Normal distribution

Exponential distribution

Gamma distribution

probability I

1398/ 2

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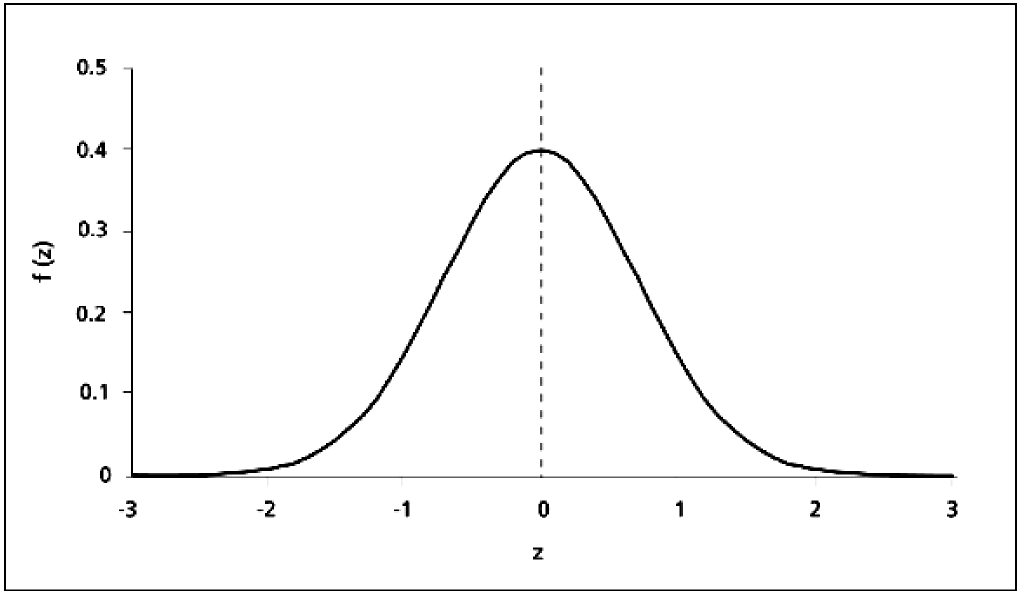
Standard Normal distribution

A continues random variable Z has ***standard normal distribution*** if

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad z \in R$$



notation

$$Z \sim N(0,1)$$

$$M(t) = E(e^{tX}) = e^{t^2/2}$$

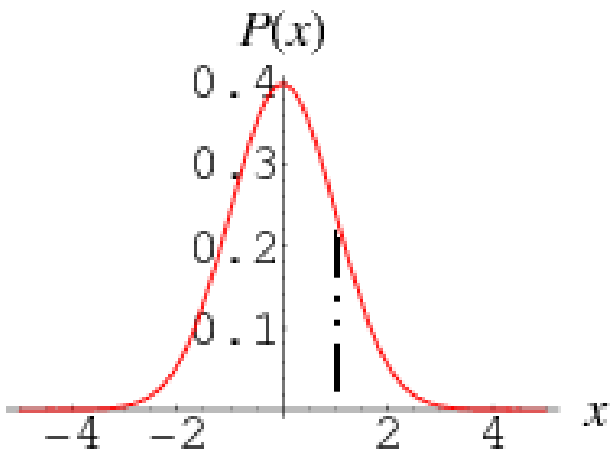
$$E(Z) = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 0 \qquad V(Z) = E(Z^2) = 1$$

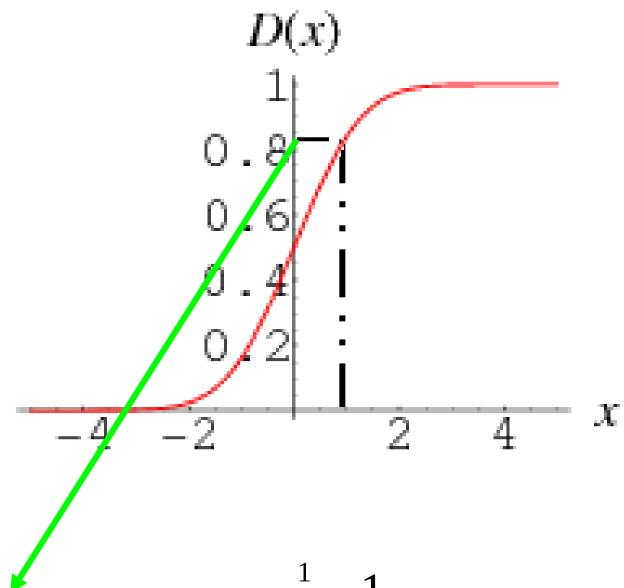
Standard Normal distribution

Cumulative Distribution function

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad z \in \quad R$$

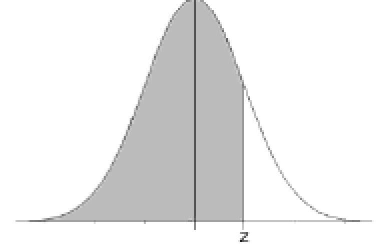
$$\varphi(\alpha) = P(Z \le \alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$





$$\varphi(1) = P(Z \le 1) = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$$\varphi(0) = P(Z \le 0) = 0.5 \qquad \varphi(1.32) = P(Z \le 1.32) = 0.9066$$

$$\boxed{\varphi(-\alpha) = 1 - \varphi(\alpha)} \qquad \varphi(-1.32) = 1 - \varphi(1.32) = 1 - 0.9066 = 0.0934$$

Z table - Normal Distribution - Windows Internet Explorer

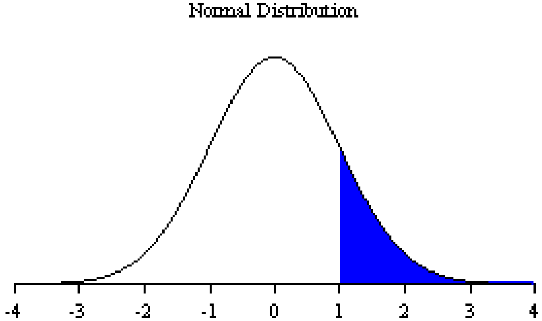
http://davidmlane.com/hyperstat/z_table.html

Z table - Normal Distribution

Normal Distribution

This page contains calculators for the normal distribution. The first calculator allows to enter the mean, standard deviation, and cutoff points; the area under the normal distribution is computed. The second calculator does just the reverse. These calculators can be used instead of a table of the normal distribution.

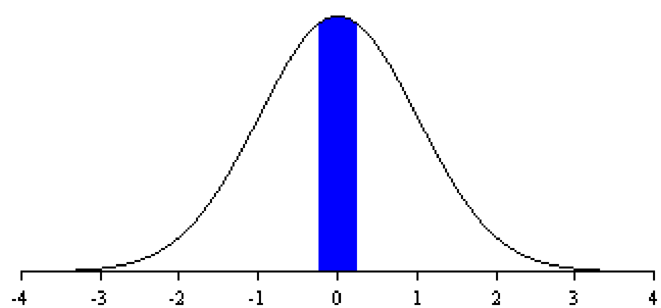
Choose parameters and the area will be computed.



Mean: 0 Sd 1

- ◉ Above 1
- ○ Below 1
- ○ Between -1 and 1
- ○ Outside -1 or 1

Shaded area: 0.158655



Mean: 0 Sd: 1

Shaded Area: 0.2

- ○ Above
- ○ Below
- ◉ Between: -0.2533 and 0.2533
- ○ Outside

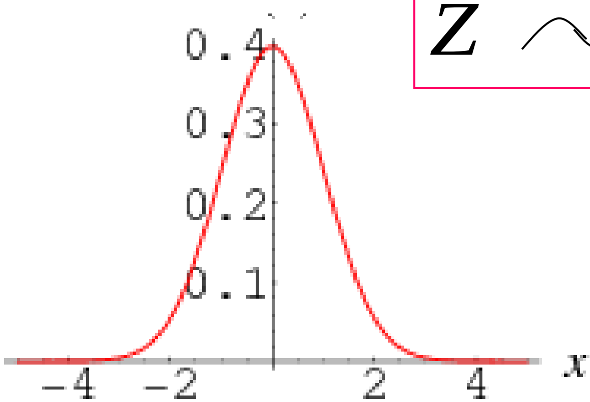
Standard Normal distribution

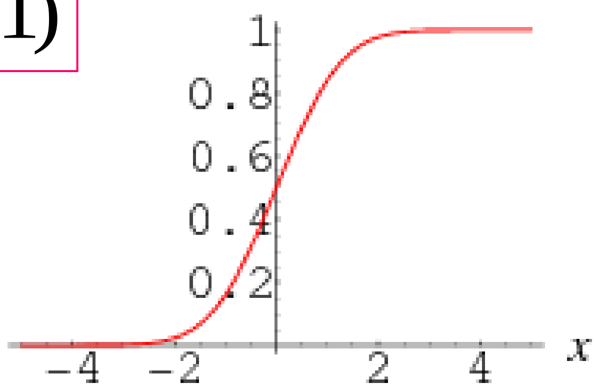
Reminder:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad z \in R$$

$$\varphi(\alpha) = P(Z \le \alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$Z \sim N(0,1)$$





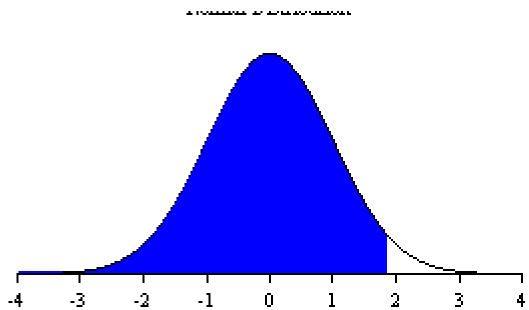
$$E(Z) = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 0$$

$$M(t) = E(e^{tX}) = e^{t^2/2}$$

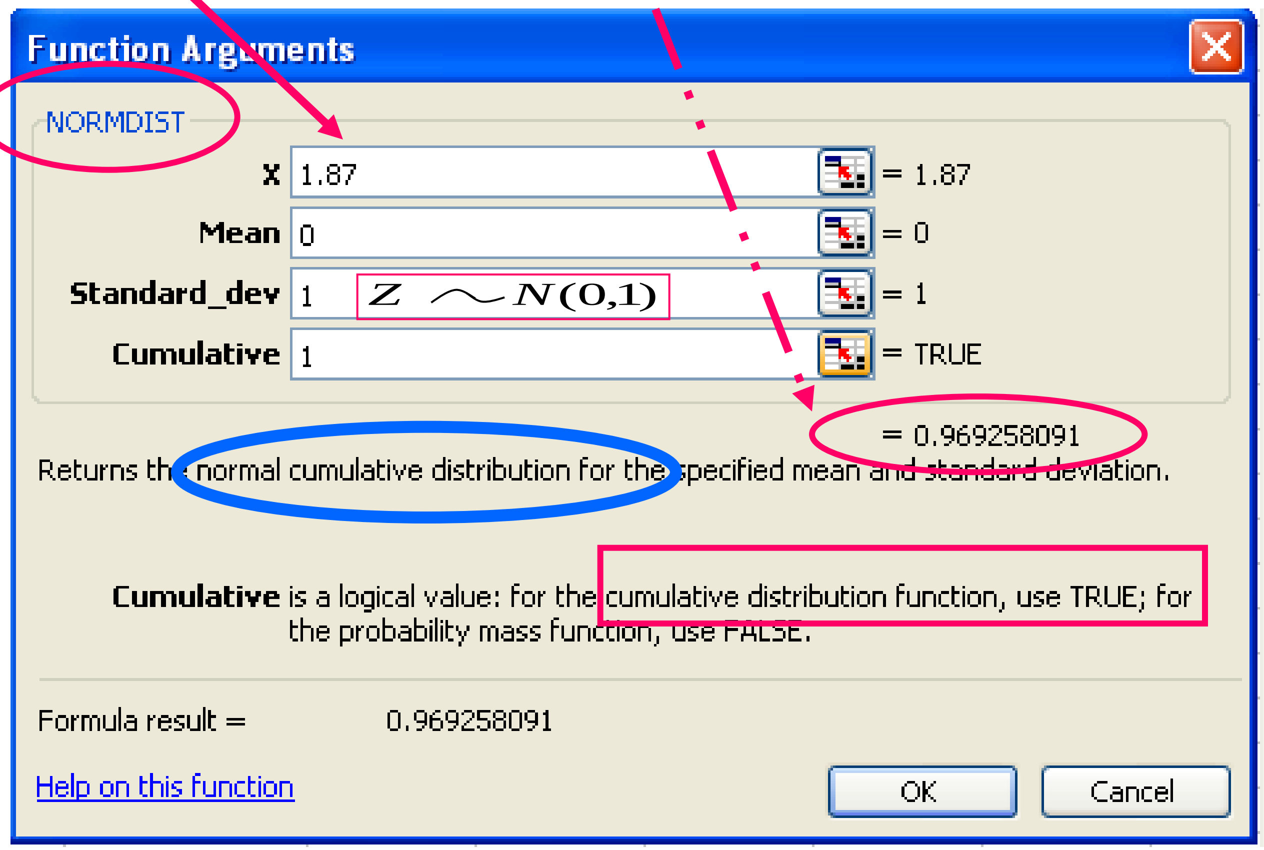
$$V(Z) = E(Z^2) = 1$$

Normal distribution and Excel

$$P(Z \le 1.87) = \varphi(1.87) = \int_{-\infty}^{1.87} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 0.96925$$

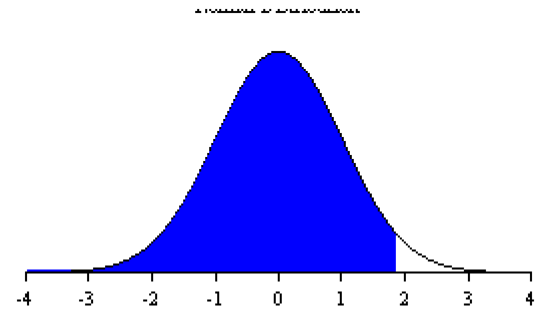


Calculation of standard normal probabilities using **Excel**



calculation of probabilities

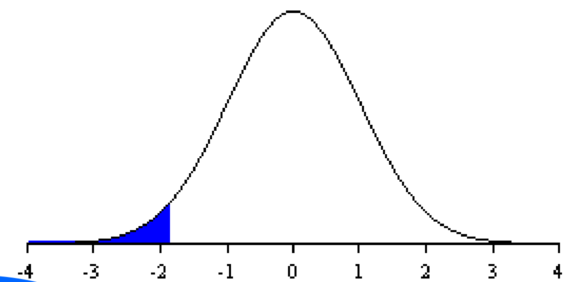
- $$P(Z \le 1.87) = \varphi(1.87) = \int_{-\infty}^{1.87} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 0.96925$$



- $$\varphi(-1.87) = P(Z \le -1.87) = \int_{-\infty}^{-1.87} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 0.0307$$

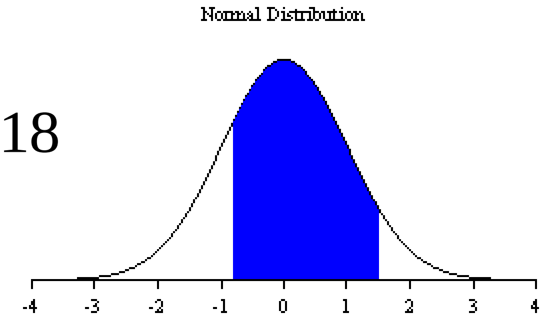
$$= \int_{1.87}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= 1 - \int_{-\infty}^{1.87} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 1 - \varphi(1.87)$$



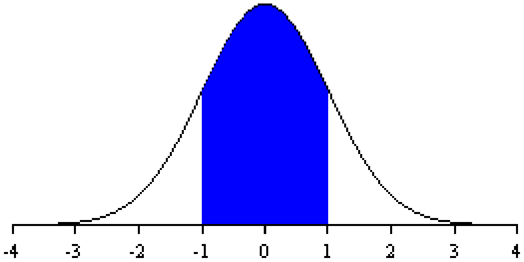
- $$P(-0.8 < Z \le 1.5) = \varphi(1.5) - \varphi(-0.8) = 0.9331 - 0.2118$$

$$= 0.7213$$

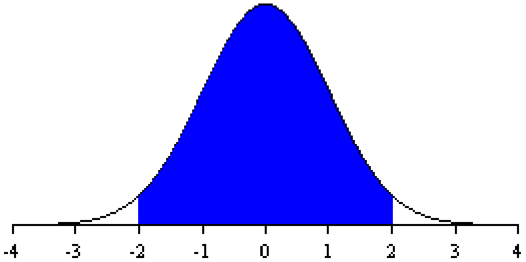


Normal distribution

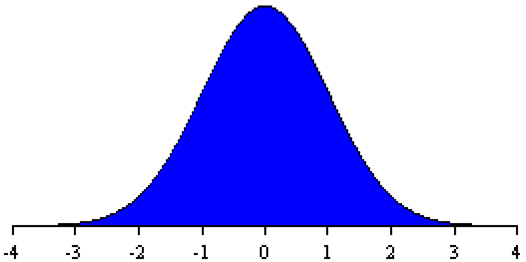
$$P(-1 < Z \le 1) = \varphi(1) - \varphi(-1) = 0.68$$



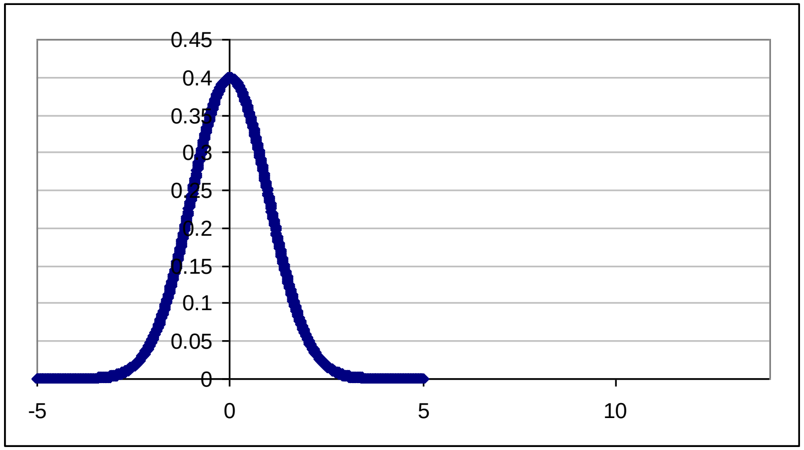
$$P(-2 < Z \le 2) = \varphi(2) - \varphi(-2) = 0.95$$

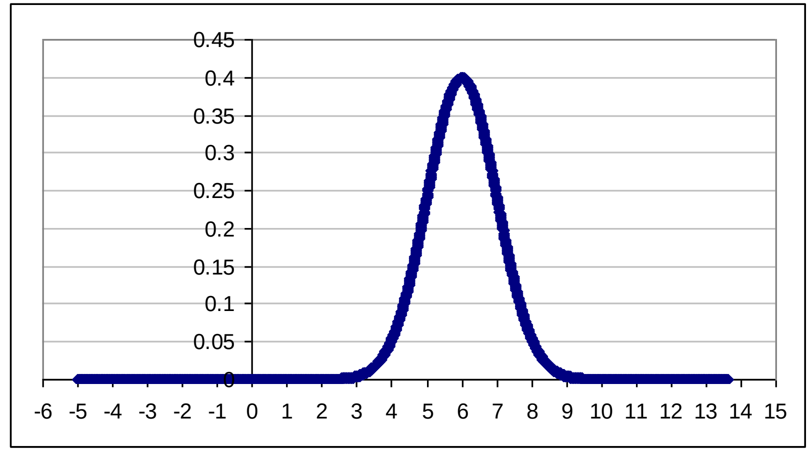


$$P(-3 < Z \le 3) = \varphi(3) - \varphi(-3) = 0.99$$



Normal dist.- different means





$$f(z) = (\sqrt{2\pi}\ )^{-1} e^{-\frac{1}{2}z^2}$$

$$f(x) = (\sqrt{2\pi}\ )^{-1}\ e^{-\frac{1}{2}(x-6)^2}$$

Standard Normal

E(Z)=0

V(X)=1

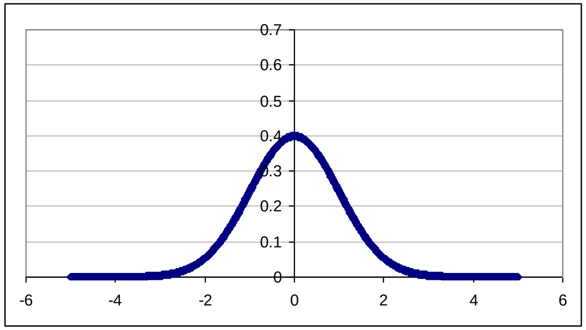
$Z \sim N(0,1)$

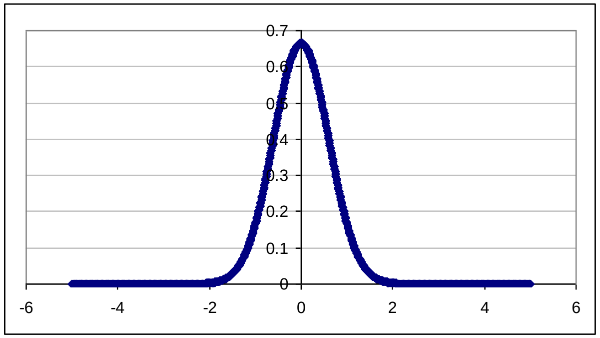
E(X)=6

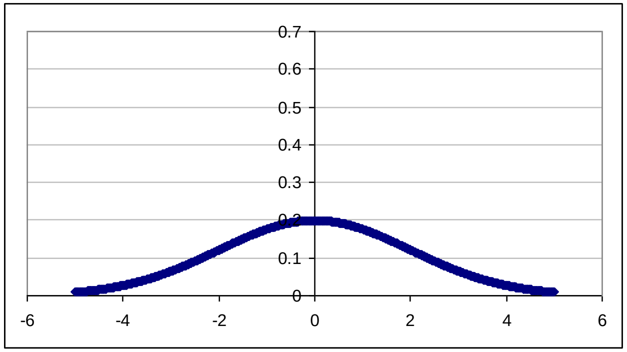
V(X)=1

$Z \sim N(6,1)$

Normal dist. – different variances







$$f(z) = (\sqrt{2\pi}\ )^{-1} e^{-\frac{1}{2}z^2}$$

Standard Normal

E(Z)=0

V(X)=1

$$Z \sim N(0,1)$$

$$(0.6\sqrt{2\pi})^{-1} e^{-\frac{1}{2}(\frac{x}{0.6})^2}$$

E(X)=0

V(X)=**(0.6)** ^{2}

$$X \sim N(0, 0.6^2)$$

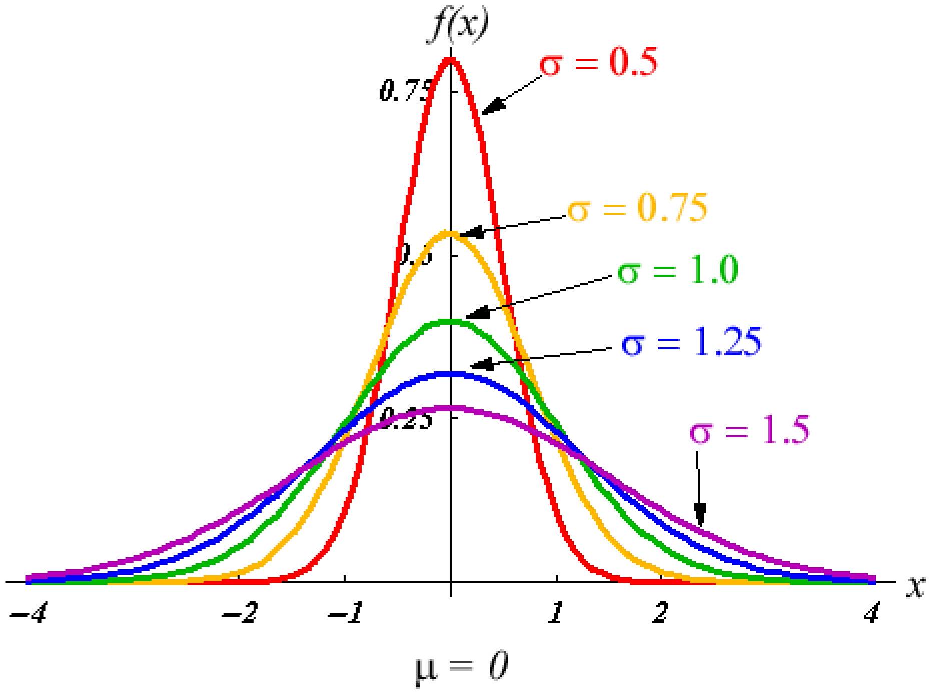
$$(2\sqrt{2\pi}\ )^{-1}\ e^{-\frac{1}{2}(\frac{y}{2})^2}$$

E(X)=0

V(X)=**(2)** ^{2}

$$Y \sim N(0, 2^2)$$

Normal distribution



Normal distribution

A continues random variable X is called ***normal random variable*** if it has a probability density function of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad x \in R \qquad \mu \in R \qquad \sigma > 0$$

where μ and σ^2 are ***mean*** and ***variance*** of X, respectively . notation $X \sim N(\mu, \sigma^2)$

Prove that:

$$M(t) = e^{t\mu + t^2\sigma^2/2}$$

$$M'(0) = \mu = E(X)$$

$$M''(0) = \mu^2 + \sigma^2$$

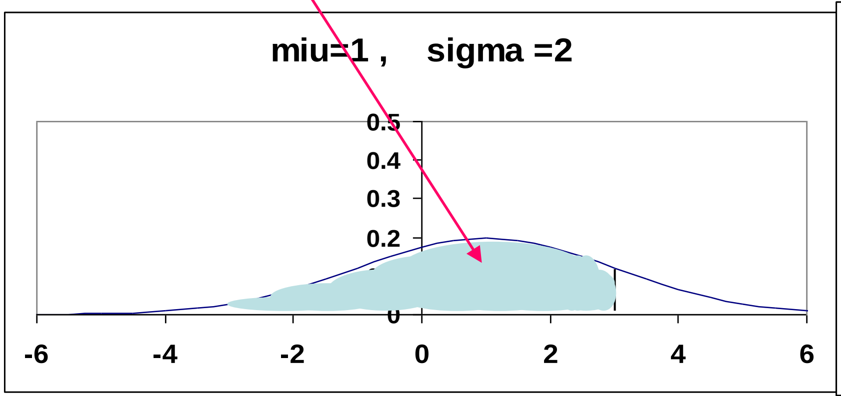
$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

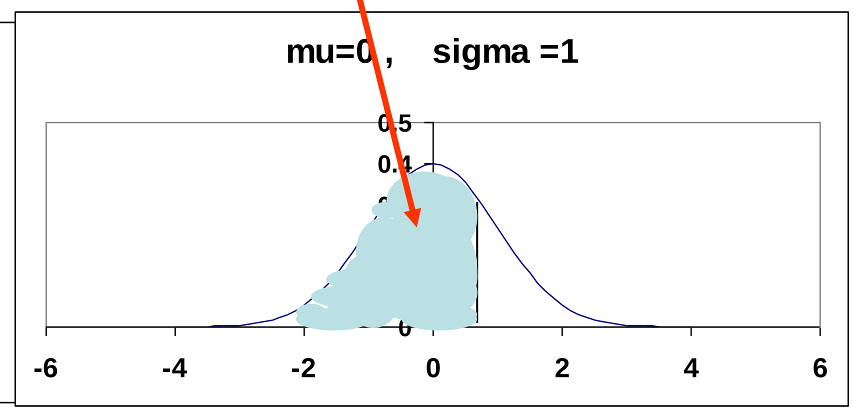
Relationship

$$\text{if} \quad X \sim N(\mu, \sigma^2) \quad \text{then} \quad [(X-\mu)/\sigma] \sim N(0,1)$$

Suppose that: $X \sim N(1,4)$

$$P(X \le 3.24) = P\left(\frac{X-1}{2} \le \frac{3.24-1}{2}\right) = P(Z \le 1.12) = \varphi(1.12) = 0.8686$$



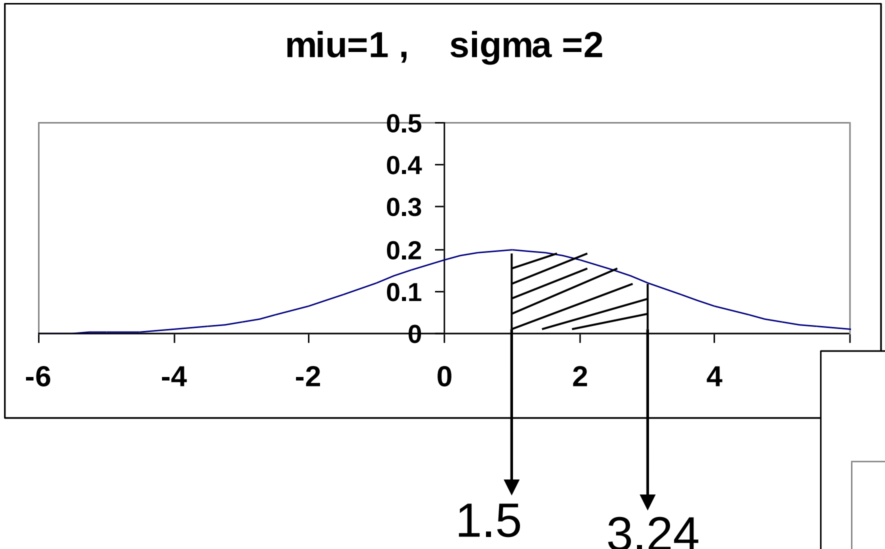


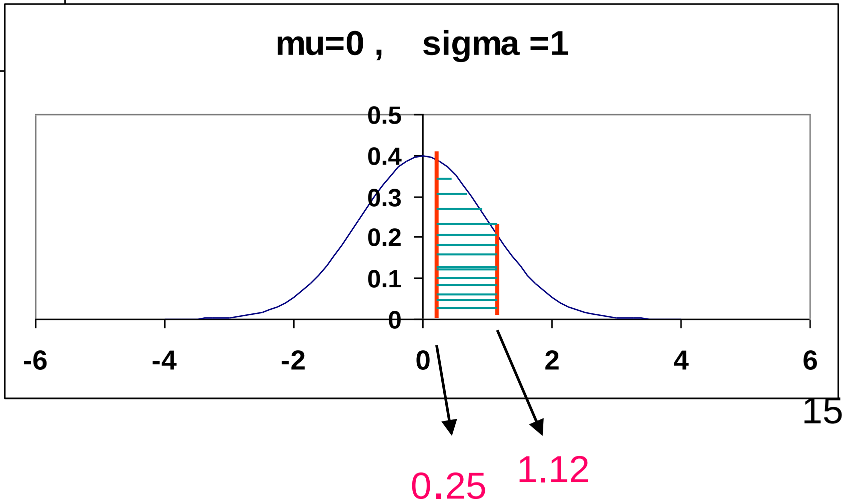
Example 2

Let X be normally distributed with parameters $\mu = 1$ and $\sigma^2 = 4$. Then

$$P(1.5 \le X \le 3.24) = P\left(\frac{1.5-1}{2} \le \frac{X-1}{2} \le \frac{3.24-1}{2}\right) = P(0.25 \le Z \le 1.12)$$

$$= \varphi(1.12) - \varphi(0.25) = 0.8686 - 0.5987$$





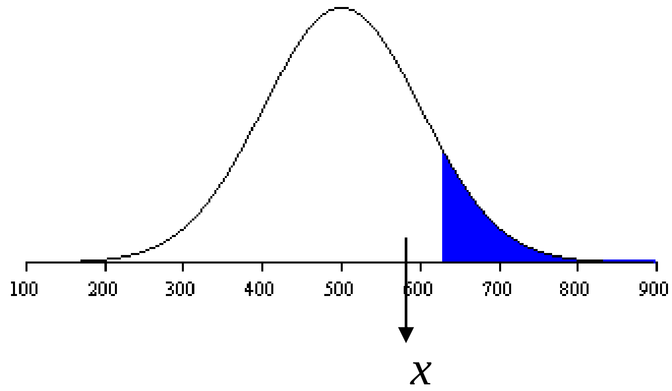
Normal probability - Example 2

The score of a test are normally distributed with mean=500 and standard deviation=100. What is the score of a student that can be regarded as 10% of excellent students. ***X***: scores of students in the exam :N(500, 100^2)

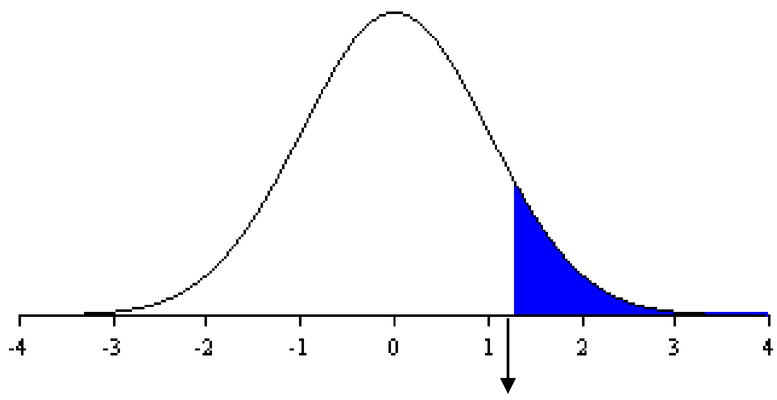
We need an x such that P(X > x)=0.1

$$P(X \le x) = 0.9 \Rightarrow P(\frac{X-500}{100} \le \frac{x-500}{100}) = 0.9$$

$$P(Z \le \frac{x-500}{100}) = 0.9$$



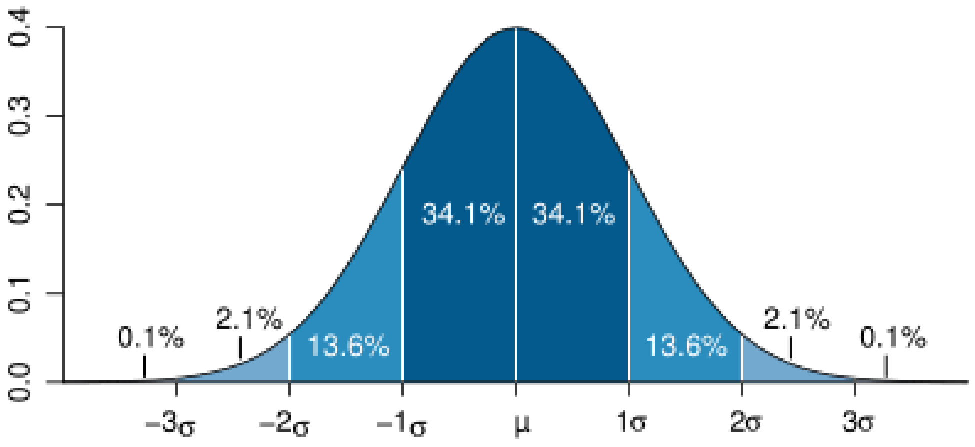
$$P(Z \le 1.2815) = \phi(1.2815) = 0.9$$



$$\Rightarrow \frac{x-500}{100} \approx 1.28 \Rightarrow x \approx 628$$

A studentshould gain 628 scores that can be regarded as upper 10 % of students

Percent of value within intervals



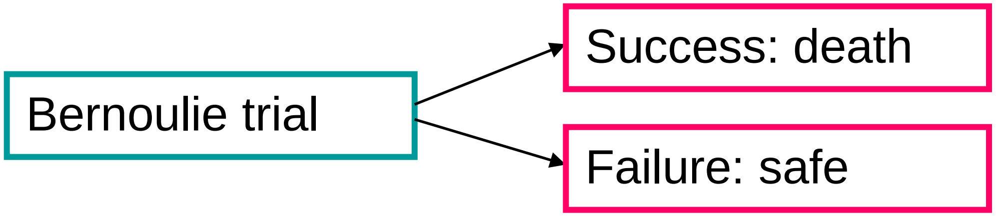
$$P(\mu \le X \le \mu+\sigma) = P(\mu-\mu \le X-\mu \le \mu+\sigma-\mu) = P(\frac{0}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\sigma}{\sigma})$$
$$= P(0 \le Z \le 1) = \varphi(1) - \varphi(0) = 0.8413 - 0.5 = 0.3413$$

$$P(\mu+\sigma \le X \le \mu+2\sigma) == P(1 \le \frac{X-\mu}{\sigma} \le 2)$$
$$= P(1 \le Z \le 2) = \varphi(2) - \varphi(1) = 0.9772 - 0.8413 = 0.1353$$

Discrete approximation – Poisson

Example : The infant mortality is 4.5 per 1000 live birth. Select 500 (randomly) live birth, what is the probability of there are

a) No infant death b) at most three infant death



$$p = 0.0045$$

X=Number of infant death in 500 live birth $X \sim B(500, 0.0045)$;

$$X \sim P(np) \; ; \qquad P(X = x) \approx \frac{e^{-np}(np)^x}{x!}$$

$$P(X = x) = \binom{500}{x}(0.0045)^x(1-0.0045)^{500-x}$$

$$a) \quad P(X = 0) = (1-0.0045)^{500} \approx e^{-2.25}(2.25)^0 / 0! = 0.105$$

$$b) \quad P(X \le 3) = \sum_{x=0}^{3}\binom{500}{x}(0.0045)^x(1-0.0045)^{500-x} \approx \sum_{x=0}^{3} e^{-2.25}(2.25)^x / x! = 0.809$$

Continuous approximation Normal distribution

Theorem: if $X \sim B(n,p)$; then

$$\lim_{n\to\infty} P\left(a < \frac{X-np}{\sqrt{npq}} < b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx$$

In the previous example

$$P(X \le 3) = P(0 \le X \le 3) = P\left(\frac{0-2.25}{\sqrt{2{,}23}} \le \frac{X-2.25}{\sqrt{2{,}23}} \le \frac{3-2.25}{\sqrt{2{,}23}}\right)$$

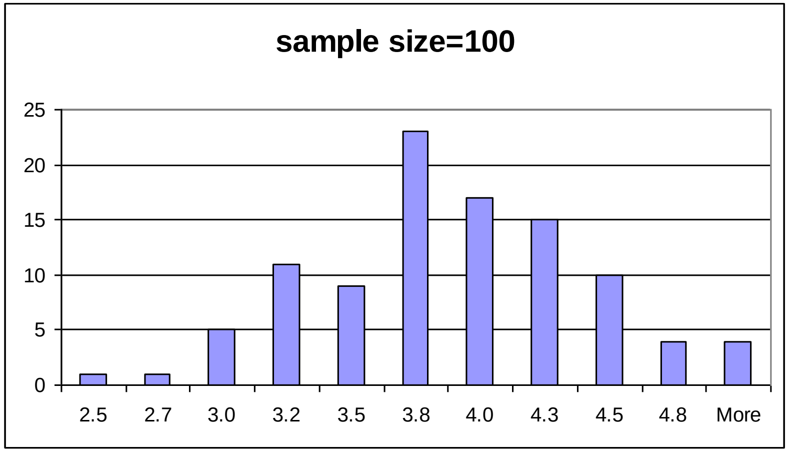
$$P\left(-1.5 \le \frac{X-2.25}{\sqrt{2{,}23}} \le 0.5\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-1.5}^{0.5} e^{-x^2/2}\,dx = ?$$

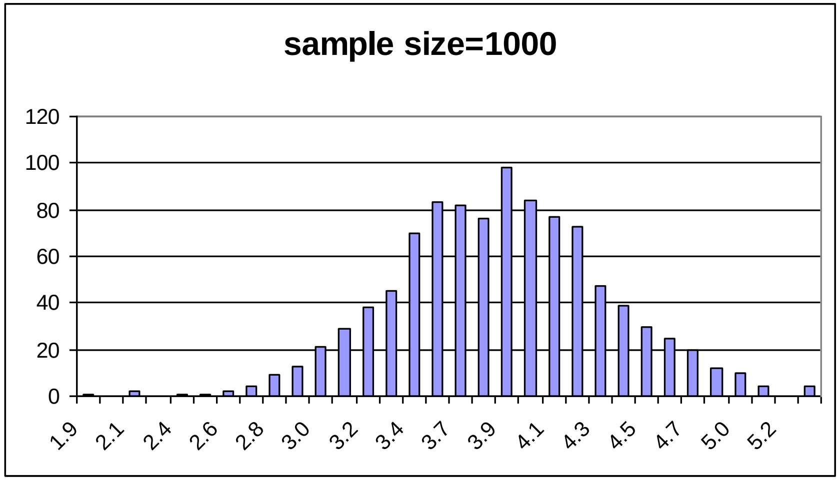
But this integral is only calculated numerically. It can be proved that (analytically)

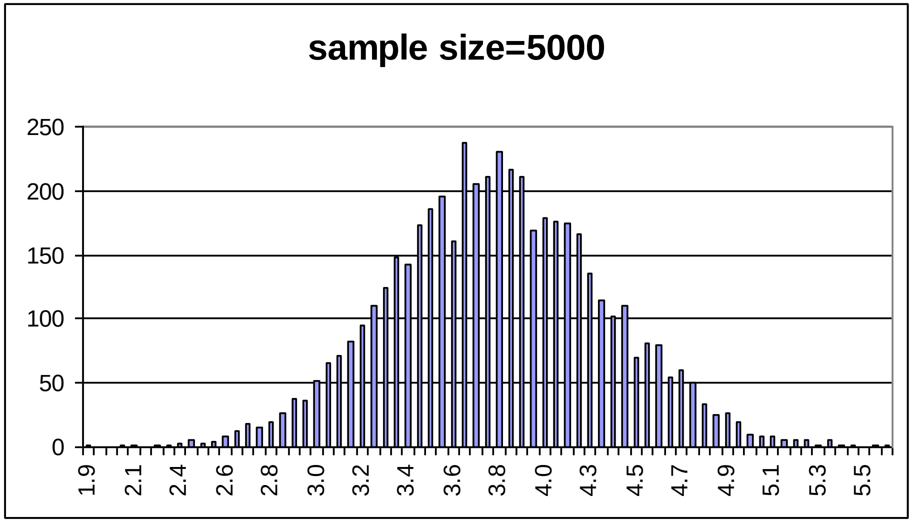
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

Note: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ can be regarded as a probability density function

Distribution of infant's heights







Exponential distribution

Consider the events that are happened in a period of time between 0 and t. We saw that the number of these events (let say N) can be modelled by Poisson distribution with parameter lambda, that is the average number of events in that period

$$P(N=n)=\frac{e^{-\lambda}\lambda^{n}}{n!} \quad n=0,1,2,....$$

If m= the average number of events in each unit of time then

$$P(N=n)=\frac{e^{-mt}(mt)^{n}}{n!} \quad n=0,1,2,....$$

Define X= happening time of the first event

$$P(X>t)=P(N=0)=e^{-mt} \quad or \quad P(X<t)=1-e^{-mt}$$

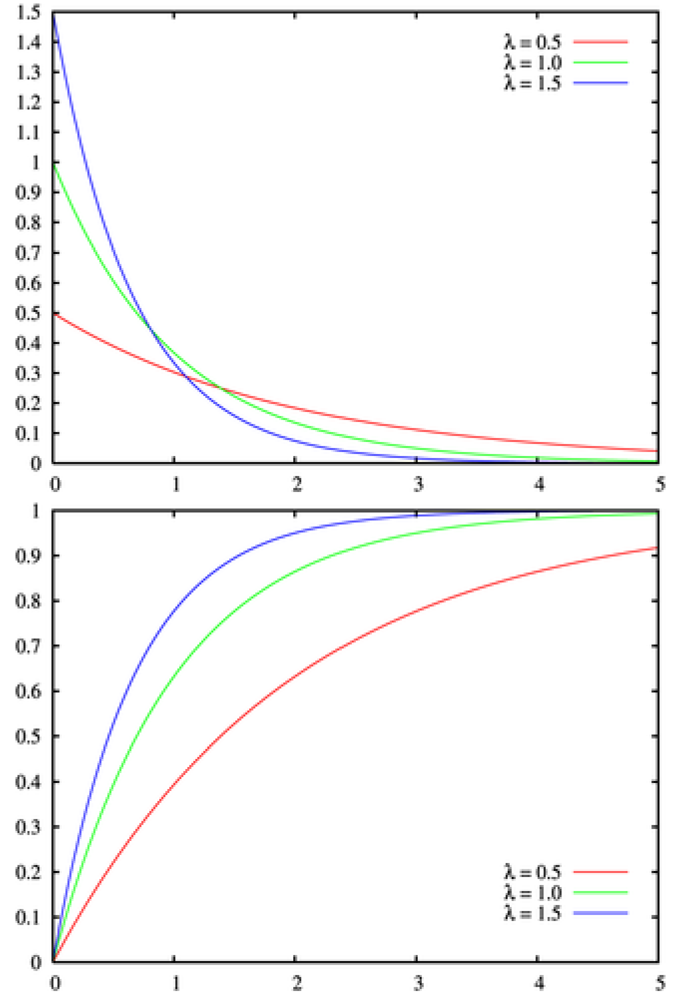
$$F_X(t)=P(X\le t)=\begin{cases}0 & t<0\\ 1-e^{-mt} & t>0\end{cases}$$

$$f(t)=\begin{cases}0 & t<0\\ me^{-mt} & t\ge 0\end{cases}$$

Exponential distribution

A continues random variable X has ***Exponential distribution*** with parameter lambda if

$$f(x)=\begin{cases}\frac{1}{\lambda}e^{-\frac{x}{\lambda}} & x>0\\ 0 & o.w.\end{cases}$$



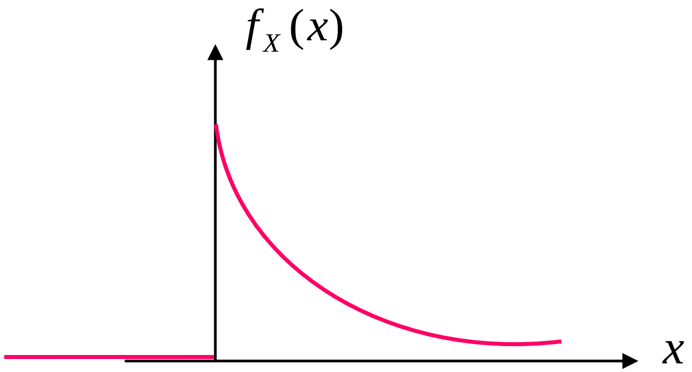
$$F(x)=\begin{cases}0 & x<0\\ 1-e^{-\frac{x}{\lambda}} & x\geq 0\end{cases}$$

Find E(x), var(X), M(t)

Example

X: the life time of an electrical component

$$f_X(x) = \begin{cases} 0.01e^{-\frac{x}{100}} & x > 0 \\ 0 & o.w \end{cases}$$



$$F(a) = P(X \le a) = \int_{-\infty}^{a} f_X(t)dt = ?$$

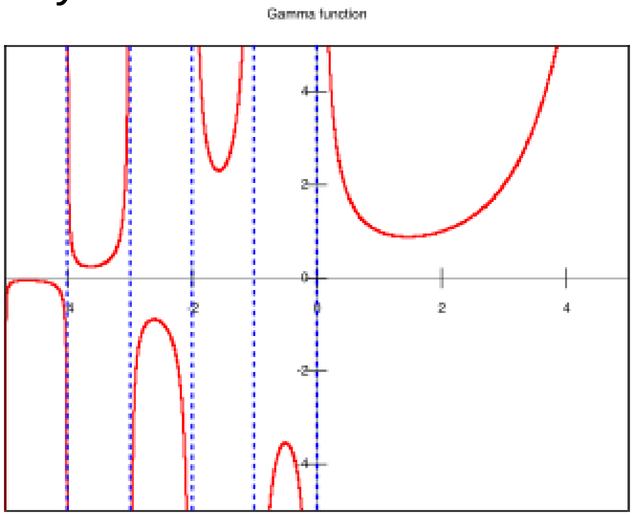
$$F(t) = \begin{cases} 0 & t < 0 \\ \int_0^t 0.01e^{-\frac{x}{100}} & t > 0 \end{cases} = \begin{cases} 0 & t < 0 \\ 1 - e^{\frac{-t}{100}} & t > 0 \end{cases}$$

P(the life time of a randomly selected component is greater than 50)=?

$$= P(X > 50) = 1 - P(X \le 50) = 1 - F(50) = e^{-1/2} = \int_{50}^{\infty} f(x)dx$$

Gamma function

The gamma function is denoted Γ and is defined by



$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \qquad t > 0$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

It can be proved that $\Gamma(t) = (t-1)\Gamma(t-1)$

If n is an integer number then

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) =$$

$$(n-1)(n-2).......1\Gamma(1) = (n-1)!$$

$$\Gamma(1) = 0! = 1$$

Gamma distribution

A continues random variable X has ***gamma distribution*** with parameters *α* and *β* if

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\beta}} & x > 0 \\ 0 & x < 0 \end{cases}$$

Find E(X), Var(X), M(t) and show that if $\alpha=1$ then the gamma distribution is an exponential distribution with parameter *β*

Approximation of Binomial distribution

Reminder: $X \sim B(n,p)$; $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$

$E(X)=np \qquad V(X)=npq$

$X \sim P(\lambda); \quad P(X=x)=e^{-\lambda}\lambda^x / x!$

When n is large then we need to approximate the binomial probability

Theorem : If ***n*** is large and ***p*** is small so that ***np*** be a constant value, then

$$X \approx P(np) \ ; \qquad P(X=x) \approx \frac{e^{-np}(np)^x}{x!}$$

Discrete approximatiom