$ and $\partial\varepsilon/\partial n = 0$, where n denotes normal to the plane or line of symmetry.
4. The free-streams are usually non turbulent and therefore $k = 0$ and $\varepsilon = 0$ are usually specified. Mean velocity and temperature may be taken equal to their atmospheric counterparts.
5. At the solid wall either the no slip condition using the low-Re version or wall function approach can be applied. We will present the features of these two approaches in the next section.

Treatment of Wall

- Turbulence major **complexities** because of zero turbulence at the wall.

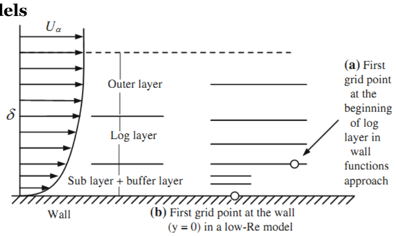
(a)Wall Functions Approach
(b) Low Reynolds Number Models

(a)Wall Functions Approach

- Standard Wall Functions

$$\overline{u}_p^+ = \frac{1}{\kappa}\ln y_p^+ + B$$

$$k_p = \frac{u_\tau^2}{C_\mu^{1/2}}, \varepsilon_p = \frac{u_\tau^3}{\kappa y_P}$$



where y_p^+ denotes the distance of the first grid point from the wall in the wall co-ordinates, $\kappa = 0.41$ the von Karman constant and $B = 5.0$ a dimensionless constant.

- It is not universal. e.g., in **separating** and **reattaching** flows and flows **with** strong **curvature**, the **law** of the wall **is invalid.**
- A user **needs** to **ensure** that the location of the **first grid** point is **in** the **logarithmic** region. However, this **cannot** be **determined** a **priori** because the value of y+ depends on the skin friction coefficient.

- Non-Equilibrium Wall Functions

It **uses** the **modified log-law** for the mean velocity **to account** for the **effects** of the **pressure gradient** (**two-layer approach** to calculate the turbulence kinetic energy in the cells close to the wall).

$$\frac{\tilde{u}C_\mu^{1/4}k^{1/2}}{\tau_w/\rho} = \frac{1}{k}\ln\left(\frac{E\rho C_\mu^{1/4}k^{1/2}y}{\mu}\right)$$

$$\tilde{u} = \bar{u}\frac{1}{2}\frac{dp}{dx}\left[\frac{y_v}{\rho k\sqrt{k}}\ln\left(\frac{y}{y_v}\right) + \frac{y-y_v}{\rho k\sqrt{k}} + \frac{y_v^2}{\mu}\right]$$

here $E = 9.793$ and y_v denotes the viscous sub-layer thickness and is obtained from the expression

$$y_v = \frac{\mu y_v^*}{\rho C_\mu^{1/4}k_P^{1/2}}$$

where $y^*_v = 11.225$. Here two-layer concept is used to calculate the turbulence kinetic energy and dissipation rate at the first grid point. If $y < y_v$ the region is called the sublayer region and $y > y_v$ the region is called the fully turbulent region. Depending on the location of the first grid point with respect to y_v values of turbulence kinetic energy and dissipation are specified.

(b) Low Reynolds Number Models

In general, in a low-Rek–ε model the eddy viscosity is computed by a slightly modified expression

$$v_t = c_\mu f_\mu \frac{k^2}{\varepsilon}$$

Here f_μ denotes a damping function and in different models it is based either on the distance from the wall or turbulence Reynolds number. For example, in the Launder and Sharma model the damping function is given as

$$f_\mu = \exp\left[-\frac{2.5}{1 + Re_T/50}\right]$$

Far away from the wall this function becomes equal to 1.0 and thus the expression reduces to the standard expression for the eddy viscosity $(v_t = C_\mu k^2/\varepsilon)$ used in the standard k–ε model. The following modified forms of the transport equations for k and ε are used in the Launder and Sharma (1974) model

$$\bar{u}\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left[\left(v + \frac{v_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + v_t\left(\frac{\partial k}{\partial y}\right)^2 - \tilde{\varepsilon} - 2v\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$

$$\bar{u}\frac{\partial\tilde{\varepsilon}}{\partial x} + v\frac{\partial\tilde{\varepsilon}}{\partial y} = \frac{\partial}{\partial y}\left[\left(v + \frac{v_t}{\sigma_k}\right)\frac{\partial\tilde{\varepsilon}}{\partial y}\right] + c_{\varepsilon 1}\frac{\tilde{\varepsilon}v_t}{k}\left(\frac{\partial\bar{u}}{\partial y}\right)^2 - c_{\varepsilon 2}\frac{\tilde{\varepsilon}^2}{k} + 2vv_t\left(\frac{\partial^2\bar{u}}{\partial y^2}\right)^2$$

The modified form of the constant $c_{\varepsilon 2}$ used by Launder and Sharma (1974) is given as

$$c_{\varepsilon 2} = 1.92[1 - 0.3\exp(-Re_T^2)]$$

- Additional terms vanish far away from the wall and the model reduces to the standard model.

Variants of k–e Model

(To enhance the range of applicability of this model).

RNG k–e Model

Yakhot and Orszag (1986) **using** a **statistical technique** (called the renormalization group).

Similar in **form** to the standard k–e model, **but** includes **refinements**:

(1) It has an **additional term** in its dissipation equation that is supposed to **improve** the accuracy for rapidly **strained flows**.
(2) The **effect** of **swirl** on turbulence is **included** in the RNG model.

The RNG theory provides an analytical formula for turbulent Prandtl numbers, (standard k–e model uses the constant values).

RNG also provides an analytically-derived differential formula for the **effective viscosity** that accounts **for low-Reynolds-number** effects.

The transport **equations** for turbulent kinetic energy and its dissipation

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_i}(\rho k \overline{u_i}) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \rho\varepsilon$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_i}(\rho \varepsilon \overline{u_i}) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c^*_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}$$

The model constants and the auxiliary relations are $C_\mu = 0.0845, C_{\varepsilon 1} = 1.42, C_{\varepsilon 2} = 1.68, \sigma_k = \sigma_\varepsilon = 0.7178, \beta = 0.012$

$$C^*_{\varepsilon 2} = C_{\varepsilon 2} + \frac{c_\mu \eta^3(1 - \eta/\eta_0)}{1 + \beta\eta^3} \quad \text{with} \quad \eta = \frac{Sk}{\varepsilon} \text{ and } \eta_0 = 4.38$$

Realizable k–e Model

- By Shih et al. (1995) and **differs** from the standard k–e model :

(1) It contains a **new formulation** for the **turbulent viscosity**

(2) it has a **new** transport **equation** for the **dissipation** rate (derived from an exact equation for the transport of the mean-square vorticity fluctuation).

For **normal Reynolds stress** in an **incompressible** strained mean flow, (using the Boussinesq relationship and the standard expression for the eddy viscosity)

$$\overline{u'^2} = \frac{2}{3}k - 2\nu_t\frac{\partial \bar{u}}{\partial x}$$

Therefore one obtains the result that the normal stress, $\overline{u'^2}$, which by definition is a positive quantity, becomes negative, i.e., "non-realizable'', when the strain is large and therefore the following equation needs to be satisfied

$$\frac{k}{\varepsilon}\frac{\partial \bar{u}}{\partial x} > \frac{1}{3C_\mu} \approx 3.7$$

The term "realizable'' means that the model satisfies certain mathematical constraints on the Reynolds stresses. Both the standard k–ε model and the RNG k–ε model are not realizable. One limitation of the realizable k–ε model is that it produces non-physical turbulent viscosities in situations when computational domain contains both rotating and stationary fluid zones.

k–ω Model

The k–ω model originally given by Wilcox (1988) is based on the modelled transport equations for the turbulence kinetic energy (k) and the specific dissipation rate (ω), which can also be thought of as the ratio of ε to k. The model incorporates modifications for low-Reynolds-number effects and is applicable to wall-bounded flows without any further modifications and free shear flows. Transport equations for this model are given as

$$\frac{\partial k}{\partial t} + \bar{u}_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma^*\frac{k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* k\omega$$

$$\frac{\partial \omega}{\partial t} + \bar{u}_j\frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma\frac{k}{\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \alpha\frac{\omega}{k}\frac{\partial \bar{u}_i}{\partial x_j}\frac{\tau_{ij}}{\rho} - \beta\omega^2$$

$$\alpha = \frac{5}{9}, \quad \beta = \frac{3}{40}, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{1}{2}, \quad \varepsilon = \beta^*\omega k$$

- This **k–ω** model is an **improved** version **of** the original model proposed by **Kolmogorov** (1942), (inclusion of the molecular diffusion and production terms).
- **Performance compare** of the k–e model and k–ω model:
- The k–e model:
  - does not accurately predict the characteristics of far wakes and mixing layers and the spreading rate of axisymmetric jets in stagnant surrounding is also overpredicted.
  - it can be improved by making ad hoc adjustments to the model constants. The model also has problems in swirling flows and flows with large strains (e.g.,highly curved boundary layers and diverging passages).
- k–ω model:
  - reproduce the behaviour within viscous sublayer without the need for any corrections.
  - However, it sensitive to the free-stream conditions for the free-shear flows.

Comparison of growth rates of four typical free shear flows predicted by k–e and k–ω models (Wilcox 2006)

Flow	Measurements	k–ε model	k–ω model
Round jet	0.08–0.09	0.12	0.07–0.37
Plane jet	0.10–0.11	0.11	0.09–0.14
Far wake	0.36	0.25	0.30–0.50
Mixing layer	0.12	0.10	0.10–0.14

V2f Model

- By Durbin (1991), based on the root mean square normal velocity fluctuations $\overline{v'^2}$, as the velocity scale rather than turbulence kinetic energy k).
- Capable of handling the wall region without the need for the additional damping functions. (because the normal velocity fluctuations are known to be quite sensitive to the presence of wall, like a natural damper).

The model employs four transport equations for the closure of the RANS equations, which include an equation for the turbulence kinetic energy (the equation is same as that for the standard k–ε model), the dissipation of turbulence kinetic energy (this equation is same as that used for the standard k–ε model but with a slightly modified value of the constant), a new transport equation for the normal r.m.s. velocity fluctuations and finally a transport equation for f that takes care of in-homogeneity and wall blocking effects in the transport equation for $\overline{v'^2}$. Since these two transport equations are used in conjunction with those for k and ε, this model is termed as the $k-\varepsilon-\overline{v'^2}-f$ model. There have been continuous improvements in the v2f model. The original v2f model was numerically unstable for segregated solvers.

The transport equations used can be written as (Kazerooni and Hannani 2009):

$$\frac{\partial k}{\partial t}+\overline{u_j}\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right)+P_k-\varepsilon$$

$$\frac{\partial \varepsilon}{\partial t}+\overline{u_j}\frac{\partial \varepsilon}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right)+\frac{c_{\varepsilon 1}P_k-c_{\varepsilon 2}}{T}$$

$$\frac{\partial \overline{v'^2}}{\partial t}+\overline{u_j}\frac{\partial \overline{v'^2}}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial \overline{v'^2}}{\partial x_j}\right)-\overline{v'^2}\frac{\varepsilon}{k}+kf$$

$$f-L^2\nabla^2 f=(C_1-1)\frac{2/3-\overline{v'^2}/k}{T}+C_2\frac{P_k}{k}$$

Here L and T denote turbulence length and time scales, respectively, and are given as

$$T=\min\left[\max\left[\frac{k}{\varepsilon},C_T\sqrt{\frac{\nu}{\varepsilon}}\right],\frac{0.6k}{2\sqrt{3}C_\mu\overline{v'^2}S}\right]$$

$$L=C_L\max\left[\min\left[\frac{k^{3/2}}{\varepsilon},\frac{k^{3/2}}{\sqrt{3}C_\mu\overline{v'^2}S}\right],C_\eta\frac{\nu^{3/4}}{\varepsilon^{1/4}}\right]$$

- The term kf redistributes turbulence kinetic energy from the streamwise velocity component. The **eddy viscosity** can be written as

$$\frac{\mu_T}{\rho}=\nu_T=C_\mu T\overline{v'^2}$$

The v2f model constants are given as (Durbin and Reif 2001)

$$c_\mu=0.22,\quad c_L=0.23,\quad c_\eta=85,\quad c_T=6,\quad c_1=0.4,\quad c_2=0.3$$

$$c_{\varepsilon 2}=1.9,\quad \sigma_k=1,\quad \sigma_\varepsilon=1.3$$

$$c_{\varepsilon 1}=1.4\left[1+0.045\sqrt{\frac{k}{\overline{v'^2}}}\right]$$

It suggested that the model has a good potential for turbo-machinery applications.

Shear Stress Transport k–ω Model

- By Menter (1994) to **take advantage** of accurate formulation of the **k–ω** model **in** the **near-wall** region **with** the **free-stream independence** of the **k–e** model in the **far field**.
- It is similar to the standard k–ω model, but includes **three refinements:**

(1) This model incorporates a damped cross-diffusion derivative term in the ω equation.
(2) The **turbulent viscosity** definition is **modified** to account for the transport of the turbulent shear stress.
(3) The model constants are different. There is **no need** for a **special** treatment for the viscosity affected **wall region** because of the low-Reynolds correction in the k–ω and k–ω SST models.

The transport equations for k and ω:

$$\frac{\partial k}{\partial t}+\overline{u_j}\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left[(\nu+\sigma_k\nu_t)\frac{\partial k}{\partial x_j}\right]+P_k-\beta^* k\omega$$

$$\frac{\partial \omega}{\partial t}+\bar{u}_j\frac{\partial \omega}{\partial x_j}=\frac{\gamma P_k}{\rho\nu_t}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[(\nu+\sigma_\omega\nu_t)\frac{\partial \omega}{\partial x_j}\right]+2(1-F_1)\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

The eddy viscosity is written as

$$\nu_T=\frac{a_1 k}{\max(a_1\omega,SF_2)}$$

The auxiliary relations and the model constants can be written as

$$F_1 = \tanh\left(\left\{\min\left[\max\left(\frac{\sqrt{k}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right), \frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}\right]\right\}^4\right)$$

$$F_2 = \tanh\left(\max\left(\frac{2\sqrt{k}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right)^2\right)$$

$$P_k = \min\left(\tau_{ij}\frac{\partial \overline{u}_i}{\partial x_j}, 10\beta^* k\omega\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i}, 10^{-20}\right)$$

The constants for example Φ are blended using the relation

$$\phi = \phi_1 F_1 + \phi_2(1 - F_1)$$

$$\gamma_1 = \frac{\beta_1}{\beta^*} - \frac{\sigma_{\omega 1}k^2}{\sqrt{\beta^*}}, \quad \gamma_1 = \frac{\beta_2}{\beta^*} - \frac{\sigma_{\omega 2}k^2}{\sqrt{\beta^*}}, \quad \beta_1 = 3/40, \quad \beta_2 = 0.0828, \quad \beta^* = 9/100$$

$$\sigma_{k1} = 0.85, \quad \sigma_{k2} = 1, \quad a_1 = 0.31, \quad \sigma_{\omega 1} = 0.5, \quad \sigma_{\omega 2} = 0.856$$

Conclusion

- **Two equation** models with the **Boussinesq** assumption are **widely used** for predicting turbulent flows encountered **in industry**.
- The **standard k–e** model is the most **widely used** turbulence model.
- The **wall functions** approach for treating turbulence in the vicinity of a solid wall **has some limitations.**
- The **wall function** approach is likely to **fail in** the following situations:

(1) Blowing/suction through the wall;
(2) Large pressure gradients;
(3) Strong body forces (e.g., flow near a rotating body);
(4) High three-dimensionality near the walls (e.g., strongly skewed flows).

Reynolds-Stress and Scalar Flux Transport Model

- **solving** transport **equations** for different **Reynolds stresses** components
- (Wilcox 2006):

$$\frac{\partial \tau_{ij}}{\partial t} + \overline{u}_k\frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + \varepsilon_{ij} - \pi_{ij} + \frac{\partial}{\partial x_k}\left[\nu\frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right]$$

The exact pressure strain correlation term is given by $\pi_{ij} = \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$

The exact dissipation term is given by $\varepsilon_{ij} = \overline{2\mu\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\right)}$

The exact turbulent transport term is given by $C_{ijk} = \overline{\rho u_i' u_j' u_k'} + \overline{p u_i'}\delta_{jk} + \overline{p u_j'}\delta_{ik}$

The pressure-strain correlation (π_{ij}), dissipation (ε_{ij}) and turbulent transport (C_{ijk}) terms need to be modeled in order to close the exact transport equation for Reynolds stress τ_{ij}.

- Modeling of Turbulent Transport

$$C_{ijk} = C_s\frac{2}{3}\frac{k^2}{\varepsilon}\left[\frac{\partial \tau_{jk}}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_k}\right]$$

Here $C_s \approx 0.11$ is a model constant.

- Modeling of Pressure Strain

$$\pi_{ij} = C_1\frac{\varepsilon}{k}\left(\tau_{ij} + \frac{2}{3}\rho k\delta_{ij}\right) - \hat{\alpha}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - \hat{\beta}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right)$$
$$- \hat{\gamma}k\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right) + \left[0.125\frac{\varepsilon}{k}\left(\tau_{ij} + \frac{2}{3}k\delta_{ij}\right) - 0.015(P_{ij} - D_{ij})\right]\frac{k^{3/2}}{\varepsilon n}$$

The model constants are given as (Launder 1989)

$$\hat{\alpha} = (8 + C_2)/11, \hat{\beta} = (8C_2 - 2)/11, \hat{\gamma} = (60C_2 - 4)/55, C_1 = 1.8, C_2 = 0.6$$

The auxiliary relations are given as

$$P_{ij} = \tau_{im}\frac{\partial \overline{u}_j}{\partial x_m} + \tau_{jm}\frac{\partial \overline{u}_i}{\partial x_m}, \quad D_{ij} = \tau_{im}\frac{\partial \overline{u}_m}{\partial x_j} + \tau_{jm}\frac{\partial \overline{u}_m}{\partial x_i}, \quad P = \frac{1}{2}P_{kk}$$

- Modeling of Dissipation $\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$

where ε denotes the scalar dissipation rate of turbulence kinetic energy $\varepsilon = \nu\overline{\frac{\partial u_i'\partial u_i'}{\partial x_k\partial x_k}}$

113

A. Sarreshtehdari

Features of Reynolds-Stress and Scalar Flux Transport Model

- This is the most complex turbulence model.
- The RSSFT model closes the Reynolds-averaged Navier–Stokes equations by solving transport equations for the Reynolds stresses and scalar fluxes, together with an equation for the dissipation rate.
- Four additional transport equations
  - two Reynolds normal stresses,
  - one Reynolds shear stress
  - one for dissipation) are **required in a two-dimensional** mean flow
  - **seven** additional transport equations (three Reynolds shear stresses, three Reynolds normal stresses, and one for dissipation) must be solved **in** a **three-dimensional** mean flow.
- Accounts for the **effects** of streamline **curvature**, **swirl**, **rotation**, and **rapid changes in strain** rate in a more physical manner **than** that by one-equation and two-equation **models**.
- Good **potential** to provide accurate **predictions** of **complex flows**.
- It is **used for** computing cyclone flows, swirling flows in combustors, rotating flow passages, and the stress-induced secondary flows in ducts.

114

A. Sarreshtehdari

- The **assumption** of **isotropic** eddy viscosity (Mixing length and k-e models) may **not** be **valid** in many situations and **therefore** the use of **RSSFT** model is **advisable in such situations**.
- Use of a **RSSFT** model **significantly** increases the **computational** cost due to the **employment** of **seven** additional partial differential equations.
- However, **RSSFT** models are **not** as **widely** used **as** the **k-e** model.

115

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Concludion

- The **second order closure** is the most **complex** and **physically** the **most realistic** among all other closure options that **can be used**.
- **However**, the additional **complexity** does **not necessarily mean** that the predictions using a Reynolds stress and scalar flux transport model will be more **accurate compared to** those obtained using **other** simpler models.
- This anomaly **arises due** to the **uncertainty** involved **in modeling** complex double and triple correlations **in** the **transport equation** for the Reynolds stress tensor and turbulent scalar flux.