Fluid Mechanics I

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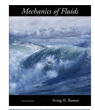
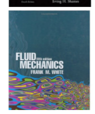
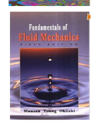
References

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Introduction

- Substances:
  - Solids & Fluids
- Fluids
  - Liquid & Gases
- Fluid mechanics : The study of fluids
  - *at rest (fluid statics)*
  - in motion (fluid *dynamics)*
  - *and the subsequent effects of the fluid upon the boundaries*

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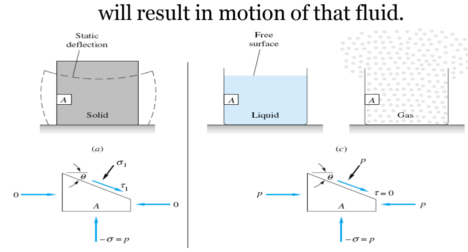
Substances & Shear Stress

- *A solid can resist a shear stress by a static deformation; a fluid cannot.*
- *Any shear* stress applied to a fluid, no matter how small, will result in motion of that fluid.



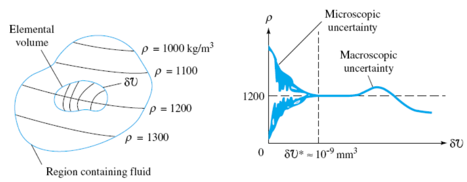
Fluid Mechanics Instances

- Life adventure
- Environment behavior
- Weather
- Daily Life
- Animals behavior
- Industries

And …

The Fluid as a Continuum

- *fluid pressure and density !?!* $\rho = \lim_{\delta\mathcal{V} \to \delta\mathcal{V}^*} \frac{\delta m}{\delta \mathcal{V}}$



Dimensions and Units

Primary dimension	SI unit	BG unit	Conversion factor
Mass {M}	Kilogram (kg)	Slug	1 slug = 14.5939 kg
Length {L}	Meter (m)	Foot (ft)	1 ft = 0.3048 m
Time {T}	Second (s)	Second (s)	1 s = 1 s
Temperature {Θ}	Kelvin (K)	Rankine (°R)	1 K = 1.8°R

❖**<u>Equation properties:</u>**

- **Consistent Units**
- **Homogeneous**

e.g.

$$h + \tfrac{1}{2}V^2 = \text{constant}$$

$$S = S_0 + V_0 t + \tfrac{1}{2} g t^2$$

❖**<u>Convenient Prefixes in Powers of 10 Part</u>**

Properties of the Velocity Field

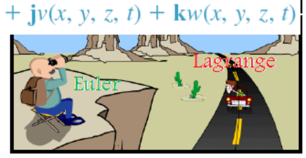
❖**Eulerian and Lagrangian Desciptions**

$p(x, y, z, t) \quad p(t)$

$$\mathbf{V}(x, y, z, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t)$$

❖<u>Examples:</u>

- Experimental Probes
- Traffic Studies



Thermodynamic Properties of a Fluid

- 1. Pressure
- 2. Density
- 3. Temperature

→ Most common properties

- 4. Internal energy
- 5. Enthalpy
- 6. Entropy
- 7. Specific heats

→ Important when work, heat, and energy balances are treated

- 8. Coefficient of viscosity
- 9. Thermal conductivity

→ *Transport properties*

Pressure

- Pressure is the (compression) stress at a point in a static fluid.
- Next to velocity, the pressure *p is the most dynamic variable in fluid mechanics.*
- *Differences or gradients in pressure often drive a fluid flow, especially in ducts.*
- *In low-speed flows,* the actual magnitude of the pressure is often not important, unless it drops so low as to cause vapor bubbles to form in a liquid.
- High-speed (compressible) gas flows, however, are indeed sensitive to the magnitude of pressure.

Temperature

- Temperature *T is a measure of the internal energy level of a fluid.*
- *It may vary considerably* during high-speed flow of a gas.
- Although engineers often use Celsius or Fahrenheit scales for convenience, many applications require *absolute* (Kelvin or Rankine) temperature scales:
  (°R = °F + 459.69 K =°C 273.16)
- If temperature differences are strong, *heat transfer may be important.*

Density

- The density of a fluid, is its mass per unit volume.
- Density is highly variable in gases and increases nearly proportionally to the pressure level.
- Density in liquids is nearly constant; (water about 1000 kg/m^3) increases only 1% if the pressure is increased by a factor of 220. (most liquid flows are treated analytically as "incompressible.")
- In general, liquids are about 3 orders of magnitude more dense than gases at atmospheric pressure.
- The heaviest liquid: mercury, the lightest gas: hydrogen They differ by a factor of 162,000!
- Various physical Properties =>*dimensional analysis*

Specific Weight

- The *specific weight of a fluid, is its weight* per unit volume.
- The units of are weight per unit volume, in lbf/ft^3 or N/m^3.
- In standard earth gravity, *the specific weights of air and water at* 20°C and 1 atm are approximately:
  - air (1.205 kg/m^3)*(9.807 m/s^2) 11.8 N/m^3 0.0752 lbf/ft^3
  - water (998 kg/m^3)*(9.807 m/s^2) 9790 N/m^3 62.4 lbf/ft^3
- Specific weight is very useful in the hydrostatic-pressure applications.

Specific Gravity

- *Specific gravity, is the ratio of a fluid density to a standard reference* fluid, water (for liquids), and air (for gases):

$$SG_{gas} = \frac{\rho_{gas}}{\rho_{air}} = \frac{\rho_{gas}}{1.205 \text{ kg/m}^3}$$

$$SG_{liquid} = \frac{\rho_{liquid}}{\rho_{water}} = \frac{\rho_{liquid}}{998 \text{ kg/m}^3}$$

- Engineers find these dimensionless ratios easier to remember than the actual numerical values of density of a variety of fluids.

Potential and Kinetic Energies

- In thermostatics the only energy in a substance is that stored in a system by molecular activity and molecular bonding forces (*internal energy*)
- *fluid flow:* 2 more energy terms which arise from newtonian mechanics: the potential energy and kinetic energy.
- The potential energy equals the work required to move the system of mass *m from* the origin to a position against a gravity field g.
- The kinetic energy equals the work required to change the speed of the mass from zero to velocity *V* $e = \hat{u} + \frac{1}{2}V^2 + gz$
- The molecular internal energy is a function of T and p for the single-phase pure substance, whereas the potential and kinetic energies are kinematic properties.

State Relations for Gases & Liquids

- All gases at high temperatures and low pressures (relative to their critical point) are in good agreement with the *perfect-gas law:*

$$p = \rho RT \qquad R = c_p - c_v = \text{gas constant} \qquad R_{gas} = \frac{\Lambda}{M_{gas}} \quad \{L^2T^{-2}\Theta^{-1}\}$$

$$\Lambda = 49{,}700 \text{ ft}^2/(\text{s}^2 \cdot {}^\circ\text{R}) = 8314 \text{ m}^2/(\text{s}^2 \cdot \text{K})$$

- There is No "perfect-liquid law" comparable to that for gases.
- Liquids are nearly incompressible and have a single reasonably constant specific heat. Thus an idealized state relation for a liquid is:

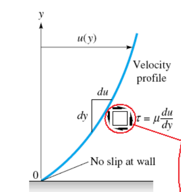
$$\rho \approx \text{const} \qquad c_p \approx c_v \approx \text{const} \qquad dh \approx c_p\, dT$$

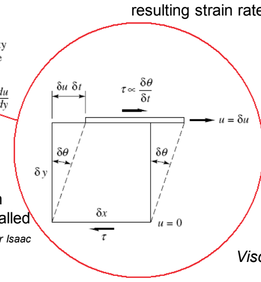
Viscosity

- There are certain secondary variables which characterize specific fluid-mechanical behavior.
- Viscosity relates the local stresses in a moving fluid to the strain rate of the fluid element.
- When a fluid is sheared, it begins to move at a strain rate inversely proportional to a property called its *coefficient of viscosity* .

Viscosity (Cont.)

Common fluids as water, oil, and air show a linear relation between applied shear and resulting strain rate:





$$\tau \propto \frac{\delta\theta}{\delta t}$$

$$\tan \delta\theta = \frac{\delta u\ \delta t}{\delta y}$$

$$\frac{d\theta}{dt} = \frac{du}{dy}$$

$$\tau = \mu\frac{d\theta}{dt} = \mu\frac{du}{dy}$$

The linear fluids which follow this equation called *newtonian fluids*, *after Sir Isaac Newton, who first postulated this resistance law in 1687.*

Viscosity Coefficient:{M/(LT)}

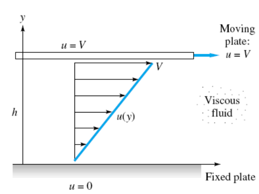
Viscosity (Cont.)

- The viscosity of newtonian fluids is a true thermodynamic property and varies with temperature and pressure. At a given state (*p, T) there is a vast range of values among the* common fluids.
- Generally speaking, the viscosity of a fluid increases only weakly with pressure. *Temperature*, however, has a strong effect, with increasing with *T for gases and decreasing* for liquids.

Student Exercises: explain behavior of sample liquid and gas for above description.

- The *kinematic viscosity:* $\nu = \frac{\mu}{\rho}$

Flow Between plates



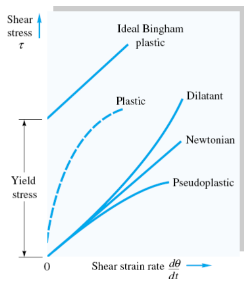
$$\frac{du}{dy} = \frac{\tau}{\mu} = \text{const}$$

$$u = a + by$$

$$u = \begin{cases} 0 = a + b(0) & \text{at } y = 0 \\ V = a + b(h) & \text{at } y = h \end{cases}$$

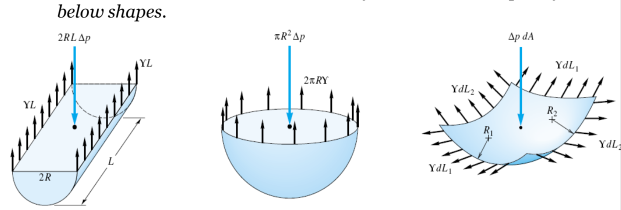
$$u = V\frac{y}{h}$$

Newtonian Fluid vs. Others



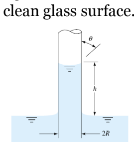
Surface Tension

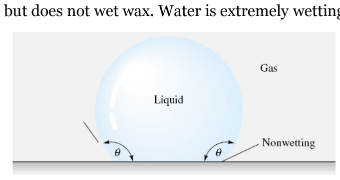
- If a cut of length *dL is made in an interfacial surface, equal and opposite forces of* magnitude *σdL are exposed normal to the cut and parallel to the surface, where is* called the *coefficient of surface tension.*
  Student Exercise: Find a relation to surface tension description for below shapes.



Contact angle

- A second important surface effect is the *contact angle which appears when a* liquid interface intersects with a solid surface.
- If the contact angle is less than 90°, the liquid is said to *wet the solid; if 90°, the liquid is termed nonwetting.*
- *e.g, water* wets soap but does not wet wax. Water is extremely wetting to a clean glass surface.





Basic Flow-Analysis Techniques

- 1. Control-volume, or *integral analysis*
- 2. Infinitesimal system, or *differential analysis*
- 3. Experimental study, or *dimensional analysis*

In all cases, the flow must satisfy the three basic laws of mechanics plus a thermodynamic state relation and associated boundary conditions:

- 1. Conservation of mass (continuity)
- 2. Linear momentum (Newton's second law)
- 3. First law of thermodynamics (conservation of energy)
- 4. A state relation like $\rho(p, T)$
- 5. Appropriate boundary conditions at solid surfaces, interfaces, inlets, and exits

Flows Classification

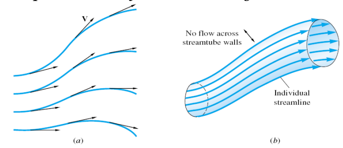
- Steady or unsteady
- Inviscid or viscous
- compressible or compressible
- Gas or liquid

Steady / Unsteady | Inviscid / Viscous | Incompressible / Compressible | Gas / Liquid

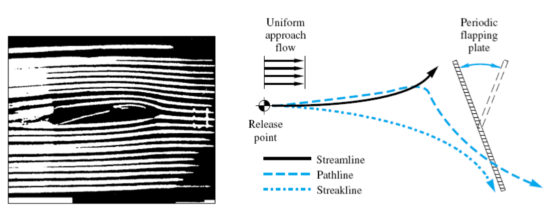
Flow Patterns

- 1. A *streamline is a line everywhere tangent to the velocity vector at a given instant.*
- 2. A *pathline is the actual path traversed by a given fluid particle.*
- 3. A *streakline is the locus of particles which have earlier passed through a prescribed* point.
- 4. A *timeline is a set of fluid particles that form a line at a given instant.*

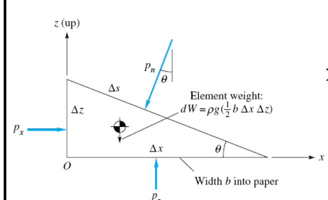
* Streamlines, pathlines, and streaklines are identical in steady flow.



Stream, Streak and Path Lines



Pressure & Pressure Gradient



Equilibrium of a small wedge of fluid at rest

$$\sum F_x = 0 = p_x b\,\Delta z - p_n b\,\Delta s \sin\theta$$

$$\sum F_z = 0 = p_z b\,\Delta x - p_n b\,\Delta s \cos\theta - \tfrac{1}{2}\gamma b\,\Delta x\,\Delta z$$

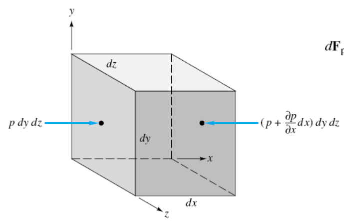
$$\Delta s \sin\theta = \Delta z \qquad \Delta s \cos\theta = \Delta x$$

$$p_x = p_n \qquad p_z = p_n + \tfrac{1}{2}\gamma\,\Delta z$$

$$p_x = p_z = p_n = p$$

$$p = -\tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Pressure & Pressure Gradient

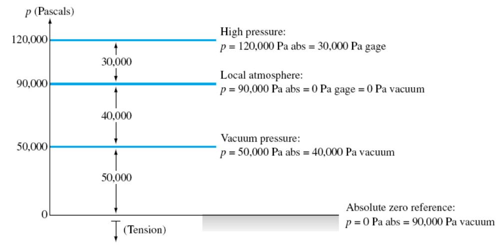


$$d\mathbf{F}_{press} = \left(-\mathbf{i}\frac{\partial p}{\partial x} - \mathbf{j}\frac{\partial p}{\partial y} - \mathbf{k}\frac{\partial p}{\partial z}\right) dx\, dy\, dz$$

$$\mathbf{f}_{press} = -\nabla p$$

$$dF_x = p\, dy\, dz - \left(p + \frac{\partial p}{\partial x} dx\right) dy\, dz = -\frac{\partial p}{\partial x} dx\, dy\, dz$$

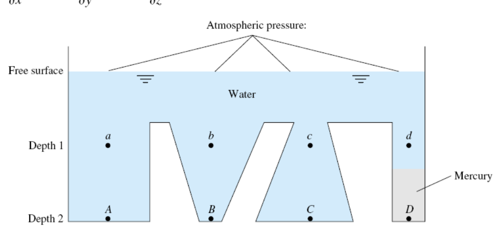
Gage Pressure and Vacuum Pressure: Relative Terms



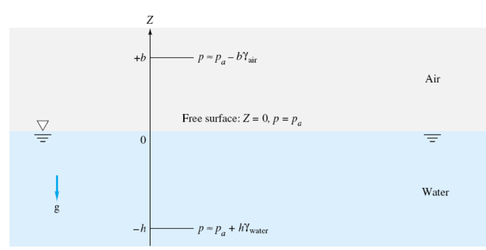
1. $p > p_a$ *Gage* pressure: $p(\text{gage}) = p - p_a$
2. $p < p_a$ *Vacuum* pressure: $p(\text{vacuum}) = p_a - p$

Hydrostatic Pressure Distributions

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g = -\gamma$$



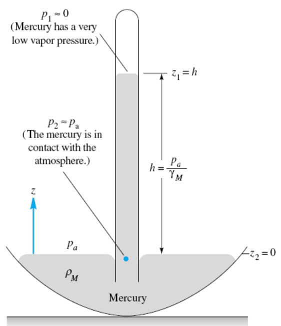
Hydrostatic Pressure in Liquids



The Mercury Barometer

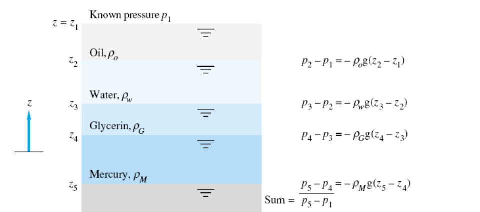
$$p_a - 0 = -\gamma_M(0 - h)$$

$$h = \frac{p_a}{\gamma_M}$$



Application to Manometry

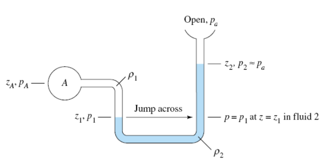
A change in elevation *of a liquid is equivalent* to a change in pressure. *Thus a static column of one or more liquids* or gases can be used to measure pressure differences between two points. Such a device is called a *manometer.*

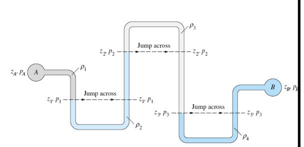


$$p_5 - p_1 = -\gamma_0(z_2 - z_1) - \gamma_w(z_3 - z_2) - \gamma_G(z_4 - z_3) - \gamma_M(z_5 - z_4)$$

$$p_{down} = p^{up} + \gamma|\Delta z|$$

Application to Manometry



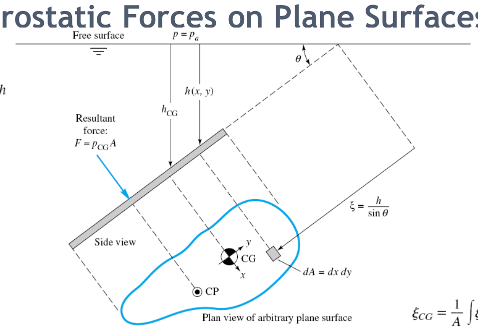


$$p_A + \gamma_1|z_A - z_1| - \gamma_2|z_1 - z_2| = p_2 \approx p_{atm}$$

Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure.

Hydrostatic Forces on Plane Surfaces

$$p = p_a + \gamma h$$



$$\xi_{CG} = \frac{1}{A}\int \xi \, dA$$

$$F = \int p\, dA = \int (p_a + \gamma h)\, dA = p_a A + \gamma \int h\, dA$$

$$F = p_a A + \gamma \sin\theta \int \xi\, dA = p_a A + \gamma \sin\theta\, \xi_{CG} A$$

Hydrostatic Forces on Plane Surfaces

$$F = p_a A + \gamma h_{CG} A = (p_a + \gamma h_{CG})A = p_{CG} A$$

$$F y_{CP} = \int yp \, dA = \int y(p_a + \gamma \xi \sin\theta)\, dA = \gamma \sin\theta \int y\xi \, dA$$

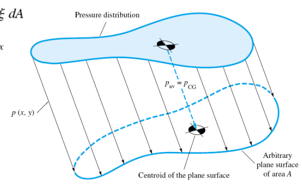
$$F y_{CP} = \gamma \sin\theta \left(\xi_{CG} \int y \, dA - \int y^2 \, dA \right) = -\gamma \sin\theta \, I_{xx}$$

$$y_{CP} = -\gamma \sin\theta \frac{I_{xx}}{p_{CG} A}$$

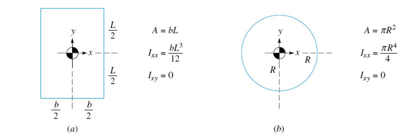
$$F x_{CP} = \int xp \, dA = \int x[p_a + \gamma(\xi_{CG} - y)\sin\theta] \, dA$$

$$= -\gamma \sin\theta \int xy \, dA = -\gamma \sin\theta \, I_{xy}$$

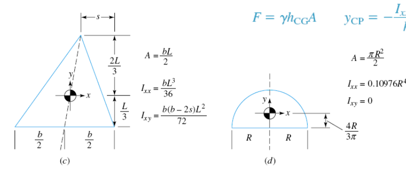
$$x_{CP} = -\gamma \sin\theta \frac{I_{xy}}{p_{CG} A}$$



Center of press.& gage press. formula

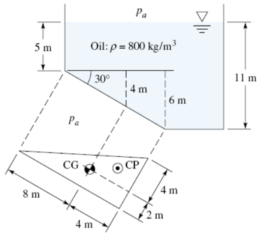


$$F = \gamma h_{CG} A \qquad y_{CP} = -\frac{I_{xx} \sin\theta}{h_{CG} A} \qquad x_{CP} = -\frac{I_{xy} \sin\theta}{h_{CG} A}$$



Example

A tank of oil has a right-triangular panel near the bottom, as in Fig. E2.6. Omitting p_a, find the (*a*) hydrostatic force and (*b*) CP on the panel.



Solution

Part (a) The triangle has properties given in Fig. 2.13*c*. The centroid is one-third up (4 m) and one-third over (2 m) from the lower left corner, as shown. The area is

$$\tfrac{1}{2}(6 \text{ m})(12 \text{ m}) = 36 \text{ m}^2$$

The moments of inertia are

$$I_{xx} = \frac{bL^3}{36} = \frac{(6 \text{ m})(12 \text{ m})^3}{36} = 288 \text{ m}^4$$

and

$$I_{xy} = \frac{b(b-2s)L^2}{72} = \frac{(6 \text{ m})[6 \text{ m} - 2(6 \text{ m})](12 \text{ m})^2}{72} = -72 \text{ m}^4$$

The depth to the centroid is $h_{CG} = 5 + 4 = 9$ m; thus the hydrostatic force from Eq. (2.44) is

$$F = \rho g h_{CG} A = (800 \text{ kg/m}^3)(9.807 \text{ m/s}^2)(9 \text{ m})(36 \text{ m}^2)$$

$$= 2.54 \times 10^6 \text{ (kg} \cdot \text{m)/s}^2 = 2.54 \times 10^6 \text{ N} = 2.54 \text{ MN} \qquad \textit{Ans. (a)}$$

Solution

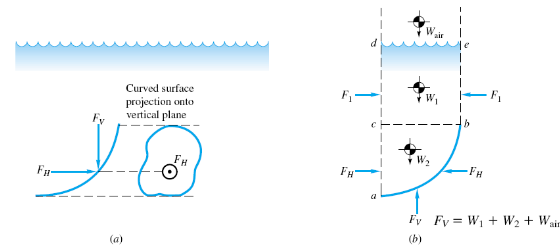
Part (b) The CP position is given by Eqs. (2.44):

$$y_{CP} = -\frac{I_{xx}\sin\theta}{h_{CG}A} = -\frac{(288\ \text{m}^4)(\sin 30°)}{(9\ \text{m})(36\ \text{m}^2)} = -0.444\ \text{m}$$

$$x_{CP} = -\frac{I_{xy}\sin\theta}{h_{CG}A} = -\frac{(-72\ \text{m}^4)(\sin 30°)}{(9\ \text{m})(36\ \text{m}^2)} = +0.111\ \text{m} \qquad \textit{Ans. (b)}$$

The resultant force $F = 2.54$ MN acts through this point, which is down and to the right of the centroid, as shown in Fig. E2.6.

Hydrostatic Forces on Curved Surfaces

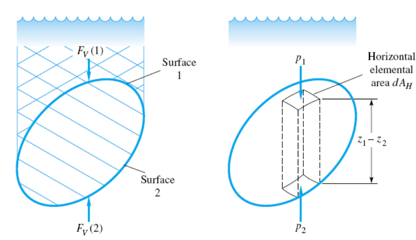


The horizontal component of force on a curved surface equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component. The vertical component of pressure force on a curved surface equals in magnitude and direction the weight of the entire column of fluid, both liquid and atmosphere, above the curved surface.

Buoyancy and Stability

- Two laws of buoyancy discovered by Archimedes in the third century B.C.:
  - 1. A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces.
  - 2. A floating body displaces its own weight in the fluid in which it floats.

Buoyancy



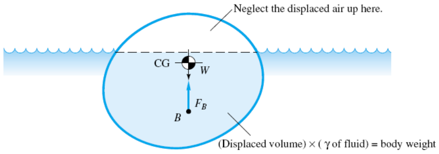
$$F_B = F_V(2) - F_V(1)$$

$$= (\text{fluid weight above 2}) - (\text{fluid weight above 1})$$

$$= \text{weight of fluid equivalent to body volume}$$

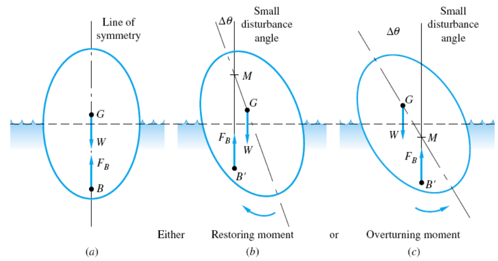
$$F_B = \int_{\text{body}} (p_2 - p_1)\, dA_H = -\gamma \int (z_2 - z_1)\, dA_H = (\gamma)(\text{body volume})$$

Floating Body

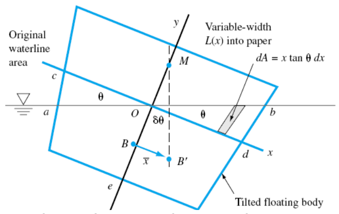


$$F_B = (\gamma)(\text{displaced volume}) = \text{floating-body weight}$$

Stability



Stability Related to Waterline Area



$$\bar{x}\, \upsilon_{abOde} = \int_{cOdea} x\, d\upsilon + \int_{Obd} x\, d\upsilon - \int_{cOa} x\, d\upsilon = 0 + \int_{Obd} x\,(L\, dA) - \int_{cOa} x\,(L\, dA)$$

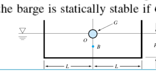
$$= 0 + \int_{Obd} x\, L\,(x \tan\theta\, dx) - \int_{cOa} xL\,(-x \tan\theta\, dx) = \tan\theta \int_{\text{waterline}} x^2\, dA_{\text{waterline}} = I_O \tan\theta$$

Stability formula

$$\frac{\bar{x}}{\ } = \overline{MB} = \frac{I_O}{\upsilon_{\text{submerged}}} = \overline{MG} + \overline{GB} \qquad \text{or} \qquad \overline{MG} = \frac{I_O}{\upsilon_{\text{sub}}} - \overline{GB}$$

If the metacentric height *MG is positive, the body is stable for small* disturbances. Note that if *GB is negative, that is, B is above G, the body is always* stable.

A barge has a uniform rectangular cross section of width $2L$ and vertical draft of height H, as in Fig. E2.10. Determine (*a*) the metacentric height for a small tilt angle and (*b*) the range of ratio L/H for which the barge is statically stable if G is exactly at the waterline as shown.



Solution

If the barge has length b into the paper, the waterline area, relative to tilt axis O, has a base b and a height $2L$; therefore, $I_O = b(2L)^3/12$. Meanwhile, $\upsilon_{\text{sub}} = 2LbH$. Equation (2.52) predicts

$$\overline{MG} = \frac{I_O}{\upsilon_{\text{sub}}} - \overline{GB} = \frac{8bL^3/12}{2LbH} - \frac{H}{2} = \frac{L^2}{3H} - \frac{H}{2} \qquad \textit{Ans. (a)}$$

The barge can thus be stable only if

$$L^2 > 3H^2/2 \qquad \text{or} \qquad 2L > 2.45H \qquad \textit{Ans. (b)}$$

The wider the barge relative to its draft, the more stable it is. Lowering G would help also.

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Pressure Distribution in Rigid-Body Motion

- In rigid-body motion, all particles are in combined translation and rotation, and there is no relative motion between particles. With no relative motion, there are no strains, strain rates, so leaving a balance between pressure, gravity, and particle acceleration

$$\nabla p = \rho(\mathbf{g} - \mathbf{a})$$

- The pressure gradient acts in the direction g -a, and lines of constant pressure (including the free surface, if any) are perpendicular to this direction.
- The general case of combined translation and rotation of a rigid body is: $\mathbf{V} = \mathbf{V}_0 + \boldsymbol{\Omega} \times \mathbf{r}_0$
- Differentiating, we obtain the most general form of the acceleration of a rigid body: $\mathbf{a} = \frac{d\mathbf{V}_0}{dt} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0) + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}_0$

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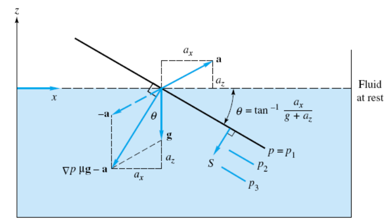
Pressure Distribution in Rigid-Body Motion

$$\theta = \tan^{-1}\frac{a_x}{g + a_z}$$



$$\frac{dp}{ds} = \rho G \qquad \text{where } G = [a_x^2 + (g + a_z)^2]^{1/2}$$

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Rigid-Body Rotation

the angular-velocity and position vectors are given by

$$\boldsymbol{\Omega} = \mathbf{k}\Omega \qquad r_0 = \mathbf{i}_r r$$

Then the acceleration is given by

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0) = -r\Omega^2\mathbf{i}_r$$

$$\nabla p = \mathbf{i}_r\frac{\partial p}{\partial r} + \mathbf{k}\frac{\partial p}{\partial z} = \rho(\mathbf{g} - \mathbf{a}) = \rho(-g\mathbf{k} + r\Omega^2\mathbf{i}_r)$$

Equating like components, we find the pressure field by solving two first-order partial differential equations

$$\frac{\partial p}{\partial r} = \rho r\Omega^2 \qquad \frac{\partial p}{\partial z} = -\gamma$$

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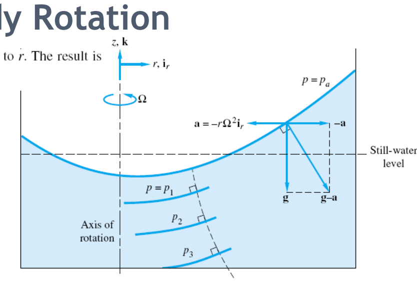
Rigid-Body Rotation

holding z constant, with respect to r. The result is

$$p = \tfrac{1}{2}\rho r^2\Omega^2 + f(z)$$

$$\frac{\partial p}{\partial z} = 0 + f'(z) = -\gamma$$

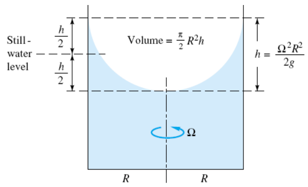
$$f(z) = -\gamma z + C$$



This is the pressure distribution in the fluid. The value of C is found by specifying the pressure at one point. If $p = p_0$ at $(r, z) = (0, 0)$, then $C = p_0$. The final desired distribution is

$$p = \text{const} - \gamma z + \tfrac{1}{2}\rho r^2\Omega^2 \qquad p = p_0 - \gamma z + \tfrac{1}{2}\rho r^2\Omega^2 \qquad z = \frac{p_0 - p_1}{\gamma} + \frac{r^2\Omega^2}{2g} = a + br^2$$

Rigid-Body Rotation



Since the volume of a paraboloid is one-half the base area times its height, the still-water level is exactly halfway between the high and low points of the free surface.

Integral Relations for a Control Volume

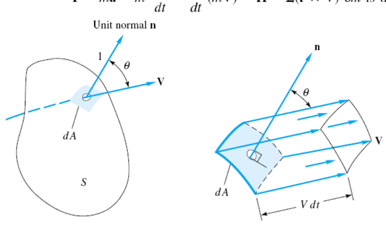
- Three basic approaches to the analysis of arbitrary flow problems:
  - 1. Control-volume, or large-scale, analysis
  - 2. Differential, or small-scale, analysis
  - 3. Experimental, or dimensional, analysis
- ***System***: *an arbitrary* quantity of mass of fixed identity. Everything external to this system is denoted by the term ***surroundings***, and the system is separated from its surroundings by its ***boundaries***.

Volume and Mass Rate of Flow

conservation of mass: $m_{syst} = \text{const}$ $\quad \frac{dm}{dt} = 0$

Newton's second law $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$

$\mathbf{M} = \frac{d\mathbf{H}}{dt}$ $\quad \mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\ \delta m$ is the angular momentum



Volume and Mass Rate of Flow

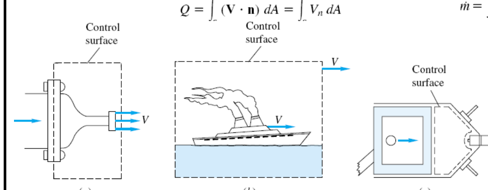
$$d\mathcal{V} = V\,dt\,dA\cos\theta = (\mathbf{V}\cdot\mathbf{n})\,dA\,dt$$

The integral of $d\mathcal{V}/dt$ is the total volume rate of flow Q through the surface S

$$Q = \int_S (\mathbf{V}\cdot\mathbf{n})\,dA = \int_S V_n\,dA$$

$$\dot{m} = \int_S \rho(\mathbf{V}\cdot\mathbf{n})\,dA = \int_S \rho V_n\,dA$$

$$\dot{m} = \rho Q$$



Fixed, **moving**, and **deformable** control volumes: (*a*) *fixed* control volume for nozzle-stress analysis; (*b*) *control volume moving* at ship speed for drag-force analysis; (*c*) *control volume deforming* within cylinder for transient pressure-variation analysis.

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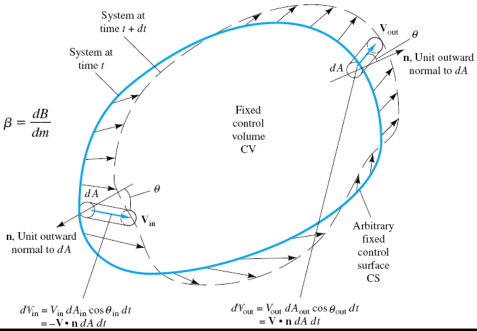
The Reynolds Transport Theorem

$$B_{CV} = \int_{CV} \beta\rho \, d\mathcal{V} \qquad \beta = \frac{dB}{dm}$$



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Reynolds Transport Theorem.

$$\frac{d}{dt}(B_{CV}) = \frac{1}{dt} B_{CV}(t + dt) - \frac{1}{dt} B_{CV}(t)$$

$$= \frac{1}{dt}[B_2(t + dt) - (\beta\rho \, d\mathcal{V})_{out} + (\beta\rho d\mathcal{V})_{in}] - \frac{1}{dt}[B_2(t)]$$

$$= \frac{1}{dt}[B_2(t + dt) - B_2(t)] - (\beta\rho AV)_{out} + (\beta\rho AV)_{in}$$

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho V \cos\theta \, dA_{out} - \int_{CS} \beta\rho V \cos\theta \, dA_{in}$$

$$\text{Flux terms} = \int_{CS} \beta\rho V_n \, dA_{out} - \int_{CS} \beta\rho V_n \, dA_{in} = \int_{CS} \beta \, d\dot{m}_{out} - \int_{CS} \beta \, d\dot{m}_{in} = \int_{CS} \beta\rho(\mathbf{V} \cdot \mathbf{n}) \, dA$$

The compact form of the Reynolds transport theorem is thus

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) + \int_{CV} \beta\rho(\mathbf{V} \cdot \mathbf{n}) \, dA$$

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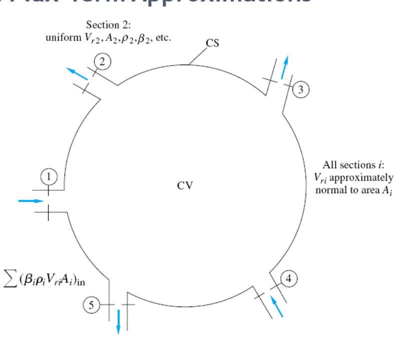
One-Dimensional Flux-Term Approximations



$$\int_{CS} \beta\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA = \sum(\beta_i\rho_i V_{ri}A_i)_{out} - \sum(\beta_i\rho_i V_{ri}A_i)_{in}$$

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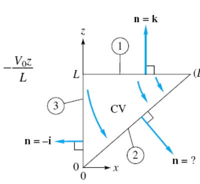
Conservation of Mass

$$\left(\frac{dm}{dt}\right)_{syst} = 0 = \frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA \qquad \int_{CV} \frac{\partial\rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA = 0$$

EXAMPLE Consider the constant-density velocity field

$$u = \frac{V_0 x}{L} \qquad v = 0 \qquad w = -\frac{V_0 z}{L}$$



Depth b into paper

$$(\mathbf{V} \cdot \mathbf{n})_1 = (\mathbf{i}u + \mathbf{k}w) \cdot \mathbf{k} = w|_1 = -\frac{V_0 z}{L}\bigg|_{z=L} = -V_0$$

$$Q_1 = \int_1 (\mathbf{V} \cdot \mathbf{n}) \, dA = \int_0^L (-V_0)b \, dx = -V_0 bL$$

$$(\mathbf{V} \cdot \mathbf{n})_3 = (\mathbf{i}u + \mathbf{k}w) \cdot (-\mathbf{i}) = -u|_3 = -\frac{V_0 x}{L}\bigg|_{s=0} = 0$$

$$(\mathbf{V} \cdot \mathbf{n})_2 = (\mathbf{i}u + \mathbf{k}w) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}) = \frac{1}{\sqrt{2}}(u - w)_2$$

$$= \frac{1}{\sqrt{2}}\left[V_0 \frac{x}{L} - \left(-V_0 \frac{z}{L}\right)\right]_{x=z} = \frac{\sqrt{2}V_0 x}{L} \quad \text{or} \quad \frac{\sqrt{2}V_0 z}{L}$$

$$Q_2 = \int_2 (\mathbf{V} \cdot \mathbf{n}) \, dA = \int_0^L \frac{\sqrt{2}V_0 x}{L}(\sqrt{2}b \, dx) = V_0 bL$$

$$Q_1 + Q_2 + Q_3 = -V_0 bL + V_0 bL + 0 = 0$$

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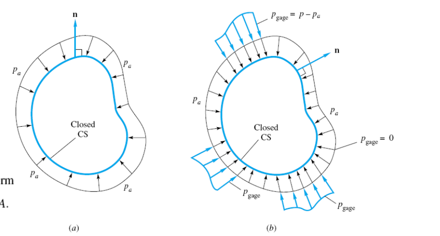
The Linear Momentum Equation

$$\frac{d}{dt}(m\mathbf{V})_{syst} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \qquad \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})\, dA$$

Net Pressure Force on a Closed Control Surface

$$\mathbf{F}_{press} = \int_{CS} p(-\mathbf{n})\, dA$$



$-p_a \int \mathbf{n}\, dA = 0$; (b) nonuniform pressure, $\mathbf{F} = -\int (p - p_a)\mathbf{n}\, dA$.

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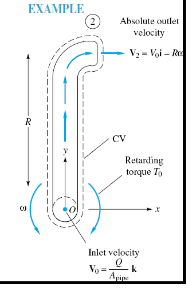
The Angular-Momentum Theorem

$$\mathbf{H}_O = \int_{syst} (\mathbf{r} \times \mathbf{V})\, dm \qquad \beta = \frac{d\mathbf{H}_O}{dm} = \mathbf{r} \times \mathbf{V}$$

$$\left.\frac{d\mathbf{H}_O}{dt}\right|_{syst} = \frac{d}{dt}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho\, d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \qquad \frac{d\mathbf{H}_O}{dt} = \sum \mathbf{M}_O = \sum (\mathbf{r} \times \mathbf{F})_O$$

$$\sum \mathbf{M}_0 = \frac{\partial}{\partial t}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho\, d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n})\, dA$$

EXAMPLE



Solution

$$\mathbf{V}_2 = V_0\mathbf{i} - R\omega\mathbf{i}$$

$$\sum \mathbf{M}_O = -T_O\mathbf{k} = (\mathbf{r}_2 \times \mathbf{V}_2)\dot{m}_{out} - (\mathbf{r}_1 \times \mathbf{V}_1)\dot{m}_{in} \qquad (1)$$

where, from continuity, $\dot{m}_{out} = \dot{m}_{in} = \rho Q$. The cross products with reference to point O are

$$\mathbf{r}_2 \times \mathbf{V}_2 = R\mathbf{j} \times (V_0 - R\omega)\mathbf{i} = (R^2\omega - RV_0)\mathbf{k}$$

$$\mathbf{r}_1 \times \mathbf{V}_1 = 0\mathbf{j} \times V_0\mathbf{k} = 0$$

Equation (1) thus becomes

$$-T_O\mathbf{k} = \rho Q(R^2\omega - RV_0)\mathbf{k}$$

$$\omega = \frac{V_O}{R} - \frac{T_O}{\rho Q R^2} \qquad \textit{Ans.}$$

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The Energy Equation

$$\beta = dE/dm = e. \qquad \frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt}\left(\int_{CV} e\rho\, d\mathcal{V}\right) + \int_{CS} e\rho(\mathbf{V} \cdot \mathbf{n})\, dA$$

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other} \qquad e = \hat{u} + \tfrac{1}{2}V^2 + gz$$

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{press} + \dot{W}_{viscous\ stresses} = \dot{W}_s + \dot{W}_p + \dot{W}_v$$

$$d\dot{W}_p = -(p\, dA)V_{n,in} = -p(-\mathbf{V} \cdot \mathbf{n})\, dA \qquad d\dot{W}_v = -\boldsymbol{\tau} \cdot \mathbf{V}\, dA$$

The total pressure work is the integral over the control surface

$$\dot{W}_v = -\int_{CS} \boldsymbol{\tau} \cdot \mathbf{V}\, dA$$

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})\, dA$$

$$\dot{W} = \dot{W}_s + \int_{CS} p(\mathbf{V} \cdot \mathbf{n})\, dA - \int_{CS} (\boldsymbol{\tau} \cdot \mathbf{V})_{SS}\, dA$$

$$\dot{Q} - \dot{W}_s - (\dot{W}_v)_{SS} = \frac{\partial}{\partial t}\left(\int_{CV} e\rho\, d\mathcal{V}\right) + \int_{CS} \left(e + \frac{p}{\rho}\right)\rho(\mathbf{V} \cdot \mathbf{n})\, dA$$

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left[\int_{CV} \left(\hat{u} + \tfrac{1}{2}V^2 + gz\right)\rho\, d\mathcal{V}\right] + \int_{CS} \left(\hat{h} + \tfrac{1}{2}V^2 + gz\right)\rho(\mathbf{V} \cdot \mathbf{n})\, dA$$

Friction Losses in Low-Speed Flow

$$\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_{in} = \left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_{out} + h_{friction} - h_{pump} + h_{turbine}$$