Chapter 8

Plasmons and Plasma Screening

In this chapter, we discuss collective excitations in the electron gas. As mentioned earlier, *collective excitations* are excitations that belong to the entire system. The collective excitations of the electron gas (= plasma) are called *plasmons*. These excitations and their effect on the dielectric constant are discussed in Chap. 1 in the framework of classical electrodynamics. In this chapter, we now develop the corresponding second-quantized formalism, which reveals that electron–electron pair excitations occur which influence the dielectric constant and other properties in fundamental ways. The excitations in the electron plasma are responsible for screening of the Coulomb potential, effectively reducing it to a potential whose interaction range is reduced with increasing plasma density. A simplified description of the screening is developed in terms of an effective collective excitation, and this is referred to as the plasmon pole approximation.

8.1 Plasmons and Pair Excitations

In order to analyze elementary excitations of the electron plasma, we compute the dynamical evolution of a charge density fluctuation. In the formalism of second quantization, we evaluate the equation of motion for the expectation value of the electron charge density operator

$$\langle \hat{\rho}_{e,\mathbf{q}} \rangle = -\frac{|e|}{L^3} \sum_{\mathbf{k},s} \langle \hat{a}^\dagger_{\mathbf{k}-\mathbf{q},s} \hat{a}_{\mathbf{k},s} \rangle \tag{8.1}$$

defined in Eq. (7.24). In a spatially homogeneous equilibrium system, this expectation value would vanish for $q \neq 0$, however, we assume here a spatially inhomogeneous charge density distribution.

To simplify the notation in the remainder of this book, we suppress from now on the superscript ^ for operators, unless this is needed to avoid misunderstandings. Furthermore, the spin index is only given where necessary. In all other cases, it can be assumed to be included in the quasi-momentum subscript.

In the following, we can also drop the subscript e of the charge density operator, since we discuss only electrons in this chapter. With this simplified notation Eq. (8.1) becomes

$$\langle \rho_{\mathbf{q}} \rangle = -\frac{|e|}{L^3} \sum_{\mathbf{k}} \langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}} \rangle \ . \tag{8.2}$$

To obtain the equation of motion for $\langle \rho_{\mathbf{q}} \rangle$, we use the Heisenberg equation for $a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}$

$$\frac{d}{dt} a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}} = \frac{i}{\hbar} [\mathcal{H}, \, a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}] \tag{8.3}$$

with the electron gas Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}} E_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2} \sum_{\substack{\mathbf{k},\mathbf{k}' \\ \mathbf{q} \neq 0}} V_q a^\dagger_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{k}'+\mathbf{q}} a_{\mathbf{k}'} a_{\mathbf{k}} \ . \tag{8.4}$$

Evaluating the commutators on the RHS of Eq. (8.3), we get for the kinetic term

$$\frac{i}{\hbar} \sum_{\mathbf{k}'} E_{k'} [a^\dagger_{\mathbf{k}'} a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}] = i(\epsilon_{k-q} - \epsilon_k) a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}} \ , \tag{8.5}$$

where we have again introduced the frequencies

$$\epsilon_k = E_k/\hbar \quad \text{and} \quad \epsilon_{k-q} = \epsilon_{|\mathbf{k}-\mathbf{q}|} = E_{|\mathbf{k}-\mathbf{q}|}/\hbar \ . \tag{8.6}$$

For the Coulomb term, we obtain

$$\begin{aligned} &\sum \frac{iV_p}{2\hbar} \ [a^\dagger_{\mathbf{k}'-\mathbf{p}} a^\dagger_{\mathbf{p}'+\mathbf{p}} a_{\mathbf{p}'} a_{\mathbf{k}'} \ , \ a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}] = \\ &= \sum \frac{iV_p}{2\hbar} \Big(a^\dagger_{\mathbf{k}-\mathbf{q}-\mathbf{p}} a^\dagger_{\mathbf{p}'+\mathbf{p}} a_{\mathbf{p}'} a_{\mathbf{k}} - a^\dagger_{\mathbf{k}'-\mathbf{p}} a^\dagger_{\mathbf{k}-\mathbf{q}+\mathbf{p}} a_{\mathbf{k}'} a_{\mathbf{k}} \\ &\qquad + a^\dagger_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{k}'-\mathbf{p}} a_{\mathbf{k}-\mathbf{p}} a_{\mathbf{k}'} - a^\dagger_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{p}'+\mathbf{p}} a_{\mathbf{p}'} a_{\mathbf{k}+\mathbf{p}} \Big) \ . \end{aligned} \tag{8.7}$$

After renaming $\mathbf{p}$ to - $\mathbf{p}$, using $V_{-p} = V_p$, and rearranging some operators, we see that the first and second term and the third and fourth term become identical.

Collecting all contributions of the commutator in (8.3), and taking the expectation value, we obtain

$$\frac{d}{dt}\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}\rangle = i(\epsilon_{k-q} - \epsilon_k)\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}\rangle + \frac{i}{\hbar}\sum_{\mathbf{p}',\mathbf{p}} V_p \left(\langle a^\dagger_{\mathbf{k}-\mathbf{q}-\mathbf{p}} a^\dagger_{\mathbf{p}'+\mathbf{p}} a_{\mathbf{p}'} a_{\mathbf{k}}\rangle + \langle a^\dagger_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{p}'-\mathbf{p}} a_{\mathbf{k}-\mathbf{p}} a_{\mathbf{p}'}\rangle\right). \quad (8.8)$$

Since we are interested in $\langle \rho_{\mathbf{q}}\rangle$, we have to solve Eq. (8.8) and sum over $\mathbf{k}$. However, we see from Eq. (8.8) that the two-operator dynamics is coupled to four-operator terms. One way to proceed therefore would be to compute the equation of motion for the four-operator term. Doing this we find that the four-operator equation couples to six-operator terms, which in turn couple to eight-operator terms, etc. Hence, if we follow this approach we obtain an infinite hierarchy of equations, which we have to truncate at some stage in order to get a closed set of coupled differential equations.

Instead of deriving such a hierarchy of equations, we make a factorization approximation directly in Eq. (8.8), splitting the four-operator expectation values into products of the relevant two-operator expectation values. For the one-component plasma under consideration, we choose the combinations

$$\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}\rangle \text{ and } \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle = f_k \quad (8.9)$$

as relevant, assuming that these terms dominate the properties of our system. This approximation scheme is often called *random phase approximation* (RPA). In (8.9), $f_{\mathbf{k}}$ denotes the carrier distribution function which is the Fermi–Dirac distribution function for electrons in thermodynamic equilibrium. However, our approximations are also valid for nonequilibrium distributions.

A hand-waving argument for the random phase approximation is to say that an expectation value $\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}'}\rangle$ has a dominant time dependence

$$\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}'}\rangle \propto e^{i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})t} \quad . \quad (8.10)$$

These expectation values occur under sums, so that expressions like

$$\sum_{\mathbf{k},\mathbf{k}'} e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}$$

have to be evaluated. Since terms with $\mathbf{k} \neq \mathbf{k}'$ oscillate rapidly they more or less average to zero, whereas the term with $\mathbf{k} = \mathbf{k}'$ gives the dominant contribution.

Technically, when we make the RPA approximation, we pick specific combinations of wave numbers from the sums on the RHS of Eq. (8.8), factorize the four-operator expectation values into the expressions (8.9), and ignore all other contributions. For example, in the term

$$T_1 = \frac{i}{\hbar} \sum_{\mathbf{p}',\mathbf{p}\neq 0} V_p \langle a^\dagger_{\mathbf{k}-\mathbf{q}-\mathbf{p}} a^\dagger_{\mathbf{p}'+\mathbf{p}} a_{\mathbf{p}'} a_{\mathbf{k}} \rangle \tag{8.11}$$

we choose $\mathbf{p} = -\mathbf{q}$ and obtain

$$T_1 \simeq \frac{iV_q}{\hbar} \sum_{\mathbf{p}'} \langle a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'} a_{\mathbf{k}} \rangle \ , \tag{8.12}$$

where $V_q = V_{-q}$ since the Coulomb potential depends only on the absolute value of $\mathbf{q}$. Now we commute $a_{\mathbf{k}}$ in (8.12) to the left:

$$T_1 \simeq \frac{iV_q}{\hbar} \sum_{\mathbf{p}'} \left(-\langle a^\dagger_{\mathbf{k}} a_{\mathbf{p}'} \rangle \delta_{\mathbf{p}'-\mathbf{q},\mathbf{k}} + \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'} \rangle \right) \ . \tag{8.13}$$

Factorizing the four-operator expectation value and using (8.9) yields

$$T_1 \simeq \frac{iV_q}{\hbar} \left(-\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}+\mathbf{q}} \rangle + f_{\mathbf{k}} \sum_{\mathbf{p}'} \langle a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'} \rangle \right) \ . \tag{8.14}$$

Similarly, for the second Coulomb term in Eq. (8.8),

$$T_2 = \frac{i}{\hbar} \sum_{\mathbf{p}',\mathbf{p}\neq 0} V_p \langle a^\dagger_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{p}'-\mathbf{p}} a_{\mathbf{k}-\mathbf{p}} a_{\mathbf{p}'} \rangle \ , \tag{8.15}$$

we select $\mathbf{p} = \mathbf{q}$ and commute the first destruction operator to the left to get

$$T_2 \simeq \frac{iV_q}{\hbar} \sum_{\mathbf{p}'} \left(\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{p}'} \rangle \delta_{\mathbf{p}',\mathbf{k}} - \langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}} a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'} \rangle \right) \ , \tag{8.16}$$

or, after factorization,

$$T_2 \simeq \frac{iV_q}{\hbar}\left(\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle - f_{\mathbf{k}-\mathbf{q}}\sum_{\mathbf{p}'}\langle a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'}\rangle\right) . \qquad (8.17)$$

Inserting the approximation (8.14) and (8.17) into Eq. (8.8) yields

$$\frac{d}{dt}\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle \simeq i(\epsilon_{k-q}-\epsilon_k)\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle + \frac{iV_q}{\hbar}(f_\mathbf{k}-f_{\mathbf{k}-\mathbf{q}})\sum_{\mathbf{p}'}\langle a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'}\rangle. \qquad (8.18)$$

In order to find the eigenfrequencies of the charge density, we use the ansatz

$$\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle(t) = e^{-i(\omega+i\delta)t}\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle(0) , \qquad (8.19)$$

in Eq. (8.18) to obtain

$$\hbar(\omega+i\delta+\epsilon_{k-q}-\epsilon_k)\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle = V_q(f_{\mathbf{k}-\mathbf{q}}-f_\mathbf{k})\sum_{\mathbf{p}'}\langle a^\dagger_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}'}\rangle . \qquad (8.20)$$

Dividing both sides by $\hbar(\omega+i\delta+\epsilon_{k-q}-\epsilon_k)$, summing the resulting equation over $\mathbf{k}$, and multiplying by $-e/L^3$, we find

$$\langle\rho_\mathbf{q}\rangle = V_q\langle\rho_\mathbf{q}\rangle\sum_\mathbf{k}\frac{f_{\mathbf{k}-\mathbf{q}}-f_\mathbf{k}}{\hbar(\omega+i\delta+\epsilon_{k-q}-\epsilon_k)} . \qquad (8.21)$$

We see that $\langle\rho_\mathbf{q}\rangle$ cancels, so that

$$V_q\sum_\mathbf{k}\frac{f_{\mathbf{k}-\mathbf{q}}-f_\mathbf{k}}{\hbar(\omega+i\delta+\epsilon_{k-q}-\epsilon_k)} = 1 . \qquad (8.22)$$

Introducing the first-order approximation $P^1(q,\omega)$ to the *polarization function* $P(q,\omega)$ as

$$P^1(q,\omega) = \sum_\mathbf{k}\frac{f_{\mathbf{k}-\mathbf{q}}-f_\mathbf{k}}{\hbar(\omega+i\delta+\epsilon_{k-q}-\epsilon_k)} , \qquad (8.23)$$

we can write Eq. (8.22) as

$$V_q\,P^1(q,\omega) = 1 . \qquad (8.24)$$

The real part of this equation determines the eigenfrequencies $\omega=\omega_q$:

$$V_q\sum_\mathbf{k}\frac{f_{\mathbf{k}-\mathbf{q}}-f_\mathbf{k}}{\hbar(\omega_q+\epsilon_{k-q}-\epsilon_k)} = 1 , \qquad (8.25)$$

where we let $\delta \to 0$.

To analyze the solutions of Eq. (8.25), we first discuss the long wave-length limit for a three dimensional plasma. Long wave length means $\lambda \to \infty$, and hence $q \propto 1/\lambda \to 0$. We expand Eq. (8.25) in terms of q and drop higher-order corrections. Using

$$E_{k-q} - E_k = \frac{\hbar^2}{2m}(k^2 - 2\mathbf{k}\cdot\mathbf{q} + q^2) - \frac{\hbar^2 k^2}{2m} \simeq -\frac{\hbar^2 \mathbf{k}\cdot\mathbf{q}}{m} \tag{8.26}$$

and

$$f_{\mathbf{k}-\mathbf{q}} - f_{\mathbf{k}} = f_{\mathbf{k}} - \mathbf{q}\cdot\nabla_{\mathbf{k}} f_{\mathbf{k}} + \cdots - f_{\mathbf{k}} \simeq -\mathbf{q}\cdot\nabla_{\mathbf{k}} f_{\mathbf{k}} \tag{8.27}$$

in Eq. (8.25) yields

$$\begin{aligned} 1 &\simeq -V_q \sum_{\mathbf{k},i} \frac{q_i \frac{\partial f}{\partial k_i}}{\hbar\omega_0 - \hbar^2\mathbf{k}\cdot\mathbf{q}/m} \\ &\simeq -\frac{V_q}{\hbar\omega_0} \sum_{\mathbf{k},i} q_i \frac{\partial f}{\partial k_i}\left(1 + \frac{\hbar\mathbf{k}\cdot\mathbf{q}}{m\omega_0}\right) \ , \end{aligned} \tag{8.28}$$

where we have set $\omega_{q\to 0} = \omega_0$. The first term vanishes since, after evaluation of the sum, it is proportional to the distribution function for $k \to \infty$. So we are left with

$$1 = -\frac{V_q}{\hbar\omega_0} \sum_{\mathbf{k},i} q_i \frac{\partial f}{\partial k_i} \frac{\hbar\mathbf{k}\cdot\mathbf{q}}{m\omega_0} \ . \tag{8.29}$$

Partial integration gives

$$1 = V_q \frac{q^2}{m\omega_0^2} \sum_{\mathbf{k}} f_{\mathbf{k}} = V_q \frac{q^2 N}{m\omega_0^2} = \frac{4\pi e^2}{\epsilon_0 q^2 L^3} \frac{q^2 N}{m\omega_0^2} \ , \tag{8.30}$$

or

$$\omega_0^2 = \frac{4\pi e^2 n}{\epsilon_0 m} = \omega_{pl}^2 \ , \tag{8.31}$$

showing that in the long wave-length limit, $q \to 0$, $\omega_{q\to 0} = \omega_{pl}$, i.e., we recover the classical result for the eigenfrequency of the electron plasma. The only difference to the plasma frequency defined in Eq. (1.26) is the factor of $1/\epsilon_0$ which results from the fact that we include the background dielectric constant in the present chapter.

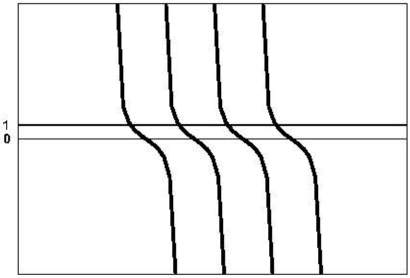

Fig. 8.1 Graphical solution of Eq. (8.25). The full lines are a schematic drawing of part of the LHS of Eq. (8.25) and the line "1" is the RHS of Eq. (8.25).

Next, we discuss the solution of Eq. (8.25) for general wave lengths. First we write the LHS of Eq. (8.25) in the form

$$V_q \sum_{\mathbf{k}} \frac{f_{\mathbf{k-q}} - f_{\mathbf{k}}}{\hbar(\omega_q + \epsilon_{k-q} - \epsilon_k)} = V_q \sum_{\mathbf{k}} \frac{f_{\mathbf{k}}}{\hbar} \left(\frac{1}{\omega_q + \epsilon_k - \epsilon_{k+q}} - \frac{1}{\omega_q + \epsilon_{k-q} - \epsilon_k} \right) . \tag{8.32}$$

This expression shows that poles occur at

$$\omega_q = \epsilon_{k+q} - \epsilon_k = \frac{\hbar k q}{m} \cos\theta + \frac{\hbar q^2}{2m} \tag{8.33}$$

and

$$\omega_q = \epsilon_k - \epsilon_{k-q} = \frac{\hbar q k}{m} \cos\theta - \frac{\hbar q^2}{2m} \ , \tag{8.34}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. As schematically shown in Fig. 8.1, we can find the solutions of Eq. (8.25) as the intersections of the LHS of Eq. (8.25) with the straight line "1", which is the RHS of Eq. (8.25). From Fig. 8.1 we see that these intersection points are close to the poles of the LHS.

For illustration, we discuss in the following the situation of a thermalized electron plasma at low-temperatures. Here, we know that the extrema of the allowed k values are $k' = \pm k_F$. Considering only $\omega_q > 0$, we obtain from Eq. (8.33)

$$\omega_q^{max} = \frac{\hbar q k_F}{m} + \frac{\hbar q^2}{2m} \ , \tag{8.35}$$

for $\cos(\theta) = 1$ and

$$\omega_q^{min} = -\frac{\hbar q k_F}{m} + \frac{\hbar q^2}{2m} \ , \tag{8.36}$$

for $\cos\theta = -1$. From Eq. (8.34) we get no solution for $\cos\theta = -1$ and for $\cos\theta = 1$ we obtain

$$\omega_q^{ext} = \frac{\hbar q k_F}{m} - \frac{\hbar q^2}{2m} \ . \tag{8.37}$$

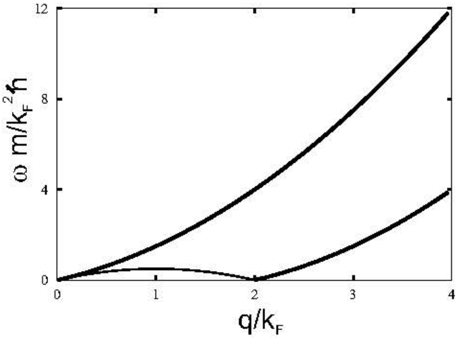

Fig. 8.2 The thick lines show the boundary of the continuum of pair excitations at $T = 0\,K$, according to Eqs. (8.35) and (8.36), respectively. The thin line is the result of Eq. (8.37).

As shown in Fig. 8.2, Eqs. (8.35) and (8.36) define two parabolas that are displaced from the origin by $\pm k_F$. The region between these parabolas for $\omega_q > 0$ is the region where we find the poles. Physically these solutions represent the transition of an electron form $\mathbf{k}$ to $\mathbf{k} \pm \mathbf{q}$, i.e., these are *pair excitations*. They are called pair excitations because the pair of states $\mathbf{k}$ and $\mathbf{k} \pm \mathbf{q}$ is involved in the transition. The region between the parabolas is therefore called the *continuum of electron–pair excitations*. These pair excitations are not to be confused with electron–hole pairs, which we discuss in later chapters of this book. Note, that the pair excitations need an empty final state to occur, and at low temperatures typically involve scattering from slightly below the Fermi surface to slightly above.

The lack of empty final states for scattering with small momentum transfer prevents conduction in an insulator, although there is no lack of electrons. When a plasma mode hits the continuum of pair excitations, it gets damped heavily (Landau damping), causing the collective plasmon excitation to decay into pair excitations. At finite temperatures the boundaries of the pair-excitation spectrum are not sharp, but qualitatively the picture remains similar to the $T = 0$ result.

8.2 Plasma Screening

One of the most important effects of the many-body interactions in an electron plasma is the phenomenon of plasma screening. To discuss plasma screening quantum mechanically, we start with the effective single particle Hamiltonian

$$\mathcal{H} = \int d^3r\, \psi^\dagger(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} \right) \psi(\mathbf{r}) + \int d^3r\, V_{eff}(r) \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \ , \tag{8.38}$$

where

$$V_{eff}(r) = V(r) + V_{ind}(r) \tag{8.39}$$

is the sum of the Coulomb potential $V(r)$ of a test charge and the induced potential $V_{ind}(r)$ of the screening particles. The effective potential V_{eff} has to be determined self-consistently. The Fourier transform of Eq. (8.38) is

$$\mathcal{H} = \sum_{\mathbf{k}} E_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{p}} V_{eff}(p) \sum_{\mathbf{k}} a_{\mathbf{k}+\mathbf{p}}^\dagger a_{\mathbf{k}} \ , \tag{8.40}$$

and the equation of motion for $a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}}$ is

$$\begin{aligned} \frac{d}{dt} a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}} &= \frac{i}{\hbar} [\mathcal{H}, a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}}] \\ &= i(\epsilon_{k-q} - \epsilon_k) a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}} \\ &\quad - \frac{i}{\hbar} \sum_{\mathbf{p}} V_{eff}(p) (a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}-\mathbf{p}} - a_{\mathbf{k}+\mathbf{p}-\mathbf{q}}^\dagger a_{\mathbf{k}}) \ . \end{aligned} \tag{8.41}$$

Using the random phase approximation in the last two terms and taking the expectation value yields

$$\frac{d}{dt}\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle = i(\epsilon_{k-q} - \epsilon_k)\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle - \frac{iV_{eff}(q)}{\hbar}(f_{\mathbf{k}-\mathbf{q}} - f_\mathbf{k}) \ . \tag{8.42}$$

We assume that the test charge varies in time as $\exp(-i(\omega+i\delta)t)$, where $\omega + i\delta$ establishes an adiabatic switch-on of the test charge potential. Making the ansatz that the driven density has the same time dependence

$$\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle \propto e^{-i(\omega+i\delta)t} \ , \tag{8.43}$$

Eq. (8.42) yields

$$\hbar(\omega + i\delta + \epsilon_{k-q} - \epsilon_k)\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle = V_{eff}(q)(f_{\mathbf{k}-\mathbf{q}} - f_\mathbf{k}) \ , \tag{8.44}$$

and therefore

$$\langle a^\dagger_{\mathbf{k}-\mathbf{q}} a_\mathbf{k}\rangle = V_{eff}(q)\frac{f_{\mathbf{k}-\mathbf{q}} - f_\mathbf{k}}{\hbar(\omega + i\delta + \epsilon_{k-q} - \epsilon_k)} \ , \tag{8.45}$$

or, after multiplication by $-|e|/L^3$ and summation over $\mathbf{k}$,

$$\langle \rho_q \rangle = -\frac{|e|}{L^3} V_{eff}(q) P^1(q,\omega) \ , \tag{8.46}$$

where P^1 again is the polarization function defined in Eq. (8.23) and ρ_q is defined in Eq. (8.2).

The potential of the screening particles obeys Poisson's equation in the form

$$\nabla^2 V_{ind}(r) = \frac{4\pi|e|\rho(r)}{\epsilon_0} \ . \tag{8.47}$$

Taking the Fourier transform and using (8.46), Poisson's equation becomes

$$\begin{aligned} V_{ind}(q) &= -\frac{4\pi|e|}{\epsilon_0 q^2}\rho_q = \frac{4\pi e^2}{\epsilon_0 q^2 L^3} V_{eff}(q) P^1(q,\omega) \\ &= V_q\, V_{eff}(q) P^1(q,\omega) \ . \end{aligned} \tag{8.48}$$

Inserting (8.48) into the Fourier-transform of Eq. (8.39):

$$V_{eff}(q) = V_q + V_{ind}(q) \ , \tag{8.49}$$

yields

$$V_{eff}(q) = V_q[1 + V_{eff}(q)P^1(q,\omega)] \tag{8.50}$$

or

$$V_{eff}(q) = \frac{V_q}{1 - V_q P^1(q,\omega)} = \frac{V_q}{\epsilon(q,\omega)} \equiv V_s(q,\omega) \ . \tag{8.51}$$

Here, we introduced $V_s(q,\omega)$ as the *dynamically screened Coulomb potential.* The dynamic dielectric function $\epsilon(q,\omega)$ is given by

$$\epsilon(q,\omega) = 1 - V_q P^1(q,\omega) \ , \tag{8.52}$$

or, using Eq. (8.23)

$$\boldsymbol{\epsilon(q,\omega) = 1 - V_q \sum_{\mathbf{k}} \frac{f_{\mathbf{k}-\mathbf{q}} - f_{\mathbf{k}}}{\hbar(\omega + i\delta + \epsilon_{k-q} - \epsilon_k)}} \ . \tag{8.53}$$

Lindhard formula for the longitudinal dielectric function

The Lindhard formula describes a complex retarded dielectric function, i.e., the poles are in the lower complex frequency plane, and it includes spatial dispersion (q dependence) and temporal dispersion (ω dependence). Eq. (8.53) is valid both in 3 and 2 dimensions. In the derivation, we sometimes used the $3D$ expressions, but that could have been avoided without changing the final result. Note, that the expectation value $f_{\mathbf{k}}$ of the particle density operator is equal to the Fermi–Dirac distribution function f_k for a thermal plasma. However, Eq. (8.53) is valid also for nonequilibrium distribution functions.

The longitudinal plasma eigenmodes are obtained from

$$\mathbf{Re}[\epsilon(q,\omega)] = 0 \quad \text{or} \quad \boldsymbol{1 = V_q \mathbf{Re}[P^1(q,\omega)]} \ . \tag{8.54}$$

longitudinal eigenmodes

This equation is identical to the plasma eigenmode equation (8.25). Hence, our discussion of plasma screening of the Coulomb potential and of the collective plasma oscillations obtained from $\epsilon(q,\omega) = 0$, shows that screening and plasmons are intimately related phenomena.