# 11

# Controllability and Observability for Linear Systems

## 11.1 INTRODUCTION

In Chapter 10 system stability was discussed. In this chapter two additional equally important system properties, controllability and observability, are defined. These were both mentioned briefly in Sec. 6.9 while discussing solution of simultaneous equations. Several criteria are presented here for determining whether a linear system possesses these properties.

Similarity transformations can be used to decompose a linear system into various canonical forms. When the modal matrix is used in this transformation, the decoupled or nearly decoupled Jordan canonical form of the system equations result. From these, controllability and observability properties of individual system modes are rather obvious. In this context, two weaker system properties can be defined which connect controllability and observability with modal stability. These properties, stabilizability and detectability, are also defined and discussed.

An alternate similarity transformation is presented, which transforms the state equations into Kalman's controllable form and/or Kalman's observable form. These transformations use orthogonal vectors obtained from a **QR** decomposition, rather than eigenvectors. This approach also provides the basis for one way of determining minimal-order state variable models from transfer function matrices or from arbitrary nonminimal realizations. This is pursued in Chapter 12.

## 11.2 DEFINITIONS

It has been shown previously that the description of a linear system, either continuous-time or discrete-time, depends upon four matrices **A, B, C,** and **D.** Depending on the choice of state variables, or alternatively on the choice of the basis for the state space

$\Sigma$, different matrices can be used to describe the same system. A particular set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is called a *system representation* or *realization*. In some cases these matrices will be constant. In other cases they will depend on time, in either a continuous fashion $\{\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)\}$ or in a discrete fashion, $\{\mathbf{A}(k), \mathbf{B}(k), \mathbf{C}(k), \mathbf{D}(k)\}$. Both the continuous-time and discrete-time cases are considered simultaneously using the notation of Chapter 3. The times of interest will be referred to as the set of scalars $\mathcal{T}$, where $\mathcal{T}$ can be a continuous interval $[t_0, t_f]$, or a set of discrete points $[t_0, t_1, \ldots, t_N]$. At any particular time $t \in \mathcal{T}$, the four system matrices are representations of transformations on the $n$-dimensional state space $\Sigma$, the $r$-dimensional input space $\mathcal{U}^r$, and the $m$-dimensional output space $\mathcal{Y}^m$. That is,

$$\mathbf{A}: \Sigma \to \Sigma$$

$$\mathbf{B}: \mathcal{U}^r \to \Sigma$$

$$\mathbf{C}: \Sigma \to \mathcal{Y}^m$$

$$\mathbf{D}: \mathcal{U}^r \to \mathcal{Y}^m$$

Figure 11.1 symbolizes these relationships.

### *11.21. Controllability*

Controllability is a property of the coupling between the input and the state, and thus involves the matrices **A** and **B**.

**Definition 11.1.** A linear system is said to be *controllable* at $t_0$ if it is possible to find some input function (or sequence in the discrete case) $\mathbf{u}(t)$, defined over $t \in \mathcal{T}$, which will transfer the initial state $\mathbf{x}(t_0)$ to the origin at some finite time $t_1 \in \mathcal{T}, t_1 > t_0$. That is, there exists some input $\mathbf{u}_{[t_0, t_1]}$, which gives $\mathbf{x}(t_1) = 0$ at a finite $t_1 \in \mathcal{T}$. If this is true for all initial times $t_0$ and all initial states $\mathbf{x}(t_0)$, the system is *completely controllable.*

Some authors define another kind of controllability involving the output $\mathbf{y}(t)$ [1]. The definition given above is referred to as state controllability. It is the most common definition, and is the only type used in this text, so the adjective "state" is omitted. Complete controllability is obviously a very important property. If a system is not completely controllable, then for some initial states no input exists which can drive the

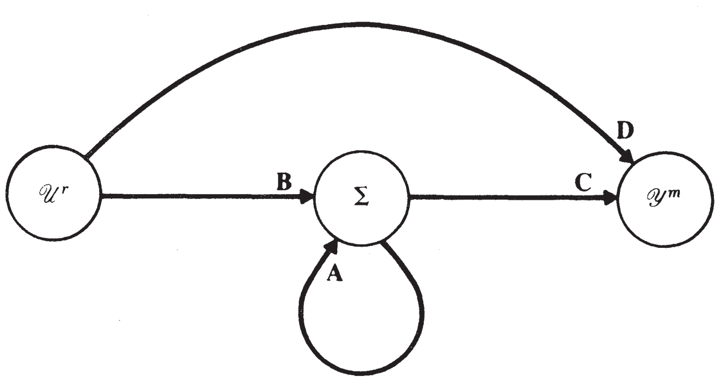


**Figure 11.1**

system to the zero state. It would be meaningless to search for an *optimal control* in this case. A trivial example of an uncontrollable system arises when the matrix **B** is zero, because then the input is disconnected from the state.

The full significance of controllability is realized in Chapters 13 and 14. It is seen there that if a linear system is controllable, it is possible to design a linear state feedback control law that will give arbitrarily specified closed-loop eigenvalues (poles). Thus an unstable system can be stabilized, a slow system can be speeded up, the natural frequencies can be changed, and so on, if the system is controllable. The existence of solutions to certain optimal control problems can be assured if the system is controllable.

### 11.2.2 Observability

Observability is a property of the coupling between the state and the output and thus involves the matrices **A** and **C.**

**Definition 11.2.** A linear system is said to be *observable* at $t_0$ if $\mathbf{x}(t_0)$ can be determined from the output function $\mathbf{y}_{[t_0,t_1]}$ (or output sequence) for $t_0 \in \mathcal{T}$ and $t_0 \leq t_1$, where $t_1$ is some *finite* time belonging to $\mathcal{T}$. If this is true for all $t_0$ and $\mathbf{x}(t_0)$, the system is said to be *completely observable.*

Clearly the observability of a system will be a major requirement in filtering and state estimation or reconstruction problems. In many feedback control problems, the controller must use output variables **y** rather than the state vector **x** in forming the feedback signals. If the system is observable, then **y** contains sufficient information about the internal states so that most of the power of state feedback can still be realized. A more complicated controller is needed to achieve these results. This is discussed fully in Chapter 13.

### 11.2.3 Dependence Upon the Model

Both controllability and observability are defined in terms of the state of the system. For a given physical system there are many ways of selecting state variables, as discussed in Chapter 3. Two such model forms for single-input–single-output systems were called the *controllable canonical form* and the *observable canonical form.* As might be expected, the state variable model will always be controllable in the first case and always observable in the second case. (See Problems 11.6 and 11.7.) It is therefore possible that a given physical system will have one state model which is controllable but not observable and another state model which is observable but not controllable. These properties are characteristics of the model {**A, B, C, D**} rather than the physical system per se. However, if one $n$th-order state variable model is both controllable and observable, then *all* possible state variable models of order $n$ will have these properties. If either property is lacking in a given $n$th-order state variable model, then *every* state variable model of that order will fail to have either one or the other property.

Ideally it would be preferred that the system model be both controllable and observable. If this is not true, then weaker conditions of stabilizability and detectability (defined in Sec. 11.8) *may* allow the control system design to proceed in an acceptable fashion. It is noted that a control system designer frequently can dictate whether the model is controllable or observable. If observability is originally lacking, this might be changed by adding additional sensors. If controllability is originally lacking, this might signal the need for additional control actuators.

If a system is not completely observable, then the initial state $\mathbf{x}(t_0)$ cannot be determined from the output, no matter how long the output is observed. The system of Example 9.1 gave an output which was identically zero for all time. That system is obviously not completely observable.

## 11.3 TIME-INVARIANT SYSTEMS WITH DISTINCT EIGENVALUES

Controllability and observability for time-invariant systems depend only on the constant matrices $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$. $\mathbf{D}$ is not included because it is a direct mapping from input space to output space, without affecting the internal states. No reference need be made to a particular interval $[t_0, t_1]$. In this section the $n$ eigenvalues of $\mathbf{A}$ are assumed to be distinct. Then the Jordan form representation for continuous-time systems is

$$\dot{\mathbf{q}} = \mathbf{\Lambda q} + \mathbf{B}_n \mathbf{u}(t) \tag{11.1}$$

$$\mathbf{y}(t) = \mathbf{C}_n \mathbf{q}(t) + \mathbf{Du}(t) \tag{11.2}$$

For discrete-time systems,

$$\mathbf{q}(k+1) = \mathbf{\Lambda q}(k) + \mathbf{B}_n \mathbf{u}(k) \tag{11.3}$$

$$\mathbf{y}(k) = \mathbf{C}_n \mathbf{q}(k) + \mathbf{Du}(k) \tag{11.4}$$

The following definitions from earlier chapters are recalled. The modal matrix is $\mathbf{M}$, and $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{AM}$, $\mathbf{B}_n = \mathbf{M}^{-1}\mathbf{B}$, $\mathbf{C}_n = \mathbf{CM}$. Clearly, if any one row of $\mathbf{B}_n$ contains only zero elements, then the corresponding mode $q_i$ is unaffected by the input. Then $\dot{q}_i = \lambda_i q_i$ or $q_i(k+1) = \lambda_i q_i(k)$. In this case the homogeneous solution for $q_i$ may eventually approach zero as $t$ (or $k$)$\rightarrow \infty$, but there is no *finite* time at which this component of $\mathbf{q}$ will be zero. Thus there is no finite time at which $\mathbf{q}$, and consequently $\mathbf{x}$, can be driven to zero.

### Controllability Criterion 1

The constant coefficient system, for which $\mathbf{A}$ has distinct eigenvalues, is completely controllable if and only if there are no zero rows of $\mathbf{B}_n = \mathbf{M}^{-1}\mathbf{B}$.

**EXAMPLE 11.1** The system of Problem 9.8, page 328, has $\mathbf{B}_n = \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix}$. Since $\mathbf{A}$ is constant and its eigenvalues are distinct, the above criterion applies. This system is completely controllable because no row of $\mathbf{B}_n$ is all zero. ∎

**EXAMPLE 11.2** The system of Problem 9.23, page 336, has $\mathbf{B}_n = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{A}$ is constant and has distinct eigenvalues, the above criterion indicates that this system is not completely controllable. Since the third row of $\mathbf{B}_n$ contains only zeros, no control can affect the third mode $q_3(k)$ of this system. Since $q_3(k) = (-\frac{1}{3})^k q_3(0)$, this component approaches zero only as $k \to \infty$. ■

Equations (*11.2*) and (*11.4*) indicate that the output $\mathbf{y}$ will not be influenced by the $i$th system mode $q_i$ if column $i$ of $\mathbf{C}_n$ contains only zero elements. If this is true, then $q_i(t_0)$ could take on any arbitrary value without influencing $\mathbf{y}$. There is no possibility of determining $\mathbf{q}(t_0)$ or $\mathbf{x}(t_0)$ in this case.

### Observability Criterion 1

The constant coefficient system, for which $\mathbf{A}$ has distinct eigenvalues, is completely observable if and only if there are no zero columns of $\mathbf{C}_n = \mathbf{CM}$.

**EXAMPLE 11.3** The system of Example 9.2 has $\mathbf{C}_n = [0 \quad 6]$. Since $\mathbf{A}$ is constant, with distinct eigenvalues, the above criterion applies. Column one of $\mathbf{C}_n$ is zero, so this system is not completely observable.

The discrete-time system of Problem 9.23, page 336, has $\mathbf{C}_n = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Since no columns are all zero, this system is completely observable. ■

The criteria for complete controllability and observability given in this section (see also Problem 11.13) are useful because of the geometrical insight they give. They also make it possible to speak of individual system modes as being controllable or uncontrollable, observable or unobservable. However, these criteria are not the most useful because they are restricted to the distinct eigenvalue case (and not merely a diagonal Jordan form). These results are generalized in Problems 11.16 and 11.17 for the case of repeated eigenvalues. There it is shown that each mode must have a direct connection to the input (nonzero row of $\mathbf{B}_n$) *or* be coupled to another mode which has such a direct control connection. This means that zero rows can be tolerated in $\mathbf{B}_n$, provided they are not the *last* row associated with a given Jordan block. A similar result (Problem 11.17) states that the system is observable provided that the *first* column associated with each Jordan block is not identically zero.

The application of the preceding criteria requires determination of the Jordan normal form, which requires finding the modal matrix $\mathbf{M}$ and its inverse. For most applications it is easier to use the results of the following section, which are stated directly in terms of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ in any arbitrary form.

## 11.4 TIME-INVARIANT SYSTEMS WITH ARBITRARY EIGENVALUES

### Controllability Criterion 2

A constant coefficient linear system with the representation $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is completely controllable if and only if the $n \times rn$ matrix of

$$\mathbf{P} \triangleq [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] \tag{11.5}$$

has rank $n$. See Problem 11.14. The form of this condition is exactly the same for both continuous-time and discrete-time systems. The sufficiency of this condition was proven in Sec. 6.9 for the discrete-time case. With the standard definition of controllability—i.e., the ability to drive any initial state to the origin in finite time—this condition is slightly stronger than necessary. If **A** has a zero eigenvalue, then certain initial conditions (those aligned with the corresponding eigenvector) can be driven to zero in one time step without **P** having full rank. When a modified definition of controllability—i.e., the ability to drive any initial state to *any* other state in finite time—is used, then the full rank of **P** is both necessary and sufficient. Problem 11.12 shows that both definitions of controllability are equivalent for continuous-time systems. Problem 11.14 proves that criterion 2 is a necessary condition for controllability of a continuous-time system. It is also a sufficient condition.

**EXAMPLE 11.4** For the continuous-time system of Problem 9.8,

$$\mathbf{A} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$\mathbf{P} = \left[\begin{array}{cc|cc|cc} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & -4 & -7 \\ 1 & 1 & -4 & -7 & 13 & 25 \end{array}\right]$$

The rank of **P** is 3, since the determinant of the first three columns is nonzero ( = −3). Therefore, this system is completely controllable. ■

**EXAMPLE 11.5** For the discrete-time system considered in Problem 9.23,

$$\mathbf{P} = \left[\begin{array}{cc|cc|cc} 3 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{9}{4} & \frac{1}{4} \\ 2 & 0 & 2 & 0 & 2 & 0 \\ -1 & 1 & -\frac{3}{2} & \frac{1}{2} & -\frac{7}{4} & \frac{1}{4} \end{array}\right]$$

The rank of **P** is not 3, since subtracting row 3 from row 1 gives twice row 2. Since rank $\mathbf{P} \neq n$, this system is not completely controllable. ■

### *Observability Criterion 2*

A constant coefficient linear system is completely observable if and only if the $n \times mn$ matrix of Eq. (*11.6*) has rank $n$:

$$\mathbf{Q} \triangleq [\overline{\mathbf{C}}^T \mid \overline{\mathbf{A}}^T\overline{\mathbf{C}}^T \mid \overline{\mathbf{A}}^{2T}\overline{\mathbf{C}}^T \mid \cdots \mid \overline{\mathbf{A}}^{n-1T}\overline{\mathbf{C}}^T] \qquad (11.6)$$

The form of observability criterion 2 is exactly the same for both continuous-time and discrete-time systems. The proof of sufficiency was given in Sec. 6.9 for the discrete-time case with real **A** and **C**.

**EXAMPLE 11.6** The continuous-time system of Example 9.1 has $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$, $\mathbf{C} = [4 \quad 1]$. These are both real, so the complex conjugate in **Q** is unnecessary for this example, and $\mathbf{Q} = \left[\begin{array}{c|c} 4 & 8 \\ 1 & 2 \end{array}\right]$. The second column is twice the first, so $r_Q = 1$. Since $r_Q \neq 2$, this system is not completely observable. Similar manipulations show that the system of Problem 9.23 is completely observable. ■

The forms of Eqs. (*11.5*) and (*11.6*) are the same. Therefore, a computer algorithm for forming $P$ and checking its rank will also form $\mathbf{Q}$ and check its rank. In forming $\mathbf{Q}$, $\overline{\mathbf{A}}^T$ replaces $\mathbf{A}$ and $\overline{\mathbf{C}}^T$ replaces $\mathbf{B}$. In the most common case where $\mathbf{A}$ and $\mathbf{B}$ are real, the complex conjugates are superfluous. In developing such a computer algorithm, the crucial consideration is accurate determination of rank. If a row-reduced echelon technique is used, the notion of "machine zero" must be used to distinguish between legitimate nonzero but small divisors and divisors which would have been zero were it not for round-off error. The discussion of the GSE method in Sec. 6.5, the **QR** decomposition based on the modified Gram-Schmidt process, and the SVD method in Problems 7.29 through 7.34 are of significance in this regard. These same considerations arise many other places as well; e.g., do nontrivial solutions exist for a set of homogeneous equations? See Reference 2 for a fuller discussion of the implications of the fact that rank of a matrix is a discontinuous function. The wrong rank can easily be computed due to very small computer errors, unless due precautions are taken.

## 11.5 YET ANOTHER CONTROLLABILITY-OBSERVABILITY CONDITION

The controllability and observability of linear, constant systems can be characterized in still another way. This characterization is not especially convenient in checking whether a given system has these properties or not. However, it will be useful in solving the pole-placement problem of Chapter 13.

The $n$th-order system realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is controllable if and only if $[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}]$ has rank $n$ *for all* values of $s$. The same system is observable if and only if $[s\mathbf{I} - \mathbf{A}^T \mid \mathbf{C}^T]$ has rank $n$ for all $s$. Since it has already been established that $\text{rank}(\mathbf{P}) = n$ is necessary and sufficient for controllability, it is next established that $\{\text{rank}(\mathbf{P}) = n\} \Rightarrow \{\text{rank}[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}] = n\}$ by proving that $\{\text{rank}[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}] \neq n \text{ for some } s\} \Rightarrow \{\text{rank}(\mathbf{P}) \neq n\}$. If $\text{rank}[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}] \neq n$ for some $s$, then there exists a nonzero vector $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}^T[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}] = 0$, so that $\boldsymbol{\eta}^T\mathbf{B} = 0$ and $\boldsymbol{\eta}^T s = \boldsymbol{\eta}^T\mathbf{A}$. Postmultiplying the last equation by $\mathbf{A}$ gives $\boldsymbol{\eta}^T\mathbf{A}s = \boldsymbol{\eta}^T\mathbf{A}^2 = \boldsymbol{\eta}^T s^2$. Likewise, $\boldsymbol{\eta}^T\mathbf{A}^3 = \boldsymbol{\eta}^T s^3, \ldots$ . Therefore,

$$\begin{aligned}\boldsymbol{\eta}^T\mathbf{P} &= [\boldsymbol{\eta}^T\mathbf{B} \quad \boldsymbol{\eta}^T\mathbf{AB} \quad \cdots \quad \boldsymbol{\eta}^T\mathbf{A}^{n-1}\mathbf{B}] \\ &= [\boldsymbol{\eta}^T\mathbf{B} \quad s\boldsymbol{\eta}^T\mathbf{B} \quad s^2\boldsymbol{\eta}^T\mathbf{B} \quad \cdots \quad s^{n-1}\boldsymbol{\eta}^T\mathbf{B}] \\ &= [\mathbf{0}]\end{aligned}$$

Thus $\mathbf{P}$ cannot have rank $n$ if $[s\mathbf{I} - \mathbf{A} \mid \mathbf{B}]$ does not have rank $n$. Minor notational changes are all that are needed to prove the corresponding observability result.

If $s$ is not an eigenvalue of $\mathbf{A}$, then it is clear that $s\mathbf{I} - \mathbf{A}$ has rank $n$ all by itself without any assistance from $\mathbf{B}$, regardless of whether the system is controllable or not. In a controllable system, $\mathbf{A}$ and $\mathbf{B}$ must work together so that when $s\mathbf{I} - \mathbf{A}$ becomes rank-deficient, $\mathbf{B}$ fills in the deficiency. The condition must be true for any scalar $s$, real or complex, including the special case $s = 0$. Thus, a simple rank test on just $[\mathbf{B} \mid \mathbf{A}]$ is sometimes useful. Note that {rank[$\mathbf{B} \quad \mathbf{A}$] $\neq n$}$\Rightarrow${system is not controllable}. The reverse implication is *not* true.

## 11.6 TIME-VARYING LINEAR SYSTEMS

### 11.6.1 Controllability of Continuous-Time Systems

A continuous-time system with the representation $\{\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)\}$ is considered. For a given input function $\mathbf{u}(t)$, the solution for the state at a fixed time $t_1$ is

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

The vector defined by $\mathbf{x}_1 = \mathbf{x}(t_1) - \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0)$ is a constant vector in $\Sigma$ for any fixed time $t_1$. The notation of Chapters 5 and 6 is used to define the linear transformation

$$\mathcal{A}_c(\mathbf{u}) \triangleq \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

The transformation $\mathcal{A}_c$ maps functions in $\mathcal{U}$ into vectors in $\Sigma$. The question of complete controllability on $[t_0, t_1]$ reduces to asking whether $\mathcal{A}_c(\mathbf{u}) = \mathbf{x}_1$ has a solution $\mathbf{u}(t)$ for arbitrary $\mathbf{x}_1 \in \Sigma$. It was shown in Problem 6.23, page 240, that a necessary and sufficient condition for the existence of such a solution is that the null space of $\mathcal{A}_c^*$ contain only the zero element $\mathcal{N}(\mathcal{A}_c^*) = \{0\}$. This is the requirement for complete controllability on $[t_0, t_1]$, but it can be put into a more useful form. Since $\mathcal{A}_c^*: \Sigma \rightarrow \mathcal{U}$, the range of $\mathcal{A}_c^*$ is an infinite dimensional function space. The following lemma allows the use of a finite dimensional transformation.

**Lemma 11.1.** The null space of $\mathcal{A}_c^*$ is the same as the null space of $\mathcal{A}_c\mathcal{A}_c^*$. That is, $\mathcal{N}(\mathcal{A}_c^*) = \mathcal{N}(\mathcal{A}_c\mathcal{A}_c^*)$.

*Proof.* Let $\mathbf{v} \in \mathcal{N}(\mathcal{A}_c^*)$. Then $\mathcal{A}_c^*(\mathbf{v}) = \mathbf{0}$. Therefore, $\mathcal{A}_c\mathcal{A}_c^*(\mathbf{v}) = \mathcal{A}_c(\mathbf{0}) = \mathbf{0}$, so $\mathbf{v} \in \mathcal{N}(\mathcal{A}_c\mathcal{A}_c^*)$ also. Now assume that $\mathbf{v} \in \mathcal{N}(\mathcal{A}_c\mathcal{A}_c^*)$. Then $\mathcal{A}_c\mathcal{A}_c^*(\mathbf{v}) = \mathbf{0}$. Therefore, $\langle \mathcal{A}_c\mathcal{A}_c^*(\mathbf{v}), \mathbf{v}\rangle = 0$ or $\langle \mathcal{A}_c^*(\mathbf{v}), \mathcal{A}_c^*(\mathbf{v})\rangle = 0$. But this indicates that $\|\mathcal{A}_c^*(\mathbf{v})\|^2 = 0$, so that $\mathcal{A}_c^*(\mathbf{v}) = \mathbf{0}$. Thus for every $\mathbf{v} \in \mathcal{N}(\mathcal{A}_c^*)$, $\mathbf{v}$ also belongs to $\mathcal{N}(\mathcal{A}_c\mathcal{A}_c^*)$ and conversely. The two null spaces are therefore equal.

To use the lemma in developing the criterion for complete controllability, an expression for the transformation $\mathcal{A}_c\mathcal{A}_c^*$ must be found:

$$\mathcal{A}_c(\mathbf{u}) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

Therefore,

$$\langle \mathbf{v}, \mathcal{A}_c(\mathbf{u})\rangle = \langle \mathcal{A}_c^*(\mathbf{v}), \mathbf{u}\rangle = \int_{t_0}^{t_1} \overline{\mathbf{v}}^T \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

so that $\mathcal{A}_c^*(\mathbf{v}) = \overline{\mathbf{B}}^T(t)\overline{\mathbf{\Phi}}^T(t_1, t)\mathbf{v}$. Then

$$\mathcal{A}_c\mathcal{A}_c^*(\mathbf{v}) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\overline{\mathbf{B}}^T(\tau)\overline{\mathbf{\Phi}}^T(t_1, \tau)\, d\tau\mathbf{v}$$

The transformation $\mathcal{A}_c \mathcal{A}_c^*$ is just an $n \times n$ matrix, redefined as $\mathbf{G}(t_1, t_0)$,

$$\mathbf{G}(t_1, t_0) \triangleq \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\overline{\mathbf{B}}^T(\tau)\overline{\mathbf{\Phi}}^T(t_1, \tau)\, d\tau \tag{11.7}$$

The null space of $\mathcal{A}_c \mathcal{A}_c^*$ will contain only the zero element if and only if $\mathbf{G}(t_1, t_0)$ does not have zero as an eigenvalue.

### *Controllability Criterion 3*

The system described by $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}(t)$ is completely controllable on the interval $[t_0, t_1]$ if any of the following equivalent conditions is satisfied:

- **(a)** The matrix $\mathbf{G}(t_1, t_0)$ is positive definite.
- **(b)** Zero is not an eigenvalue of $\mathbf{G}(t_1, t_0)$.
- **(c)** $|\mathbf{G}(t_1, t_0)| \neq 0$.

The time-varying controllability criterion 3 can be shown to reduce to criterion 2 in the special case where **A** and **B** are constant. Because of the quadratic form of the integrand of **G**, it is clear that **G** is always at least positive semidefinite. Therefore, **G** having full rank and being positive definite are equivalent conditions. First it will be proven that if **G** has full rank, then **P** must also have full rank. The Cayley-Hamilton theorem gives

$$\mathbf{\Phi}(t_1, \tau)\mathbf{B} = \alpha_0(\tau)\mathbf{B} + \alpha_1(\tau)\mathbf{AB} + \cdots + \alpha_{n-1}(\tau)\mathbf{A}^{n-1}\mathbf{B}$$

$$= [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]\begin{bmatrix} \alpha_0 \mathbf{I}_r \\ \alpha_1 \mathbf{I}_r \\ \vdots \\ \alpha_{n-1}\mathbf{I}_r \end{bmatrix} = \mathbf{PS}(\tau)$$

Because **A**, **B**, and (hence) **P** are constant,

$$\mathbf{G}(t_1, t_0) = \mathbf{P}\int_{t_0}^{t_1} \mathbf{S}(\tau)\overline{\mathbf{S}}^T(\tau)\, d\tau \overline{\mathbf{P}}^T = \mathbf{PR}\overline{\mathbf{P}}^T$$

Sylvester's law says that $\text{rank}(\mathbf{G}) \leq \min\{\text{rank}(\mathbf{P}), \text{rank}(\mathbf{R})\}$. Therefore, $\{\mathbf{G}$ is positive definite$\} \Rightarrow \{\text{rank}(\mathbf{G}) = n\} \Rightarrow \{\text{rank}(\mathbf{P}) = n\}$. The reverse implication is also true. Assume $\text{rank}(\mathbf{P}) = n$. Then $\mathbf{P}^T$ has $n$ independent columns. Since, by the Cayley-Hamilton theorem, for any $m \geq n$, $\mathbf{A}^m$ can be expressed as a linear combination of lower powers $\mathbf{A}^j$, $j < n$, the matrix

$$\begin{bmatrix} \overline{\mathbf{B}}^T \\ \overline{\mathbf{B}}^T\overline{\mathbf{A}}^T \\ \overline{\mathbf{B}}^T(\overline{\mathbf{A}}^T)^2 \\ \vdots \\ \overline{\mathbf{B}}^T(\overline{\mathbf{A}}^T)^m \\ \vdots \end{bmatrix}$$

still has $n$ linearly independent columns, even as $m \to \infty$. This remains true when groups of rows are weighted by various powers of $(t_1 - t_0)$ and summed, so $\overline{\mathbf{B}}^T\overline{\mathbf{\Phi}}^T(t_1, t_0)$

also has $n$ independent columns and has full rank $n$. This is the operator $\mathcal{A}_c^*$. Therefore $\mathcal{A}_c^* \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} \equiv \mathbf{0}$, i.e., $\mathcal{N}(\mathcal{A}_c^*) = \{0\}$. By Lemma 11.1,

$$\mathcal{N}(\mathcal{A}_c \mathcal{A}_c^*) \equiv \mathcal{N}(\mathcal{A}_c^*)$$

The conclusion is that $\mathbf{G}(t_1, t_0) = \mathcal{A}_c \mathcal{A}_c^*$ has only the zero vector in its null space, which means $\text{rank}(\mathbf{G}) = n$ if $\text{rank}(\mathbf{P}) = n$.

### 11.6.2 Observability of Continuous-Time Systems

The general form for the output $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t)$$

Since the input $\mathbf{u}(t)$ is assumed known, the two terms containing the input could be combined with the output function $\mathbf{y}(t)$ to give a modified function $\mathbf{y}_1(t)$. Alternatively, only the unforced solution could be considered. In either case complete observability requires that a knowledge of $\mathbf{y}(t)$ (or $\mathbf{y}_1(t)$) be sufficient for the determination of $\mathbf{x}(t_0)$. Defining the linear transformation $\mathcal{A}_0(\mathbf{x}(t_0)) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}(t_0)$, the requirement for complete observability is that a unique $\mathbf{x}(t_0)$ can be associated with each output function $\mathbf{y}(t)$. This requires that $\mathcal{N}(\mathcal{A}_0) = \{0\}$ (see Problem 6.22, page 240). Using only minor changes in the previous lemma, it can be shown that $\mathcal{N}(\mathcal{A}_0^* \mathcal{A}_0) = \mathcal{N}(\mathcal{A}_0)$. To find the adjoint transformation, consider

$$\langle \mathbf{w}(t), \mathcal{A}_0(\mathbf{x}(t_0)) \rangle = \langle \mathcal{A}_0^* \mathbf{w}(t), \mathbf{x}(t_0) \rangle = \int_{t_0}^{t_1} \overline{\mathbf{w}}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau, t_0)\, d\tau\, \mathbf{x}(t_0)$$

Thus

$$\mathcal{A}_0^*(\mathbf{w}) = \int_{t_0}^{t_1} \overline{\mathbf{\Phi}}^T(\tau, t_0)\overline{\mathbf{C}}^T(\tau)\mathbf{w}(\tau)\, d\tau$$

and

$$\mathcal{A}_0^* \mathcal{A}_0(\mathbf{x}(t_0)) = \int_{t_0}^{t_1} \overline{\mathbf{\Phi}}^T(\tau, t_0)\overline{\mathbf{C}}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau, t_0)\, d\tau\, \mathbf{x}(t_0)$$

The transformation $\mathcal{A}_0^* \mathcal{A}_0 \colon \Sigma \to \Sigma$ is just an $n \times n$ matrix, redefined as $\mathbf{H}(t_1, t_0)$,

$$\mathbf{H}(t_1, t_0) \triangleq \int_{t_0}^{t_1} \overline{\mathbf{\Phi}}^T(\tau, t_0)\overline{\mathbf{C}}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau, t_0)\, d\tau \qquad (11.8)$$

#### Observability Criterion 3

The system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

is completely observable at $t_0$ if there exists some finite time $t_1$ for which any one of the following equivalent conditions holds:

**(a)** The matrix $\mathbf{H}(t_1, t_0)$ is positive definite.
**(b)** Zero is not a an eigenvalue of $\mathbf{H}(t_1, t_0)$.
**(c)** $|\mathbf{H}(t_1, t_0)| \neq 0$.

The proof that $\{\text{rank}[\mathbf{H}(t_1, t_0)] = n\} \rightleftarrows \{\text{rank}(\mathbf{Q}) = n\}$ is the same as the proof that $\{\text{rank}[\mathbf{G}(t_1, t_0)] = n\} \rightleftarrows \{\text{rank}(\mathbf{P}) = n\}$ given in the last section, with notational changes.

### 11.6.3 Discrete-Time Systems

The corresponding forms of the controllability and observability criteria 3 for discrete systems are derived in the same way. However, since the input and output spaces have sequences, rather than functions, as their elements, the appropriate inner product is a summation rather than an integral:

$$\langle \mathbf{w}(k), \mathbf{y}(k) \rangle = \sum_{k=0}^{N} \overline{\mathbf{w}}^T(k)\mathbf{y}(k)$$

The criteria may be stated as follows.

#### Controllability and Observability Criteria 3, Discrete Systems

The system

$$\mathbf{x}(k + 1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k)$$

is completely controllable at $k = 0$ if and only if for some finite time index $N$, the $n \times n$ matrix

$$\sum_{k=0}^{N} \mathbf{\Phi}(N, k)\mathbf{B}(k)\overline{\mathbf{B}}^T(k)\overline{\mathbf{\Phi}}^T(N, k)$$

is positive definite (or does not have zero as an eigenvalue, or has a nonzero determinant). This system is completely observable at $k = 0$ if and only if there exists some finite index $N$ such that the $n \times n$ matrix

$$\sum_{k=0}^{N} \overline{\mathbf{\Phi}}^T(k, 0)\overline{\mathbf{C}}^T(k)\mathbf{C}(k)\mathbf{\Phi}(k, 0)$$

is positive definite (or does not have zero as an eigenvalue, or has a nonzero determinant).

## 11.7 KALMAN CANONICAL FORMS

It can be seen from Problem 11.12 that any vector $\mathbf{x}(t_0)$ that belongs to the subspace spanned by the columns of $\mathbf{P}$ can be driven to zero, that is, these states are controllable. If the columns of $\mathbf{P}$ span the entire $n$-dimensional state space, then the system is controllable. When rank($\mathbf{P}$) $= r_P < n$, state space can be decomposed into two orthogonal subspaces, $\Sigma = \Sigma_1 \oplus \Sigma_2$, with $\Sigma_1$ being the subspace spanned by the columns of $\mathbf{P}$. $\Sigma_1$ is called the controllable subspace for obvious reasons. Select a set of basis vector for $\Sigma$ consisting of $r_P$ orthogonal vectors belonging to $\Sigma_1$ and the remaining $n - r_P$ vectors orthogonal to these. Let these vectors form the columns of a transformation matrix $\mathbf{T} = [\mathbf{T}_1 \mid \mathbf{T}_2]$. The original state equations are transformed to the Kalman controllable canonical form by letting $\mathbf{x} = \mathbf{Tw}$. Then, the orthogonal basis set gives $\mathbf{T}^{-1} = \mathbf{T}^T$, so that

$$\dot{\mathbf{w}} = \mathbf{T}^T\mathbf{ATw} + \mathbf{T}^T\mathbf{Bu} \quad \text{and} \quad \mathbf{y} = \mathbf{CTw} + \mathbf{Du}$$

Partitioning these equations according to the dimensions of $\Sigma_1$ and $\Sigma_2$ gives

$$\begin{bmatrix} \dot{\mathbf{w}}_1 \\ \dot{\mathbf{w}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{T}_1^T\mathbf{AT}_1 & \mathbf{T}_1^T\mathbf{AT}_2 \\ \hline \mathbf{T}_2^T\mathbf{AT}_1 & \mathbf{T}_2^T\mathbf{AT}_2 \end{array}\right] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \left[\begin{array}{c} \mathbf{T}_1^T\mathbf{B} \\ \hline \mathbf{T}_2^T\mathbf{B} \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y} = [\mathbf{CT}_1 \mid \mathbf{CT}_2]\mathbf{w} + \mathbf{Du}$$

Since columns of $\mathbf{T}_2$ were selected orthogonal to all columns in $\mathbf{P}$ (this includes the columns in $\mathbf{B}$), it is clear that $\mathbf{T}_2^T\mathbf{B} = [\mathbf{0}]$. The control variables have no direct input to the states $\mathbf{w}_2$. Also, the term $\mathbf{T}_2^T\mathbf{AT}_1 = [\mathbf{0}]$ (proven in Problem 11.22a), so that the states $\mathbf{w}_2$ are not coupled to states $\mathbf{w}_1$. The Kalman controllable canonical decomposition is thus

$$\begin{bmatrix} \dot{\mathbf{w}}_1 \\ \dot{\mathbf{w}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{T}_1^T\mathbf{AT}_1 & \mathbf{T}_1^T\mathbf{AT}_2 \\ \hline [\mathbf{0}] & \mathbf{T}_2^T\mathbf{AT}_2 \end{array}\right] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \left[\begin{array}{c} \mathbf{T}_1^T\mathbf{B} \\ \hline [\mathbf{0}] \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y} = [\mathbf{CT}_1 \mid \mathbf{CT}_2]\mathbf{w} + \mathbf{Du} \tag{11.9}$$

State variables $\mathbf{w}_2$ are not connected to the input, neither directly nor indirectly through $\mathbf{w}_1$ coupling. Thus they are uncontrollable.

A straightforward method of finding the required orthogonal basis vectors in $\mathbf{T}_1$ and $\mathbf{T}_2$ is to perform a $\mathbf{QR}$ decomposition on $\mathbf{P}$, (using the modified Gram-Schmidt procedure because it has better numerical properties than the standard Gram-Schmidt procedure.) The $\mathbf{QR}$ decomposition is a reliable method of determining the rank of $\mathbf{P}$, so it will often be carried out anyway in the test of controllability criterion 2. If rank($\mathbf{P}$) $= r_P < n$, then a random-number generator is used to select components of additional vectors to augment the columns of $\mathbf{P}$ until a full set of $n$ orthogonal basis vectors is found. The randomly selected augmenting vectors are transformed into columns of $\mathbf{T}_2$ as the modified Gram-Schmidt process is carried out.

**EXAMPLE 11.7** A state variable model has

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.666667 & 0.333333 \\ 0.333333 & -0.666667 \\ 0.333333 & 0.333333 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 2 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}$$

Evaluate the controllability matrix $\mathbf{P}$ and then use it to find the Kalman controllable canonical form.

Using Eq. (*11.5*) gives

$$\mathbf{P}=\left[\begin{array}{cc|cc|cc} -0.666667 & 0.333333 & 1.333333 & -1.666667 & -2.666667 & 6.333333\\ 0.333333 & -0.666667 & -0.666667 & 2.333333 & 1.333333 & -7.666667\\ 0.333333 & 0.333333 & -0.666667 & -0.666667 & 1.333333 & 1.333333 \end{array}\right]$$

The **QR** decomposition of this matrix is $\mathbf{P}=\mathbf{T}_1\mathbf{R}$, with

$$\mathbf{T}_1=\begin{bmatrix} -0.816497 & 0\\ 0.408248 & -0.707107\\ 0.408248 & 0.707107 \end{bmatrix}$$

$$\mathbf{R}=\begin{bmatrix} 0.816496 & -0.408248 & -1.63299 & 2.04124 & 3.265986 & -7.75671\\ 0 & 0.707107 & 0 & -2.12132 & 0 & 6.36396 \end{bmatrix}$$

From this, rank($\mathbf{P}$) = 2, so the system is uncontrollable. The two columns of $\mathbf{T}_1$ are an orthogonal basis for the controllable subspace. A third orthogonal basis vector is found and forms the column of $\mathbf{T}_2=[0.57735 \quad 0.57735 \quad 0.57735]^T$. If $\mathbf{P}$ is augmented with an additional independent column, then $\mathbf{T}_2$ is found automatically via the modified Gram-Schmidt **QR** decomposition. The **R** factor will be found to have an additional all-zero row. Using $\mathbf{T}=[\mathbf{T}_1 \mid \mathbf{T}_2]$ gives the Kalman controllable canonical form

$$\dot{\mathbf{w}}=\left[\begin{array}{cc|c} -2 & 1.73204 & -5.65684\\ 0 & -3 & -19.5959\\ \hline 0 & 0 & 5 \end{array}\right]\mathbf{w}+\left[\begin{array}{cc} 0.816495 & -0.408247\\ 0 & 0.707107\\ \hline 0 & 0 \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y}=\left[\begin{array}{cc|c} 1.224756 & 0.707107 & 5.19615\\ 0 & 1.41421 & 3.46410 \end{array}\right]\mathbf{w}$$

The third component of **w** is uncontrollable. ■

An exactly analogous Kalman observable canonical form can be found by selecting an orthonormal basis set from columns of **Q** (these form columns of a matrix $\mathbf{V}_1$) and, if needed, augmenting vectors which form the columns of $\mathbf{V}_2$. Then the transformation $\mathbf{x}=\mathbf{V}\mathbf{v}$ leads to the desired form

$$\begin{bmatrix} \dot{\mathbf{v}}_1\\ \dot{\mathbf{v}}_2 \end{bmatrix}=\left[\begin{array}{c|c} \mathbf{V}_1^T\mathbf{A}\mathbf{V}_1 & [\mathbf{0}]\\ \hline \mathbf{V}_2^T\mathbf{A}\mathbf{V}_1 & \mathbf{V}_2^T\mathbf{A}\mathbf{V}_2 \end{array}\right]\begin{bmatrix} \mathbf{v}_1\\ \mathbf{v}_2 \end{bmatrix}+\left[\begin{array}{c} \mathbf{V}_1^T\mathbf{B}\\ \hline \mathbf{V}_2^T\mathbf{B} \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y}=[\mathbf{C}\mathbf{V}_1 \mid [\mathbf{0}]]\begin{bmatrix} \mathbf{v}_1\\ \mathbf{v}_2 \end{bmatrix}+\mathbf{D}\mathbf{u} \qquad (11.10)$$

The facts that $\mathbf{V}_2^T\mathbf{C}^T=[\mathbf{0}]$ by construction and that $\mathbf{V}_1^T\mathbf{A}\mathbf{V}_2=[\mathbf{0}]$ (see Problem 11.22b) have been used in arriving at this so-called Kalman observable canonical form of the state equations. Note that states $\mathbf{v}_2$ are not directly contributing to the output **y**. Information about $\mathbf{v}_2$ is also not available indirectly in **y** through the $\mathbf{v}_1$ variables because $\mathbf{v}_2$ has no effect upon the $\mathbf{v}_1$ states.

The same **QR** decomposition method, this time applied to the matrix **Q** of Eq. (*11.6*), can be used to find the required transformation matrix **V**.

**EXAMPLE 11.8** A state-variable model has the same matrix **A** as in Example 11.7, and

$$\mathbf{B} = \begin{bmatrix} 0.333333 & 1.333333 \\ 1.333333 & 0.333333 \\ -0.666667 & 0.333333 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}$$

Calculate the observability matrix **Q** and then use it to evaluate system observability and to find the Kalman observable canonical form.

From Eq. (*11.6*),

$$\mathbf{Q} = \left[\begin{array}{cc|cc|cc} 1 & 3 & 0 & -6 & -6 & 12 \\ 2 & 3 & -3 & -6 & 3 & 12 \\ 3 & 6 & -3 & -12 & -3 & 24 \end{array}\right] = \mathbf{V}_1\mathbf{R}$$

where

$$\mathbf{V}_1 = \begin{bmatrix} 0.26726 & 0.771517 \\ 0.53452 & -0.617213 \\ 0.801784 & 0.154303 \end{bmatrix}$$

and

$$\mathbf{R} = \begin{bmatrix} 3.741657 & 7.21605 & -4.0089 & -14.43211 & -2.40535 & 28.86421 \\ 0 & 1.38873 & 1.38873 & -2.77746 & -6.94365 & 5.55492 \end{bmatrix}$$

Since there are only two columns in $\mathbf{V}_1$, rank(**Q**) is 2 and the system is not observable. Using these results, augmented by $\mathbf{V}_2 = [-0.57735 \quad -0.57735 \quad 0.57735]^T$, Eq. (*11.9*) gives

$$\dot{\mathbf{v}} = \left[\begin{array}{cc|c} -1.07143 & 0.371154 & 0 \\ -4.82499 & -3.92857 & 0 \\ \hline -19.4422 & -3.74171 & 5 \end{array}\right]\mathbf{v} + \left[\begin{array}{cc} 0.26726 & 0.80178 \\ -0.66865 & 0.87439 \\ \hline -1.3472 & -0.76980 \end{array}\right]\mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cc|c} 3.74165 & 0 & 0 \\ 7.21604 & 1.38873 & 0 \end{array}\right]\mathbf{v}$$

The third component of **v** is unobservable. ■

## 11.8 STABILIZABILITY AND DETECTABILITY

In general the properties of controllability and observability, discussed in this chapter, and stability, discussed in Chapter 10, are independent. None of these properties implies or is implied by any of the others. In this section two additional system properties are defined, which are useful whenever a system fails to be either completely controllable or completely observable.

**Definition 11.3.** A linear system is said to be *stabilizable* if all its unstable modes, if any, are controllable.

If a system is stable it is stabilizable. If a system is completely controllable, it is stabilizable. In the general case, the subsystem defined by the modes or states in $\mathbf{w}_2$ must be stable in order that the system be stabilizable. For a linear, constant system the requirement is that all eigenvalues of $\mathbf{T}_2^T\mathbf{A}\mathbf{T}_2$ must lie in the stable region, that is, the left-half of the $s$-plane (for continuous-time systems). The system of Example 11.7 is not stabilizable because the uncontrollable mode $\mathbf{w}_3$ has an eigenvalue of 5. The same definitions, decompositions, and analysis can be applied to discrete-time systems. Everything remains the same except that the stable region is the interior of the unit circle of the complex $Z$-plane in that case. The significance of stabilizability is that even though certain modes cannot be controlled by choice of input or feedback, if they are stable (better yet asymptotically stable), these modes will stay bounded (or better yet decay to zero). This modal behavior can often be tolerated in the overall control system.

**Definition 11.4.** A linear system is said to be *detectable* if all of its unstable modes, if any, are observable.

If the system is stable, it is detectable. If it is observable, it is also detectable. In general, the condition is met if the subsystem described by modes $\mathbf{v}_2$ are stable. In the linear constant case, the requirement is that all eigenvalues of $\mathbf{V}_2^T\mathbf{A}\mathbf{V}_2$ fall in the stable region of the complex plane ($s$ or $Z$). the system of Example 11.8 is not detectable because the unobservable state $v_3$ is unstable (eigenvalue of 5). The significance of the detectability property is that if certain modes are unstable and hence subject to growth without bound, at least this undesirable behavior will be obvious from the output signals $\mathbf{y}$. No "hidden modes" such as those contained in $\mathbf{v}_2$ can be allowed to grow secretly in an unstable fashion.

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## ILLUSTRATIVE PROBLEMS

### *Application of the Criteria*

**11.1** Is the following system completely controllable and completely observable?

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t), \qquad y(t) = [4 \quad 2]x(t)$$

Using criteria 2, $\mathbf{P} = [\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has rank 1 and $\mathbf{Q} = [\mathbf{C}^T \mid \mathbf{A}^T\mathbf{C}^T] = \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}$ has rank 1. Therefore, the system is neither completely controllable nor completely observable.

11.2 Is the following discrete-time system completely controllable and completely observable?

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}u(k), \qquad y(k) = [5 \quad 1]\mathbf{x}(k)$$

Using criteria 2, $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, rank $\mathbf{P} = 1$ but $n = 2$. Therefore, the system is *not* completely controllable. $\mathbf{Q} = \begin{bmatrix} 5 & \frac{9}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ has rank 2. The system is completely observable.

11.3 Investigate the controllability and observability of the systems in Figure 11.2(*a*) and (*b*) individually and when connected in series as in (*c*).

For system (*a*), $\dot{y}_1 + \beta y_1 = \dot{u}_1 + \alpha u_1$. Letting $x_1 = y_1 - u_1$ gives the state equation $\dot{x}_1 = -\beta x_1 + (\alpha - \beta)u_1$. This system is completely controllable if $\alpha \neq \beta$. It is always completely observable.

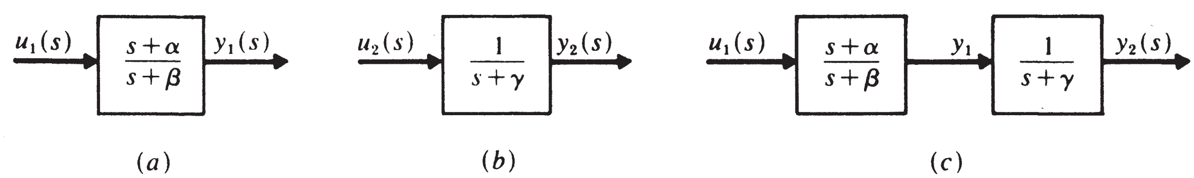



**Figure 11.2**

For system (*b*), $\dot{y}_2 + \gamma y_2 = u_2$. Letting $x_2 = y_2$ gives the state equation $\dot{x}_2 = -\gamma x_2 + u_2$. This system is completely controllable and observable.

Using the same definition for $x_1$ and $x_2$ in system (*c*), and noting that $y_1 = x_1 + u_1$ replaces $u_2$, the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\beta & 0 \\ 1 & -\gamma \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha - \beta \\ 1 \end{bmatrix}u_1$$

The controllability matrix is

$$\mathbf{P} = \begin{bmatrix} \alpha - \beta & -\beta(\alpha - \beta) \\ 1 & \alpha - \beta - \gamma \end{bmatrix} \quad \text{and} \quad |\mathbf{P}| = (\alpha - \beta)(\alpha - \gamma)$$

The rank of $\mathbf{P}$ is 2 and system (*c*) is completely controllable, unless $\alpha = \beta$ or $\alpha = \gamma$. If either of these conditions is satisfied, the pole-zero cancellation leads to an uncontrollable system. The observability matrix is $\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$; and since its rank is 2, system (*c*) is completely observable. With other choices of states, this system is controllable but not observable.

11.4 Investigate the controllability and observability of the two systems shown in Figure 11.3.

System (*a*) is described by $\dot{x}_1 = -\alpha x_1 + Ku$, $y = x_1$, and is completely controllable and observable.

System (*b*) can be described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}u, \qquad y = [C_1 \quad C_2]\mathbf{x}$$

The controllability and observability matrices of criterion 2 are

$$\mathbf{P} = \begin{bmatrix} K_1 & -\alpha K_1 \\ K_2 & -\beta K_2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} C_1 & -\alpha C_1 \\ C_2 & -\beta C_2 \end{bmatrix}$$

System $(b)$ is completely controllable and observable except when $\alpha = \beta$.

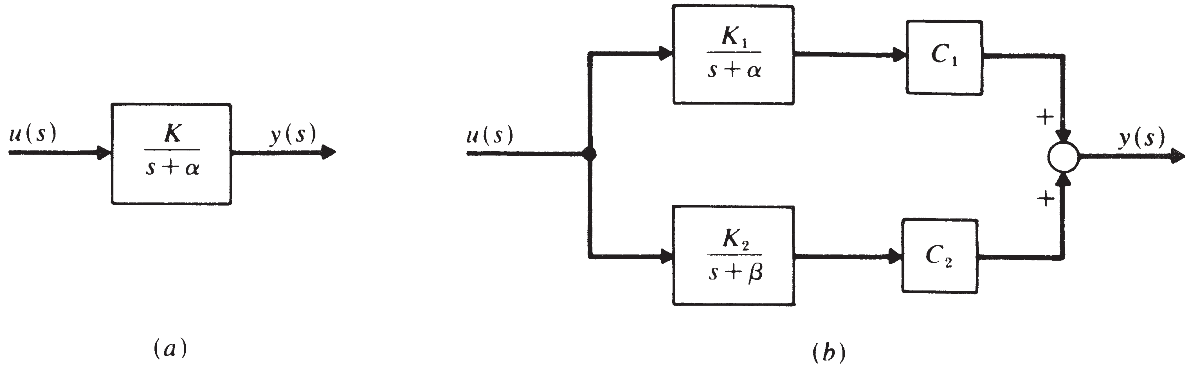



**Figure 11.3**

**11.5** Use the system of Problem 9.6 to draw conclusions regarding the observability of a discrete-time system obtained by sampling a completely observable continuous-time system. Assume that the output is $y = x_1$.

To consider observability, it is only necessary to consider the unforced system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}\mathbf{x}$, $\mathbf{y} = [1 \quad 0]\mathbf{x}$. This system is completly observable since $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -\Omega \end{bmatrix}$ has rank 2.

Using the state transition matrix $\mathbf{\Phi}(t_{k+1}, t_k) = \mathbf{\Phi}(\Delta t, 0)$, with $\Delta t \triangleq t_{k+1} - t_k$, the discrete-time equations are

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos \Omega\,\Delta t & -\sin \Omega\,\Delta t \\ \sin \Omega\,\Delta t & \cos \Omega\,\Delta t \end{bmatrix}\mathbf{x}(k), \qquad \mathbf{y}(k) = [1 \quad 0]\mathbf{x}(k)$$

The discrete observability matrix is $\mathbf{Q} = \begin{bmatrix} 1 & \cos \Omega\,\Delta t \\ 0 & -\sin \Omega\,\Delta t \end{bmatrix}$. The rank is 2 unless the sampling period $\Delta t$ is an integer multiple of $\pi/\Omega$. The property of complete observability is lost if an oscillatory system is sampled at its natural frequency.

**11.6** Prove that the name attached to the *controllable* canonical form of the state equations for a single input system is justified.

The controllable canonical form of the single-input state equations were given in Chapter 3 and always have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ a & b & c & \cdots & d \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $a, b, c, \ldots, d$ are arbitrary coefficients. Direct application of controllability criterion 2 shows that

$$\mathbf{AB} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ d \end{bmatrix} \qquad \mathbf{A}^2\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ d \\ d^2 \end{bmatrix} \; \mathbf{A}^3\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ d \\ d^2 \\ d^3 \end{bmatrix} \qquad \mathbf{A}^{n-1}\mathbf{B} = \begin{bmatrix} 1 \\ d \\ d^2 \\ d^3 \\ \vdots \\ d^{n-1} \end{bmatrix}$$

so that for any $n > 0$, the $n \times n$ matrix $\mathbf{P}$ has a nonzero determinant independent of the system coefficient values $a, b, c, \ldots, d$. Therefore, this form of the state equations is always controllable, and the name is aptly chosen.

**11.7** Prove that the name attached to the *observable* canonical form of the state equations for a single output system is justified.

The observable canonical form for the single-input, single-output state equations were given in Chapter 3 and always have

$$\mathbf{A} = \begin{bmatrix} a & 1 & 0 & \cdots & 0 \\ b & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ c & 0 & 0 & \cdots & 1 \\ d & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \mathbf{C}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{B} = [\text{not important}]$$

where $a, b, c, \ldots, d$ are arbitrary coefficients. Direct application of observability criterion 2 shows that the $n \times n$ matrix $\mathbf{Q}$ always has a nonzero determinant, and therefore this form of the state equations is always observable.

**11.8** Is the following time-variable system completely controllable?

$$\dot{\mathbf{x}} = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\mathbf{x} + e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$$

Since $\mathbf{B}(t)$ is time-varying, criterion 3 is used. The controllability matrix of Eq. (*11.7*) can be written as

$$\mathbf{G}(t_1, t_0) = \mathbf{\Phi}(t_1, 0)\int_{t_0}^{t_1} \mathbf{\Phi}^{-1}(\tau, 0)\mathbf{B}(\tau)\mathbf{B}^T(\tau)[\mathbf{\Phi}^{-1}(\tau, 0)]^T\, d\tau\, \mathbf{\Phi}^T(t_0, 0)$$

The transition matrix $\mathbf{\Phi}(t, 0)$ can be found by any of the methods of Chapter 8, and then

$$\mathbf{\Phi}^{-1}(\tau, 0) = \mathbf{\Phi}(-\tau, 0) = \frac{1}{2}\begin{bmatrix} e^{-\tau/2} + e^{-\tau/3} & e^{-\tau/2} - e^{-\tau/3} \\ e^{-\tau/2} - e^{-\tau/3} & e^{-\tau/2} + e^{-\tau/3} \end{bmatrix}$$

Therefore, $\mathbf{\Phi}^{-1}(\tau, 0)\mathbf{B}(\tau) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so that

$$|\mathbf{G}(t_1, t_0)| = |\mathbf{\Phi}(t_1, 0)| \left| \begin{bmatrix} t_1 - t_0 & t_1 - t_0 \\ t_1 - t_0 & t_1 - t_0 \end{bmatrix} \right| |\mathbf{\Phi}^T(t_1, 0)| = 0$$

This is true for all $t_0, t_1$. The system is not completely controllable.

**11.9** An approximate linear model of the lateral dynamics of an aircraft, for a particular set of flight conditions, has [7] the state and control vectors in the perturbation quantities

$$\mathbf{x} = [p \quad r \quad \beta \quad \phi]^T \quad \text{and} \quad \mathbf{u} = [\delta_a \quad \delta_r]^T$$

where $p$ and $r$ are incremental roll and yaw rates, $\beta$ is an incremental sideslip angle, and $\phi$ is an incremental roll angle. The control inputs are the incremental changes in the aileron angle $\delta_a$ and in the rudder angle $\delta_r$, respectively. These variables are shown in Figure 11.4. In a consistent set of units this linearized model has

$$\mathbf{A} = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose a malfunction prevents manipulation of the input $\delta_r$. Is it possible to control the aircraft using only $\delta_a$? Is the aircraft controllable with just $\delta_r$? Verify that it is controllable with both inputs operable.

When $\delta_a$ is the only input, just the first column of the $\mathbf{B}$ matrix must be used in checking for controllability. The $\mathbf{P}$ matrix is determined to be

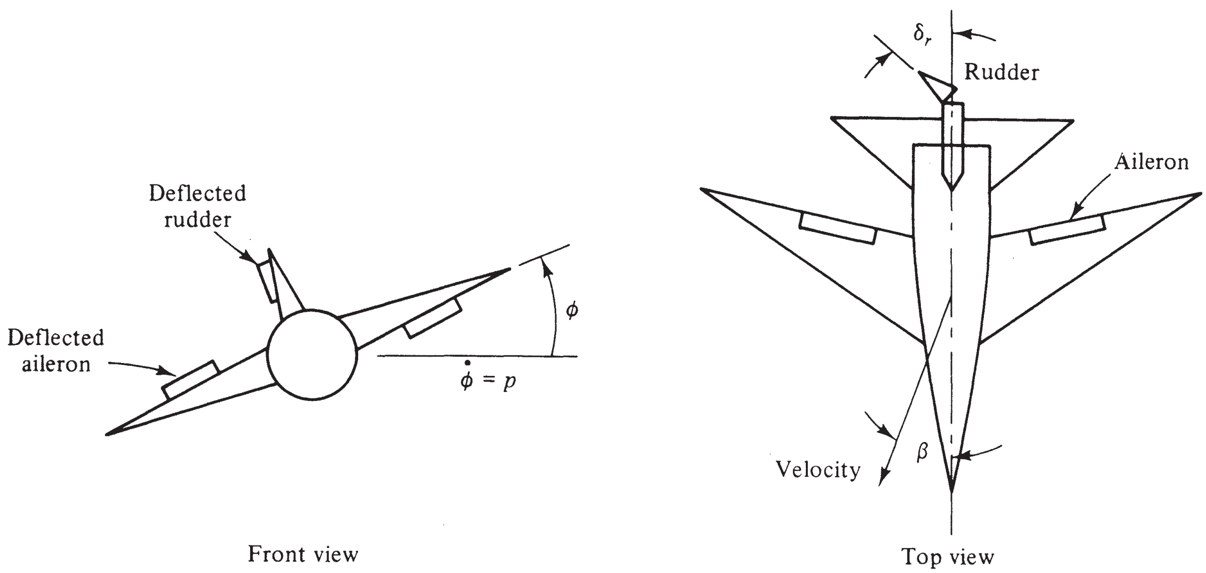



**Figure 11.4**

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} 20 & -200 & 2000 & -2\times 10^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 20 & -200 & 2\times 10^3 \end{array}\right]$$

and its rank is $2 < n$. Thus it is not controllable. The aircraft can be made to roll using only the ailerons, but it cannot be made to turn, at least insofar as this linearized model is concerned.

With $\delta_r$ as the only input, column 2 of **B** is used to compute another **P** matrix:

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} 2.8 & -28 & 248.7 & -2443.18 \\ -3.13 & 2.191 & 26.636 & -58.08 \\ 0 & 3.13 & -4.382 & -23.57 \\ 0 & 2.8 & -28 & 248.7 \end{array}\right]$$

The rank is now 4, and the system is controllable. The controllability index (the number of partitions in **P** that are required before full rank is achieved) is 4. The maneuverability of the aircraft would be greatly degraded, and a very sloppy flight profile would be expected under such conditions. Controllability does not guarantee a high-quality control, it just guarantees that all the states can be manipulated to zero in some fashion in some finite time. Adding the second input only adds more columns to **P** so its rank is also 4. Now, however, a rank 4 matrix can be formed from just the first two partitions of **P.** The controllability index is 2, indicating a stronger degree of controllability in some sense.

**11.10** If the only output is a measurement of the roll rate $p$ (provided by a rate gyro) in the previous problem, is the system observable?

The output matrix is $\mathbf{C} = [1 \quad 0 \quad 0 \quad 0]$. Using this, the matrix **Q** is

$$\mathbf{Q} = \left[\begin{array}{c|c|c|c} 1 & -10 & 100 & -1000 \\ 0 & 0 & 10 & -114 \\ 0 & -10 & 107 & -984.9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rank is $3 < 4$, so the system is not observable. Measurements of roll rate allow the *change* in roll angle to be monitored, but will never allow determination of the roll angle itself because its initial value is unknown. In fact, if any one or all of the states except $\phi$ are measured outputs, this system remains unobservable. A bank indicator or some other means of measuring $\phi$ is required in order to obtain an observable system. If $\phi$ is the only measurement, then $\mathbf{C} = [0 \quad 0 \quad 0 \quad 1]$, which leads to

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & -10 & 100 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & -10 & 107 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This has rank 4, so the system is observable. The observability index is 4, meaning that all four partitions in **Q** are required to give a rank 4 result. If other states can also be measured and combined with $\phi$ to form a vector rather than scalar output, the observability index will improve (decrease).

**11.11** **(a)** Show that a system governed by

$$\mathbf{x}(k+1) = \mathbf{x}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$$

with **C** constant, is never observable unless $\text{rank}(\mathbf{C}) = n$.

**(b)** Show that if $\mathbf{C}(k)$ is time-varying, observability criterion 3 leads to the condition that the normal equations of least squares must eventually become invertible for some finite time $N$ if the system is to be observable.

**(a)** With $\mathbf{A} = \mathbf{I}$, criterion 2 gives

$$\mathbf{Q} = [\mathbf{C}^T \mid \mathbf{C}^T \mid \cdots \mid \mathbf{C}^T]$$

Therefore, $\text{rank}(\mathbf{Q}) = \text{rank}(\mathbf{C}^T) = \text{rank}(\mathbf{C})$. Observability requires that $\text{rank}(\mathbf{C}) = n$, where $n$ is the number of components in **x**.

**(b)** Again, $\mathbf{A} = \mathbf{I}$, so the observability matrix of criterion 3 becomes

$$\mathbf{Q}' = \sum_{k=0}^{N} \mathbf{C}^T(k)\mathbf{C}(k) = [\mathbf{C}^T(0) \quad \mathbf{C}^T(1) \quad \cdots \quad \mathbf{C}^T(N)] \begin{bmatrix} \mathbf{C}(0) \\ \mathbf{C}(1) \\ \vdots \\ \mathbf{C}(N) \end{bmatrix}$$

Define the stacked-up measurement vector and measurement matrix as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(N) \end{bmatrix}, \qquad \mathcal{H} = \begin{bmatrix} \mathbf{C}(0) \\ \mathbf{C}(1) \\ \vdots \\ \mathbf{C}(N) \end{bmatrix}$$

Then the entire group of measurements, as it would be processed in batch least squares, is $\mathbf{Y} = \mathcal{H}\mathbf{x}(0)$, and $\mathbf{Q}' = \mathcal{H}^T\mathcal{H}$. The normal equation is $\mathcal{H}^T\mathcal{H}\mathbf{x}(0) = \mathcal{H}^T\mathbf{Y}$, and is invertible if and only if $\mathcal{H}$ has rank $n$. This is the same as requiring that $\mathbf{Q}'$ have rank $n$. Thus, observability is seen to be the same as having a *unique* least-squares solution in this case of a constant state vector.

### *Extensions and Proofs*

**11.12** Show that if a continuous-time linear system is completely controllable at $t_0$, then any initial state $\mathbf{x}(t_0)$ can be transferred to any other state $\mathbf{x}(t_1)$ at some finite time $t_1$.

Complete controllability means that any $\mathbf{x}(t_0)$ can be transferred to the origin $\mathbf{x}(t_1) = \mathbf{0}$. The solution for a given input is of the form

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \qquad (1)$$

This could be written as

$$\mathbf{0} = \mathbf{\Phi}(t_1, t_0)[\mathbf{x}(t_0) - \mathbf{\Phi}(t_0, t_1)\mathbf{x}(t_1)] + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \qquad (2)$$

Since $\mathbf{x}' \triangleq \mathbf{x}(t_0) - \mathbf{\Phi}(t_0, t_1)\mathbf{x}(t_1)$ belongs to $\Sigma$, it is a possible initial state. Complete controllability at $t_0$ guarantees that $\mathbf{x}'$ can be driven to the origin (Eq. (*2*)), which means that $\mathbf{x}(t_0)$ can be driven to any arbitrary $\mathbf{x}(t_1)$ (Eq. (*1*)).

**11.13** The arguments in Sec. 11.3 establish the necessity of controllability criterion 1. Show that this criterion is sufficient by assuming no zero rows of $\mathbf{B}_n$ and deriving the control which drives an arbitrary initial state to the origin.

The normal form description of the system is $\dot{\mathbf{q}} = \mathbf{\Lambda q} + \mathbf{B}_n\mathbf{u}$ and the solution at $t_1$ is

$$\mathbf{q}(t_1) = e^{\mathbf{\Lambda} t_1}\mathbf{q}(0) + \int_0^{t_1} e^{\mathbf{\Lambda} t_1} e^{-\mathbf{\Lambda}\tau}\mathbf{B}_n\mathbf{u}(\tau)\, d\tau$$

Let the *i*th row of $\mathbf{B}_n$ define the *row* vector $\mathbf{b}_i$, and let $\mathbf{u}(t) = \sum_{j=1}^{n} \beta_j e^{-\bar{\lambda}_j t}\bar{\mathbf{b}}_j^T$. The coefficients $\beta_j$ are unknown constants. It is to be shown that these constants can be selected in such a way that $\mathbf{q}(t_1) = \mathbf{0}$ if none of the rows $\mathbf{b}_i$ are identically zero. Using the assumed form for $\mathbf{u}(t)$, a typical component of $\mathbf{q}(t_1)$ is

$$q_i(t_1) = e^{\lambda_i t_1} q_i(0) + \sum_{j=1}^{n} e^{\lambda_i t_1} \int_0^{t_1} e^{-\lambda_i \tau}\mathbf{b}_i\bar{\mathbf{b}}_j^T e^{-\bar{\lambda}_j\tau}\, d\tau\, \beta_j$$

or

$$e^{-\lambda_i t_1} q_i(t_1) - q_i(0) = \sum_{j=1}^{n} \langle \boldsymbol{\theta}_i(\tau), \boldsymbol{\theta}_j(\tau)\rangle \beta_j$$

The integral inner product (Problem 5.22, page 197) of the functions $\boldsymbol{\theta}_j(\tau) = e^{-\bar{\lambda}_j\tau}\bar{\mathbf{b}}_j^T$ is used. The unknown coefficients can be obtained by solving $n$ simultaneous equations, and are given by

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = [\langle \boldsymbol{\theta}_i(\tau), \boldsymbol{\theta}_j(\tau)\rangle]^{-1}[e^{-\mathbf{\Lambda} t_1}\mathbf{q}(t_1) - \mathbf{q}(0)]$$

The indicated inverse is guaranteed to exist if $\mathbf{b}_i \neq \mathbf{0}$ for all $i$ and if all the $\lambda_i$ are distinct. This is true because under these conditions the set of functions $\{\boldsymbol{\theta}_j(\tau)\}$ is linearly independent over every finite interval $[0, t_1]$. The matrix $[\langle \boldsymbol{\theta}_i(\tau), \boldsymbol{\theta}_j(\tau)\rangle]$, which can also be written as $\int_0^{t_1} e^{-\mathbf{\Lambda}\tau}\mathbf{B}_n\bar{\mathbf{B}}_n^T e^{-\bar{\mathbf{\Lambda}}\tau}\, d\tau$, is the Grammian matrix and is nonsingular. The conditions of controllability criterion 1 are sufficient to guarantee that any $\mathbf{q}(0)$ can be driven to any $\mathbf{q}(t_1)$, including $\mathbf{q}(t_1) = \mathbf{0}$. An input function which drives $\mathbf{q}_1(0)$ to the origin at $t_1$ is

$$\mathbf{u}(t) = \bar{\mathbf{B}}_n^T e^{\bar{\mathbf{\Lambda}} t}\boldsymbol{\beta} = -\bar{\mathbf{B}}_n^T e^{-\bar{\mathbf{\Lambda}} t}\left[\int_0^{t_1} e^{-\mathbf{\Lambda}\tau}\mathbf{B}_n\bar{\mathbf{B}}_n^T e^{-\bar{\mathbf{\Lambda}}\tau}\, d\tau\right]^{-1}\mathbf{q}(0)$$

**11.14** Assume that the time-invariant system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ is completely controllable. Prove that the controllability criterion 2 is a necessary condition.

Complete controllability means that for every $\mathbf{x}_0$ there is some finite time $t_1$ and some input function $\mathbf{u}(t)$ such that

$$\mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{Bu}(\tau)\, d\tau \quad \text{or} \quad -\mathbf{x}_0 = \int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{Bu}(\tau)\, d\tau$$

Using the remainder form from the matrix exponential,

$$e^{-\mathbf{A}\tau} = \alpha_0(\tau)\mathbf{I} + \alpha_1(\tau)\mathbf{A} + \alpha_2(\tau)\mathbf{A}^2 + \cdots + \alpha_{n-1}(\tau)\mathbf{A}^{n-1}$$

gives

$$-\mathbf{x}_0 = \sum_{j=0}^{n-1} \mathbf{A}^j \mathbf{B} \int_0^{t_1} \alpha_j(\tau)\mathbf{u}(\tau)\, d\tau$$

Each integral term is an $r \times 1$ constant vector, defined as

$$\mathbf{v}_j = \int_0^{t_1} \alpha_j(\tau)\mathbf{u}(\tau)\, d\tau$$

Then

$$-\mathbf{x}_0 = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{v}_0 \\ \hline \mathbf{v}_1 \\ \hline \vdots \\ \hline \mathbf{v}_{n-1} \end{bmatrix}$$

This result states that every *vector* $-\mathbf{x}_0$ can be expressed as some linear combination of the columns of $\mathbf{P} = [\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$. These columns must span the $n$-dimensional state space $\Sigma$, that is, it is necessary that rank $\mathbf{P} = n$. The necessity of the observability condition 2 can be established in a similar manner.

**11.15** Assume that the following system is completely controllable and completely observable over the interval $[t_0, t_1]$:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

**(a)** Derive an explicit expression for an input which transfers the state from $\mathbf{x}(t_0)$ to $\mathbf{x}(t_1)$.

**(b)** If the input is zero, find an explicit expression for $\mathbf{x}(t_0)$ in terms of the output function $\mathbf{y}(t)$, $t_0 \le t \le t_1$.

**(a)** The solution for the state at $t_1$ can be written in terms of the transformation $\mathcal{A}_c: \mathcal{U} \to \Sigma$,

$$\mathbf{x}(t_1) - \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0) = \mathcal{A}_c(\mathbf{u})$$

Let $\mathbf{u}(t) = \mathcal{A}_c^*(\mathbf{w})$, where $\mathbf{w}$ is an unknown vector in $\Sigma$. Then $\mathbf{x}(t_1) - \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0) = \mathcal{A}_c\mathcal{A}_c^*(\mathbf{w})$. The condition for complete controllability is that the $n \times n$ matrix $\mathcal{A}_c\mathcal{A}_c^*$ has an inverse. Inverting this matrix to solve for $\mathbf{w}$ leads to

$$\mathbf{u}(t) = \mathcal{A}_c^*(\mathcal{A}_c\mathcal{A}_c^*)^{-1}[\mathbf{x}(t_1) - \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0)]$$

$$= \overline{\mathbf{B}}^T(t)\overline{\mathbf{\Phi}}^T(t_1, t)\left[\int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\overline{\mathbf{B}}^T(\tau)\overline{\mathbf{\Phi}}^T(t_1, \tau)\, d\tau\right]^{-1}[\mathbf{x}(t_1) - \mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0)]$$

**(b)** In terms of the transformation $\mathcal{A}_0$, the output of the unforced system is $\mathbf{y}(t) = \mathcal{A}_0(\mathbf{x}(t_0))$. Operating on both sides with the adjoint transformation $\mathcal{A}_0^*$ gives $\mathcal{A}_0^*(\mathbf{y}(t)) = \mathcal{A}_0^*\mathcal{A}_0(\mathbf{x}(t_0))$. The criterion for complete observability ensures that the matrix $\mathcal{A}_0^*\mathcal{A}_0$ has an inverse, so

$$\mathbf{x}(t_0) = (\mathcal{A}_0^*\mathcal{A}_0)^{-1}\mathcal{A}_0^*(\mathbf{y}(t))$$

$$= \left[\int_{t_0}^{t_1} \overline{\mathbf{\Phi}}^T(\tau, t_0)\overline{\mathbf{C}}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau, t_0)\, d\tau\right]^{-1} \int_{t_0}^{t_1} \overline{\mathbf{\Phi}}(\tau, t_0)\overline{\mathbf{C}}^T(\tau)\mathbf{y}(\tau)\, d\tau$$

**11.16** A system with $n$ state variables and $r$ inputs is expressed in Jordan form

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_p \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \\ \hline \vdots \\ \hline \mathbf{B}_p \end{bmatrix}\mathbf{u} \qquad (1)$$

Show that the controllability of this system is determined entirely by the last rows $\mathbf{b}_{il}^T$ of each $\mathbf{B}_i$ submatrix. (The subscript $l$ signifies the last row in a given block and is not a fixed integer.) In particular, show that the system is completely controllable if and only if

1. $\{\mathbf{b}_{il}, \mathbf{b}_{jl}, \dots, \mathbf{b}_{kl}\}$ is a linearly independent set if $\mathbf{J}_i, J_j, \dots, \mathbf{J}_k$ are Jordan blocks *with the same eigenvalue* $\lambda_i$, and
2. $\mathbf{b}_{pl} \neq \mathbf{0}$ if $\mathbf{J}_p$ is the *only* Jordan block with eigenvalue $\lambda_p$.

Note that if all $\mathbf{J}_i$ blocks are $1 \times 1$ blocks so that $\mathbf{A}$ is diagonal, the controllability criterion 1 of Sec. 11.3 requires that all $\mathbf{b}^T \neq \mathbf{0}$ for all rows of $\mathbf{B}$. Controllability criterion 2 is used to investigate the more general case. For simplicity, assume there are just three blocks,

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \qquad \mathbf{J}_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \qquad \mathbf{J}_3 = \begin{bmatrix} \lambda_3 & 1 \\ 0 & \lambda_3 \end{bmatrix}$$

with $\mathbf{B}^T = [\mathbf{b}_{11} \quad \mathbf{b}_{12} \quad \mathbf{b}_{13} \mid \mathbf{b}_{21} \quad \mathbf{b}_{22} \mid \mathbf{b}_{31} \quad \mathbf{b}_{32}]$. The controllability matrix is

$$\mathbf{P} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \mathbf{A}^3\mathbf{B} \mid \mathbf{A}^4\mathbf{B} \mid \mathbf{A}^5\mathbf{B} \mid \mathbf{A}^6\mathbf{B}]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{B}_1 & \mathbf{J}_1\mathbf{B}_1 & \mathbf{J}_1^2\mathbf{B}_1 & \mathbf{J}_1^3\mathbf{B}_1 & \mathbf{J}_1^4\mathbf{B}_1 & \mathbf{J}_1^5\mathbf{B}_1 & \mathbf{J}_1^6\mathbf{B}_1 \\ \hline \mathbf{B}_2 & \mathbf{J}_2\mathbf{B}_2 & \mathbf{J}_2^2\mathbf{B}_2 & \mathbf{J}_2^3\mathbf{B}_2 & \mathbf{J}_2^4\mathbf{B}_2 & \mathbf{J}_2^5\mathbf{B}_2 & \mathbf{J}_2^6\mathbf{B}_2 \\ \hline \mathbf{B}_3 & \mathbf{J}_3\mathbf{B}_3 & \mathbf{J}_3^2\mathbf{B}_3 & \mathbf{J}_3^3\mathbf{B}_3 & \mathbf{J}_3^4\mathbf{B}_3 & \mathbf{J}_3^5\mathbf{B}_3 & \mathbf{J}_3^6\mathbf{B}_3 \end{array}\right]$$

The results of Problem 8.4, page 293, are used for the various powers $\mathbf{J}_i^k$. Then

$$\mathbf{J}_1^k\mathbf{B}_1 = \begin{bmatrix} \lambda_1^k\mathbf{b}_{11}^T + k\lambda_1^{k-1}\mathbf{b}_{12}^T + \tfrac{1}{2}k(k-1)\lambda_1^{k-2}\mathbf{b}_{13}^T \\ \lambda_1^k\mathbf{b}_{12}^T + k\lambda_1^{k-1}\mathbf{b}_{13}^T \\ \lambda_1^k\mathbf{b}_{13}^T \end{bmatrix}$$

$$\mathbf{J}_2^k\mathbf{B}_2 = \begin{bmatrix} \lambda_1^k\mathbf{b}_{21}^T + k\lambda_1^{k-1}\mathbf{b}_{22}^T \\ \lambda_1^k\mathbf{b}_{22}^T \end{bmatrix}$$

$$\mathbf{J}_3^k\mathbf{B}_3 = \begin{bmatrix} \lambda_3^k\mathbf{b}_{31}^T + k\lambda_3^{k-1}\mathbf{b}_{32}^T \\ \lambda_3^k\mathbf{b}_{32}^T \end{bmatrix}$$

At this point the necessity of condition (2) is obvious since, for example, if $\mathbf{b}_{32} = \mathbf{0}$, the entire seventh row of $\mathbf{P}$ would be zero and rank $\mathbf{P} < n$. To see the necessity of condition (1), let $\mathbf{b}_{22} = \alpha\mathbf{b}_{13}$. Then an elementary row operation ($-\alpha$ times row 3 added to row 5) would make row 5 zero. This would again give rank $\mathbf{P} < n$, so the system would be uncontrollable.

It is tedious but trivial to show that a sequence of elementary column operations can be used to reduce $\mathbf{P}$ to $\mathbf{P}'$. Specifically, subtract $\lambda_1$ times each of the first $r$ columns from the corresponding column in the second group of $r$ columns. Then subtract $\lambda_1^2$ times column 1 and $2\lambda_1$ times the modified $(r+1)$st column from the $(2r+1)$st column and so on. Continuing this process leads to $\mathbf{P}'$:

$$\mathbf{P}' = \left[\begin{array}{c|c|c|c} \mathbf{b}_{11}^T & \mathbf{b}_{12}^T & \mathbf{b}_{13}^T & \\ \mathbf{b}_{12}^T & \mathbf{b}_{13}^T & \mathbf{0} & \cdots \\ \mathbf{b}_{13}^T & \mathbf{0} & \mathbf{0} & \\ \hline \mathbf{b}_{21}^T & \mathbf{b}_{22}^T & \mathbf{0} & \cdots \\ \mathbf{b}_{22}^T & \mathbf{0} & \mathbf{0} & \\ \hline \mathbf{b}_{31}^T & (\lambda_3-\lambda_1)\mathbf{b}_{31}^T + \mathbf{b}_{32}^T & (\lambda_3^2-\lambda_1^2)\mathbf{b}_{31}^T + 2(\lambda_3-\lambda_1)\mathbf{b}_{32}^T - 2\lambda_1(\lambda_3-\lambda_1)\mathbf{b}_{31}^T & \cdots \\ \mathbf{b}_{32}^T & (\lambda_3-\lambda_1)\mathbf{b}_{32}^T & (\lambda_3^2-\lambda_1^2)\mathbf{b}_{32}^T - 2\lambda_1(\lambda_3-\lambda_1)\mathbf{b}_{32}^T & \end{array}\right]$$

The last two rows of $\mathbf{P}'$ should be recognized as consisting of $\mathbf{B}_3, (\mathbf{J}_3 - \mathbf{I}\lambda_1)\mathbf{B}_3, (\mathbf{J}_3 - \mathbf{I}\lambda_1)^2\mathbf{B}_3, (\mathbf{J}_3 - \mathbf{I}\lambda_1)^3\mathbf{B}_3, \dots$. Finally, a series of elementary row operations (row interchanges) gives

$$\mathbf{P}'' = \left[\begin{array}{c|c|c|c|c|c}
\mathbf{b}_{13}^T & & & & & \\
\mathbf{b}_{22}^T & & & \mathbf{0} & & \\ \cline{1-2}
\mathbf{b}_{12}^T & \mathbf{b}_{13}^T & & & & \\
\mathbf{b}_{21}^T & \mathbf{b}_{22}^T & & & & \\ \cline{1-3}
\mathbf{b}_{11}^T & \mathbf{b}_{12}^T & \mathbf{b}_{13}^T & & & \\ \hline
\mathbf{B}_3 & (\mathbf{J}_2 - \mathbf{I}\lambda_1)\mathbf{B}_3 & (\mathbf{J}_3 - \mathbf{I}\lambda_1)^2\mathbf{B}_3 & (\mathbf{J}_3 - \mathbf{I}\lambda_1)^3\mathbf{B}_3 & (\mathbf{J}_3 - \mathbf{I}\lambda_1)^4\mathbf{B}_3 & \cdots
\end{array}\right]$$

By definition, the rank of $\begin{bmatrix}\mathbf{b}_{13}^T \\ \mathbf{b}_{22}^T\end{bmatrix}$ is 2 if and only if $\mathbf{b}_{13}$ and $\mathbf{b}_{22}$ are linearly independent. If they are, then a $2 \times 2$ nonzero determinant can be formed by deleting columns. Linear independence requires $\mathbf{b}_{13} \neq \mathbf{0}$. It is easy to show that if any one element of $\mathbf{b}_{32}$ is nonzero, then a nonzero $2 \times 2$ determinant can be obtained from $[(\mathbf{J}_3 - \mathbf{I}\lambda_1)^3\mathbf{B}_3 \quad (\mathbf{J}_3 - \mathbf{I}\lambda_1)^4\mathbf{B}_3]$. Thus an $n \times n$ lower block triangular matrix can be formed from $\mathbf{P}''$, whose determinant is nonzero. Rank $\mathbf{P}''$ = rank $\mathbf{P} = n$ implies complete controllability. The procedure used for this example quickly becomes unwieldy, but the result generalizes to any number of Jordan blocks [3].

**11.17** If the output equation for the system in Problem 11.16 is

$$\mathbf{y} = [\mathbf{C}_1 \mid \mathbf{C}_2 \mid \cdots \mid \mathbf{C}_p]\mathbf{x} + \mathbf{D}\mathbf{u}$$

show that observability depends only on the first columns $\mathbf{c}_{i1}$ of each $\mathbf{C}_i$ block. In particular, show that the system is completely observable if and only if

1. $\{\mathbf{c}_{i1}, \mathbf{c}_{j1}, \ldots, \mathbf{c}_{k1}\}$ is a linearly independent set if $\mathbf{J}_i, \mathbf{J}_j, \ldots, \mathbf{J}_k$ are Jordan blocks *with the same eigenvalue* $\lambda_i$, and
2. $\mathbf{c}_{p1} \neq \mathbf{0}$ if $\mathbf{J}_p$ is the *only* Jordan block with eigenvalue $\lambda_p$.

Employ the same seventh-order system of Problem 11.16 for simplicity, and use $\mathbf{C} = [\mathbf{c}_{11} \quad \mathbf{c}_{12} \quad \mathbf{c}_{13} \mid \mathbf{c}_{21} \quad \mathbf{c}_{22} \mid \mathbf{c}_{31} \quad \mathbf{c}_{32}]$. The observability matrix is

$$\mathbf{Q} = [\overline{\mathbf{C}}^T \mid \overline{\mathbf{A}}^T\overline{\mathbf{C}}^T \mid (\overline{\mathbf{A}}^T)^2\overline{\mathbf{C}}^T \mid \cdots \mid (\overline{\mathbf{A}}^T)^6\overline{\mathbf{C}}^T]$$

Using the same kind of elementary column and row operations as in Problem 11.16, $\mathbf{Q}$ can be reduced to $\mathbf{Q}''$, a matrix of a similar form to $\mathbf{P}''$. Because now $\overline{\mathbf{A}}^T$ is used instead of $\mathbf{A}$, the *first* rows of $\overline{\mathbf{C}}_i^T$ play the role of the *last* rows of $\mathbf{B}_i$. Thus the conclusions regarding the leading columns of $\mathbf{C}_i$ follow directly from the results of Problem 11.16.

### *Decomposition*

**11.18** Show that at any time $t_1 > t_0$, the state space $\Sigma$ can be expressed as the direct sum $\Sigma = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1$ is the subspace which contains all the controllable initial states and $\mathcal{X}_2$ is the null space of $\mathbf{G}'(t_1, t_0)$ of Problem 11.33.

Let $\mathbf{x}(t_0)$ be a controllable initial state. Then by definition, there exists an input such that

$$-\mathbf{x}(t_0) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \triangleq \mathcal{A}(\mathbf{u})$$

All controllable initial states belong to $\mathcal{R}(\mathcal{A}) \triangleq \mathcal{X}_1$. Since $\mathcal{A}: \mathcal{U} \to \Sigma$, the results of Problem 6.21 give $\Sigma = \mathcal{R}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A}^*)$. From Sec. 11.6, $\mathcal{N}(\mathcal{A}^*) = \mathcal{N}(\mathcal{A}\mathcal{A}^*)$. For this example, $\mathcal{A}^* = \overline{\mathbf{B}}^T(t)\overline{\mathbf{\Phi}}^T(t_0, t)$ so that $\mathcal{A}\mathcal{A}^* = \mathbf{G}'(t_1, t_0)$. Therefore, $\mathcal{N}(\mathcal{A}^*) = \mathcal{N}(\mathbf{G}'(t_1, t_0))$. Note that this result implies that for each $\mathbf{x}(t_0) \in \Sigma$, $\mathbf{x}(t_0) = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in \mathcal{X}_1$ (controllable) and $\mathbf{x}_2 \in \mathcal{X}_2$ (not controllable). However, it does not imply that the set of all uncontrollable states is the subspace $\mathcal{X}_2$.

Let $\mathbf{x}_a(t_0) = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_b(t_0) = \mathbf{x}_1 - \mathbf{x}_2$. Since $\mathbf{x}_a(t_0)$ and $\mathbf{x}_b(t_0)$ do not belong to $\mathcal{X}_1$ they are not controllable. They do not belong to $\mathcal{X}_2$ either. Still, $\mathbf{x}_a(t_0) + \mathbf{x}_b(t_0) = 2\mathbf{x}_1$ belongs to $\mathcal{X}_1$ and is therefore controllable. The set of all uncontrollable initial states is *not* a subspace.

**11.19** Let $\mathcal{A}_0 : \Sigma \to \mathcal{Y}$ be the output transformation defined in Sec. 11.6. Then the results of Problem 6.21 allow the decomposition $\Sigma = \mathcal{N}(\mathcal{A}_0) \oplus \mathcal{R}(\mathcal{A}_0^*)$. It has been shown that $\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(\mathcal{A}_0^* \mathcal{A}_0) = \mathcal{N}(\mathbf{H}(t_1, t_0))$. Define this null space as $\mathcal{X}_3$. Then every $\mathbf{x}(t_0) \in \mathcal{X}_3$ contributes nothing to the output $\mathbf{y}(t)$, and these are referred to as unobservable states. Use this and the results of Problem 11.18 to show that for all $\mathbf{x}(t_0) \in \Sigma$, $\mathbf{x}(t_0) = \mathbf{x}_a + \mathbf{x}_b + \mathbf{x}_c + \mathbf{x}_d$, where $\mathbf{x}_a$ is controllable but unobservable, $\mathbf{x}_b$ is controllable and observable, $\mathbf{x}_c$ is uncontrollable but observable, and $\mathbf{x}_d$ is uncontrollable and unobservable.

Define $\mathcal{X}_4 = \mathcal{X}_3^\perp = \mathcal{R}(\mathcal{A}_0^*)$. Each $\mathbf{x}(t_0) \in \mathcal{X}_4$ is observable in the sense that a unique $\mathbf{x}(t_0) \in \mathcal{X}_4$ can be associated with a given unforced output record $\mathbf{y}(t)$. (Of course, $\mathbf{x}'(t_0) = \mathbf{x}(t_0) + \mathbf{x}_3$ will give the same $\mathbf{y}(t)$ if $\mathbf{x}_3 \in \mathcal{X}_3$, so it is not possible to determine whether $\mathbf{x}(t_0)$ or $\mathbf{x}'(t_0)$ is the actual initial state.)

Every $\mathbf{x}(t_0)$ can be written as $\mathbf{x}(t_0) = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in \mathcal{X}_1$, $\mathbf{x}_2 \in \mathcal{X}_2$. The orthogonal projection of $\mathbf{x}_1$ into $\mathcal{X}_3$ gives $\mathbf{x}_a$. The projection into $\mathcal{X}_4$ gives $\mathbf{x}_b$. Similarly, projecting $\mathbf{x}_2$ into $\mathcal{X}_4$ gives $\mathbf{x}_c$ and projecting $\mathbf{x}_2$ into $\mathcal{X}_3$ gives $\mathbf{x}_d$.

**11.20** Indicate how a time-invariant linear system with distinct eigenvalues can be decomposed into four possible subsystems with the respective properties (1) controllable but unobservable, (2) controllable and observable, (3) uncontrollable but observable, and (4) uncontrollable and unobservable.

The system can be put into normal form, giving $\mathbf{x}(t_0) = q_1(t_0)\xi_1 + q_2(t_0)\xi_2 + \cdots + q_n(t_0)\xi_n$.

For this class of systems, controllability and observability criteria 1 apply. The controllability and observability can be ascertained for each mode individually. The modes are each assigned to one of the four categories. The resulting decomposition is illustrated in Figure 11.5.

Notice that there is no signal path from the input to an uncontrollable subsystem, either directly or through other subsystems. Also, there is no signal path from an unobservable subsystem to the output. The decomposition of Figure 11.5 can be accomplished for any linear system, but the process is not always this simple [4].

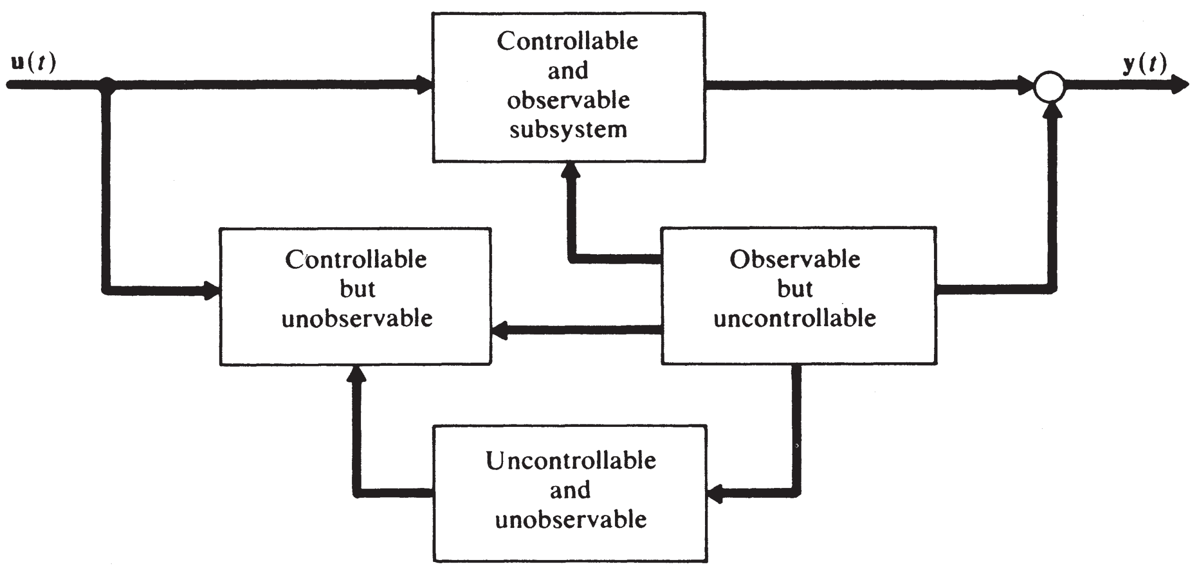



**Figure 11.5**

**11.21** Subdivide the following system into subsystems as discussed in Problem 11.20:

$$\dot{\mathbf{x}} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\mathbf{x}$$

The eigenvalues of **A** are $\lambda_i = -1, -3$, and $-5$. The Jordan normal form will be used, since the controllability and observability criteria 1 apply. The modal matrix containing the eigenvectors is

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

so that

$$\dot{\mathbf{q}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{q}$$

The first mode is uncontrollable and the third mode is unobservable. The second mode is both controllable and observable. There is no mode which is both uncontrollable and unobservable. Figure 11.6 illustrates the three subsystems.

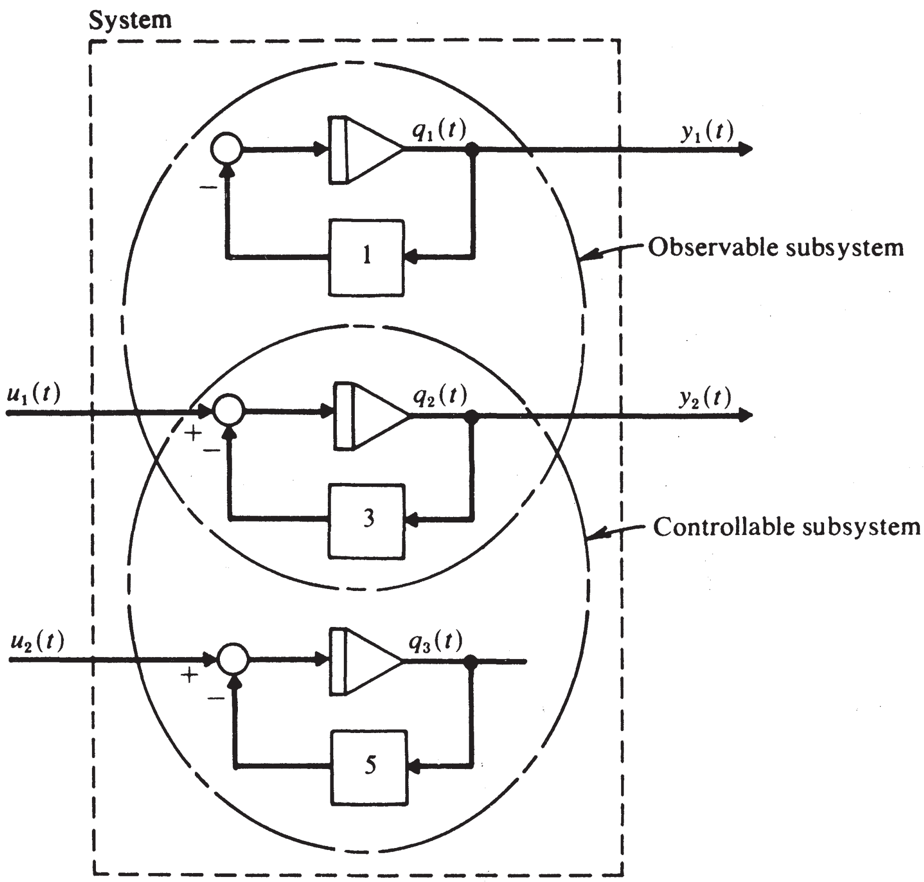



**Figure 11.6**

**11.22** **(a)** Let matrix $\mathbf{T}_2$ have columns which are orthogonal to the subspace spanned by the columns of **P**, and let columns of $\mathbf{T}_1$ form an orthogonal basis for this same subspace. Show that $\mathbf{T}_2^T\mathbf{A}\mathbf{T}_1 = [\mathbf{0}]$.

**(b)** Let matrix $\mathbf{V}_2$ have columns which are orthogonal to the subspace spanned by the columns of **Q**, and let columns of $\mathbf{V}_1$ form an orthogonal basis for this same subspace. Show that $\mathbf{V}_1^T\mathbf{A}\mathbf{V}_2 = [\mathbf{0}]$.

**(a)** The controllability matrix can be written as $\mathbf{P} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{T}_1\mathbf{R}$ by using a **QR** decomposition. This implies that $\mathbf{B} = \mathbf{T}_1\mathbf{R}_1$, $\mathbf{AB} = \mathbf{T}_1\mathbf{R}_2$, $\mathbf{A}^2\mathbf{B} = \mathbf{T}_1\mathbf{R}_3, \ldots$, $\mathbf{A}^{n-1}\mathbf{B} = \mathbf{T}_1\mathbf{R}_n$, where the $\mathbf{R}_i$ terms are partitions of **R**. By construction, $\mathbf{T}_2^T\mathbf{P} = [\mathbf{0}]$. By looking at individual partitions of this equation and using the previous results for the $\mathbf{A}^i\mathbf{B}$ terms, it is found that

$$\mathbf{T}_2^T\mathbf{B} = [\mathbf{0}] \Rightarrow \mathbf{T}_2^T\mathbf{T}_1\mathbf{R}_1 = [\mathbf{0}]$$

$$\mathbf{T}_2^T\mathbf{AB} = [\mathbf{0}] \Rightarrow \mathbf{T}_2^T\mathbf{AT}_1\mathbf{R}_1 = [\mathbf{0}]$$

$$\mathbf{T}_2^T\mathbf{A}^2\mathbf{B} = [\mathbf{0}] \Rightarrow \mathbf{T}_2^T\mathbf{A}(\mathbf{AB}) = \mathbf{T}_2^T\mathbf{AT}_1\mathbf{R}_2 = [\mathbf{0}]$$

and so on. The first of these expressions just gives the lower partition of the new matrix **B** in the Kalman canonical form. The second expression gives the desired result *if* $\mathbf{R}_1$ is nonsingular. If $\mathbf{R}_2$ is nonsingular, the third equation proves the desired result, and so on. Each $\mathbf{R}_i$ partition is square, of dimension equal to the rank of **P**, so it is assured that there is a nonsingular matrix that can be formed from one $\mathbf{R}_i$ or from a combination of $\mathbf{R}_i$ columns.

**(b)** Since $\mathbf{Q} = [\mathbf{C}^T \mid \mathbf{A}^T\mathbf{C}^T \mid (\mathbf{A}^2)^T\mathbf{C}^T \mid \cdots \mid (\mathbf{A}^{n-1})^T\mathbf{C}^T]$, repeating this procedure with notational changes proves that $\mathbf{V}_2^T\mathbf{A}^T\mathbf{V}_1 = [\mathbf{0}]$. The transpose of this gives the desired result.

**11.23** **(a)** Find the Kalman controllable canonical form for the system of Problem 11.21.

**(b)** Find the Kalman observable canonical form for the system of Problem 11.21.

**(a)** The controllability matrix of criterion 2 is

$$\mathbf{P} = \left[\begin{array}{cc|cc|cc} 1 & 1 & -3 & -5 & 9 & 25 \\ 1 & -1 & -3 & 5 & 9 & -25 \\ 1 & 0 & -3 & 0 & 9 & 0 \end{array}\right]$$

Using **QR** decomposition, this matrix can be expressed as

$$\mathbf{P} = \begin{bmatrix} 0.57735 & 0.707107 \\ 0.57735 & -0.707107 \\ 0.57735 & 0 \end{bmatrix} \begin{bmatrix} 1.732 & 0 & -5.19615 & 0 & 15.588 & 0 \\ 0 & 1.4142 & 0 & -7.07107 & 0 & 35.355 \end{bmatrix}$$

This shows that $\text{rank}(\mathbf{P}) = 2$, and the system is uncontrollable. The two-dimensional controllable subspace is spanned by the columns of $\mathbf{T}_1$, the first factor in the preceding decomposition. A third orthogonal vector makes up $\mathbf{T}_2$ and is found to be $[-0.40825 \quad -0.40825 \quad 0.816496]^T$. The orthogonal transformation $\mathbf{T} = [\mathbf{T}_1 \mid \mathbf{T}_2]$ then gives

$$\dot{\mathbf{w}} = \mathbf{T}^T\mathbf{ATw} + \mathbf{T}^T\mathbf{Bu}$$

$$= \left[\begin{array}{cc|c} -3 & 0 & 5.6569 \\ 0 & -5 & 6.9282 \\ \hline 0 & 0 & -1 \end{array}\right]\mathbf{w} + \left[\begin{array}{cc} 1.73205 & 0 \\ 0 & 1.4142 \\ \hline 0 & 0 \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y} = \mathbf{CTw} = \left[\begin{array}{cc|c} 0 & 0 & 2.4495 \\ 0.57735 & 0 & -1.6330 \end{array}\right]\mathbf{w}$$

This form verifies that the system is not controllable. It is stabilizable, since the uncontrollable mode has its eigenvalue located at $-1$, and thus is stable.

**(b)** The observability matrix of criterion 2 is

$$\mathbf{Q} = \left[\begin{array}{cc|cc|cc} -1 & 1 & 1 & -3 & -1 & 9 \\ -1 & 1 & 1 & -3 & -1 & 9 \\ 2 & -1 & -2 & 3 & 2 & -9 \end{array}\right] = \mathbf{V}_1\mathbf{R}_1$$

where

$$\mathbf{V}_1 = \begin{bmatrix} -0.40825 & 0.57735 \\ -0.40825 & 0.57735 \\ 0.81650 & 0.57735 \end{bmatrix}$$

and

$$\mathbf{R}_1 = \begin{bmatrix} 2.4495 & -1.6330 & -2.4495 & 4.89898 & 2.4495 & -14.6969 \\ 0 & 0.57735 & 0 & -1.73205 & 0 & 5.19615 \end{bmatrix}$$

This shows that rank(**Q**) = 2, and the system is unobservable. The two-dimensional controllable subspace is spanned by the columns of $\mathbf{V}_1$. A third orthogonal vector makes up $\mathbf{V}_2 = [-0.707107 \quad 0.707107 \quad 0]^T$. The orthogonal transformation $\mathbf{V} = [\mathbf{V}_1 \mid \mathbf{V}_2]$ then gives

$$\dot{\mathbf{v}} = \mathbf{V}^T\mathbf{A}\mathbf{V}\mathbf{v} + \mathbf{V}^T\mathbf{B}\mathbf{u}$$

$$= \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 5.6569 & -3 & 0 \\ \hline -6.9282 & 0 & -5 \end{array}\right]\mathbf{v} + \left[\begin{array}{cc} 0 & 0 \\ 1.73205 & 0 \\ \hline 0 & -1.4142 \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{v} = \left[\begin{array}{cc|c} 2.44949 & 0 & 0 \\ -1.6330 & 0.57735 & 0 \end{array}\right]\mathbf{v}$$

This form verifies that the system is not observable. It is detectable, since the unobservable mode has its eigenvalue located at $-5$ and thus is stable.

**11.24** Consider a system which has the **B** and **C** matrices of Example 11.7 and has

$$\mathbf{A} = \begin{bmatrix} -6 & -3 & -5 \\ 0 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

Find the Kalman controllable canonical form. Is this system controllable? Is it stabilizable?

The controllability matrix is

$$\mathbf{P} = \left[\begin{array}{cc|cc|cc} -0.66667 & 0.33333 & 1.33333 & -1.66667 & -2.66667 & 6.33333 \\ 0.33333 & -0.66667 & -0.66667 & 2.33333 & 1.33333 & -7.66667 \\ 0.33333 & 0.33333 & -0.66667 & -0.66667 & 1.33333 & 1.33333 \end{array}\right]$$

This can be decomposed into the product of

$$\mathbf{T}_1 = \begin{bmatrix} -0.81650 & 0 \\ 0.40825 & -0.707107 \\ 0.40825 & 0.707107 \end{bmatrix}$$

and

$$\mathbf{R}_1 = \begin{bmatrix} 0.81649 & -0.40824 & -1.6330 & 2.04124 & 3.2660 & -7.75671 \\ 0 & 0.70711 & 0 & -2.12132 & 0 & 6.36396 \end{bmatrix}$$

The rank of **P** is 2, so the system is not controllable. A third orthogonal basis vector is $[0.57735 \quad 0.57735 \quad 0.57735]^T$, and this is used for $\mathbf{T}_2$ in the orthogonal matrix $\mathbf{T} = [\mathbf{T}_1 \mid \mathbf{T}_2]$. Using this, the Kalman controllable canonical form is

$$\dot{\mathbf{w}} = \left[\begin{array}{cc|c} -2 & 1.73205 & 0.707107 \\ 0 & -3 & 2.4495 \\ \hline 0 & 0 & -4 \end{array}\right]\mathbf{w} + \left[\begin{array}{cc} 0.81650 & -0.40825 \\ 0 & 0.707107 \\ \hline 0 & 0 \end{array}\right]\mathbf{u}$$

and

$$\mathbf{y} = \begin{bmatrix} 1.2248 & 0.707107 & 5.1962 \\ 0 & 1.4142 & 3.4641 \end{bmatrix}\mathbf{w}$$

Since the eigenvalue of the uncontrollable mode is $-4$, this system is stabilizable.

**11.25** The state variable system matrices are

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 1 & 2 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 3 & 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix}$$

Find the Kalman observable canonical form.

The observability matrix is

$$\mathbf{Q} = \left[\begin{array}{cc|cc|cc} 1 & 3 & 2 & 6 & -2 & -6 \\ 1 & 3 & 0 & 0 & -6 & -18 \\ -1 & -3 & -4 & -12 & -2 & -6 \end{array}\right] = \mathbf{V}_1 \mathbf{R}_1$$

where

$$\mathbf{V}_1 = \begin{bmatrix} 0.57735 & 0 \\ 0.57735 & -0.707107 \\ -0.57735 & -0.707107 \end{bmatrix}$$

and

$$\mathbf{R}_1 = \begin{bmatrix} 1.723 & 5.196 & 3.464 & 10.392 & -3.464 & -10.392 \\ 0 & 0 & 2.828 & 8.485 & 5.657 & 16.971 \end{bmatrix}$$

Since $\mathbf{Q}$ has rank 2, the system is not observable. By augmenting $\mathbf{V}_1$ with a third orthonormal column, $[0.816496 \quad -0.408248 \quad 0.408248]^T$, the requested canonical form is found to be

$$\dot{\mathbf{v}} = \left[\begin{array}{cc|c} 2 & 1.6330 & 0 \\ -3.6742 & 0 & 0 \\ \hline 0.7011 & -1.1547 & 0 \end{array}\right] \mathbf{v} + \left[\begin{array}{cc} 1.1547 & 0 \\ -4.2426 & -2.1213 \\ \hline 1.6330 & 1.2247 \end{array}\right] \mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cc|c} 1.7321 & 0 & 0 \\ 5.1962 & 0 & 0 \end{array}\right] \mathbf{v}$$

The eigenvalue of the unobservable mode is at zero. Note that even though the second mode, $\mathbf{v}_2$, does not directly affect $\mathbf{y}$, it does so indirectly by virtue of its coupling into $\mathbf{v}_1$, and it is observable.

## PROBLEMS

**11.26** A continuous-time system is represented by $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ -4 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{C} = [1 \quad 1]$. Is this system completely controllable and completely observable?

**11.27** Investigate the controllability properties of time-invariant systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ if $u(t)$ is a scalar, and

**(a)** $\mathbf{A} = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; **(b)** $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;

**(c)** $\mathbf{A} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

**11.28** Investigate the controllability and observability of the following systems. Note the results are unaffected by whether or not the system has a nonzero $\mathbf{D}$ matrix.

(a) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}, \quad \mathbf{C} = [1 \ \ 0 \ \ 0]$

(c) $\mathbf{A} = \begin{bmatrix} -1 & 3 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{C} = \frac{1}{25}\begin{bmatrix} -3 & -4 & 3 & 250 \\ 4 & -3 & -4 & 100 \end{bmatrix}$

**11.29** A factor in determining useful life of a flexible structure, such as a ship, a tall building, or a large airplane, is the possibility of fatigue failures due to structural vibrations. Each vibration mode is described by an equation of the form $m\ddot{x} + kx = u(t)$, where $u(t)$ is the input force. Is it possible to find an input which will drive both the deflection $x(t)$ and the velocity $\dot{x}(t)$ to zero in finite time for arbitrary initial conditions?

**11.30** Investigate the controllability and observability of the mechanical system of Figure 11.7. Use $x_1$ and $x_2$ as state variables, $u(t)$ as the input force, and $y(t) = x_1(t)$ as the output. Assume the masses $m_1$ and $m_2$ are negligible [5].

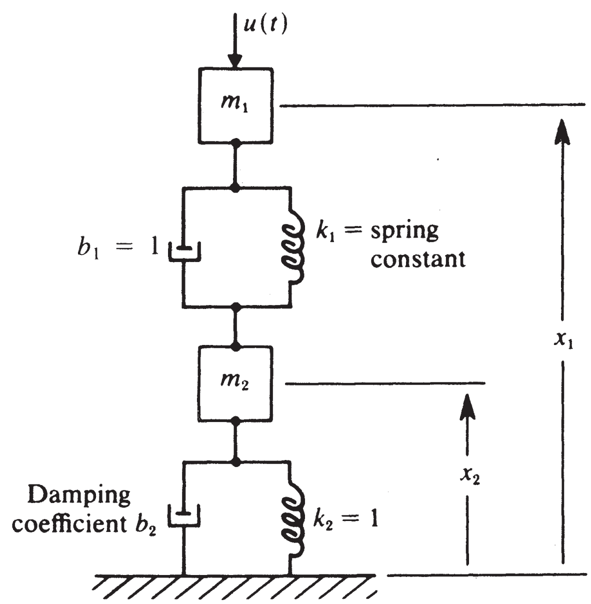



**Figure 11.7**

**11.31** Is the system of Figure 11.8 completely controllable and completely observable?

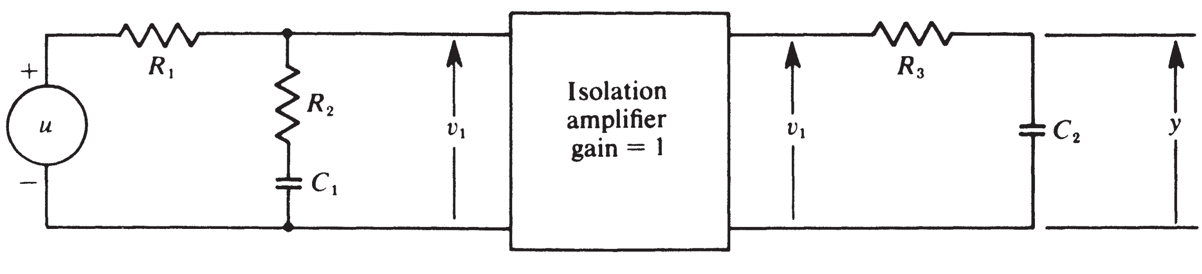



**Figure 11.8**

**11.32** Determine whether the circuits of Figure 11.9 are completely controllable and completely observable.

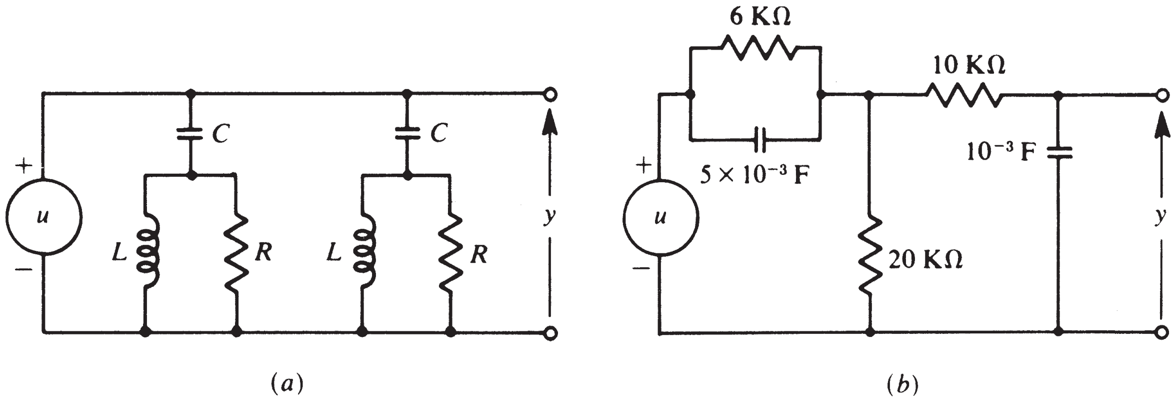



**Figure 11.9**

**11.33** The criterion for complete controllability is often stated as the requirement that

$$\mathbf{G}'(t_1, t_0) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, t)\mathbf{B}(t)\mathbf{B}^T(t)\mathbf{\Phi}^T(t_0, t)\, dt$$

be positive definite for some finite $t_1 > t_0$. Show that this is equivalent to the controllability criterion 3, provided that the system matrices **A** and **B** are real.

**11.34** Is the system of Problem 11.25 controllable?

**11.35** Change the output matrix of Problem 11.25 to $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and find the new Kalman observable canonical form. Is the system observable?

**11.36** Consider a system which has the matrix **A** of Problem 11.24 along with the matrices **B** and **C** of Example 11.8.

**(a)** Find the Kalman controllable canonical form.

**(b)** Find the Kalman observable canonical form.

# 12

# The Relationship Between State Variable and Transfer Function Description of Systems

## 12.1 INTRODUCTION

Classical control theory makes extensive use of input-output transfer function models to describe physical systems. When such a system has multiple inputs and/or multiple outputs, a matrix array of transfer function elements $H_{ij}$ can be used to relate the $j$th input to the $i$th output. Since transfer functions are largely restricted to linear constant coefficient systems, this chapter is similarly restricted.

The objective of this chapter is to explore the relationships between the transfer function matrix and the state variable model for the same system. Treatment of continuous-time and discrete-time systems proceed with only minor notational differences.

## 12.2 TRANSFER FUNCTION MATRICES FROM STATE EQUATIONS

For a given state variable model there is one unique transfer function matrix. The most general state space description of a linear, constant system with $r$ inputs $\mathbf{u}(t)$, $m$ outputs $\mathbf{y}(t)$, and $n$ state variables $\mathbf{x}(t)$ is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{12.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{12.2}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices of dimensions $n \times n$, $n \times r$, $m \times n$, and $m \times r$, respectively. The Laplace transforms of Eqs. (*12.1*) and (*12.2*) are

$$s\mathbf{x}(s) - \mathbf{x}(t=0) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s)$$

$$\mathbf{y}(s) = \mathbf{C}\mathbf{x}(s) + \mathbf{D}\mathbf{u}(s) \tag{12.3}$$

As usual when dealing with transfer functions, the initial conditions $\mathbf{x}(t=0)$ will be ignored. Solving for $\mathbf{x}(s)$ gives

$$\mathbf{x}(s) = [s\mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}\mathbf{u}(s)$$

Using this result leads to Eq. (*12.4*), the input-output relationship for the transformed variables:

$$\mathbf{y}(s) = \{\mathbf{C}[s\mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}\}\mathbf{u}(s) \tag{12.4}$$

The $m \times r$ matrix which premultiplies $\mathbf{u}(s)$ is the *transfer matrix* $\mathbf{H}(s)$,

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \tag{12.5}$$

This result was also given as Eq. (*4.5*). The discrete-time equivalent of this result was derived in Problem 9.26. A typical element $H_{ij}(s)$ of $\mathbf{H}(s)$ is the transfer function relating the $j$th input component $u_j$ to the $i$th output component $y_i$, $H_{ij}(s) = y_i(s)/u_j(s)$, with all inputs equal to zero except $u_j$ and all initial conditions being zero.

It is convenient to define $\Delta(s) = |s\mathbf{I}_n - \mathbf{A}|$.‡ Then $[s\mathbf{I}_n - \mathbf{A}]^{-1} = \text{Adj}[s\mathbf{I}_n - \mathbf{A}]/\Delta(s)$ and the transfer matrix can be rewritten as

$$\mathbf{H}(s) = \frac{\mathbf{C}\ \text{Adj}[s\mathbf{I}_n - \mathbf{A}]\mathbf{B} + \mathbf{D}\Delta(s)}{\Delta(s)} \tag{12.6}$$

Each element of the adjoint matrix is a polynomial in $s$ of degree less than or equal to $n-1$. Since $\Delta(s)$ is an $n$th degree polynomial in $s$, each element $H_{ij}(s)$ is a ratio of polynomials in $s$, with the degree of the denominator at least as great as the degree of the numerator. Such an $\mathbf{H}$ matrix is called a *proper* rational matrix (loosely, at least as many poles as zeros). If $\mathbf{D} = [\mathbf{0}]$, then the numerator of every element in $\mathbf{H}(s)$ will be of degree less than the denominator. In this case, $\mathbf{H}(s)$ is a *strictly proper* rational matrix (loosely, more poles than zeros). Clearly, from Eq. (*12.6*),

$$\mathbf{D} = \lim_{s\to\infty} \mathbf{H}(s) \tag{12.7}$$

Equation (*12.5*) indicates that a proper transfer function matrix can be written as the sum of a strictly proper transfer function matrix plus a constant matrix **D.**

When a state variable description of a system is given, the unique corresponding transfer function matrix is given by Eq. (*12.5*) or (*12.6*). The reverse process is *not* unique. If the matrix $\mathbf{H}(s)$ (or $\mathbf{H}(z)$) is given, the matrix $\mathbf{D}$ is immediately given by Eq. (*12.7*). However, the determination of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ from a knowledge of $\mathbf{H}$ is not a unique process. Many different state variable realizations $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ yield the same transfer function. The determination of state variable models from a given transfer function is treated in the following sections.

**EXAMPLE 12.1** A single-input, single-output system is described in state variable form with

$$\mathbf{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0], \qquad D = 0$$

‡ Recall that in Chapter 7, page 246, $\Delta(\lambda) \triangleq |\mathbf{A} - \mathbf{I}\lambda|$. The above definition is more convenient for this chapter and the next. They differ by the inconsequential factor $(-1)^n$.

Then

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\begin{bmatrix} (s+2)^2 & s+2 & 1 \\ 0 & (s+2)(s+5) & (s+5) \\ 0 & 0 & (s+2)(s+5) \end{bmatrix}}{(s+5)(s+2)^2}$$

From Eq. (*12.5*),

$$H(s) = \frac{s+3}{(s+2)^2(s+5)}$$

This agrees with the results of Example 3.8, where the same problem was worked in reverse order. Note that $\lim_{s \to \infty} H(s) = 0$ as expected, since $D = 0$ in this case. ■

## 12.3 STATE EQUATIONS FROM TRANSFER MATRICES: REALIZATIONS

Several methods of selecting state variables were presented in Sec. 3.4. These methods are entirely satisfactory for single-input, single-output systems but were also applied to multiple input-output systems without dwelling on the consequences. Interconnections of such systems were discussed in Sec. 3.5. Multivariable systems are now discussed in detail, and the consequences of previous methods will be explored.

If a pair of equations (*12.1*) and (*12.2*) can be found which has a specified transfer matrix $\mathbf{H}(s)$, then those equations are called a *realization* of $\mathbf{H}(s)$. For brevity, it is common to refer to the matrices $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ as the realization of $\mathbf{H}(s)$. Specifying $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is equivalent to giving a prescription for synthesizing a system with a given transfer matrix. Figure 3.3, for example, can be mechanized with physical devices such as operational amplifiers and other standard analog computer equipment. The discrete equivalent in Figure 3.4 would most often use digital hardware.

Perhaps the simplest method of realizing a given transfer matrix $\mathbf{H}(s)$ is illustrated in Figure 12.1. Each scalar component $H_{ij}(s)$ is considered individually, and the methods discussed for scalar transfer function in Sec. 3.4 can be used on each. This method is directly related to the techniques given earlier (Sec. 3.5) for dealing with composite systems.

**EXAMPLE 12.2** A system with two inputs and two outputs has the transfer matrix

$$\mathbf{H}(s) = \begin{bmatrix} 1/(s+1) & 2/[(s+1)(s+2)] \\ 1/[(s+1)(s+3)] & 1/(s+3) \end{bmatrix}$$

Using the approach suggested in Figure 12.1, four separate scalar transfer functions are simulated as shown in Figure 12.2.

Using the state variables $x_1$ through $x_6$ as defined in Figure 12.2, a realization of $\mathbf{H}(s)$ is given by

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

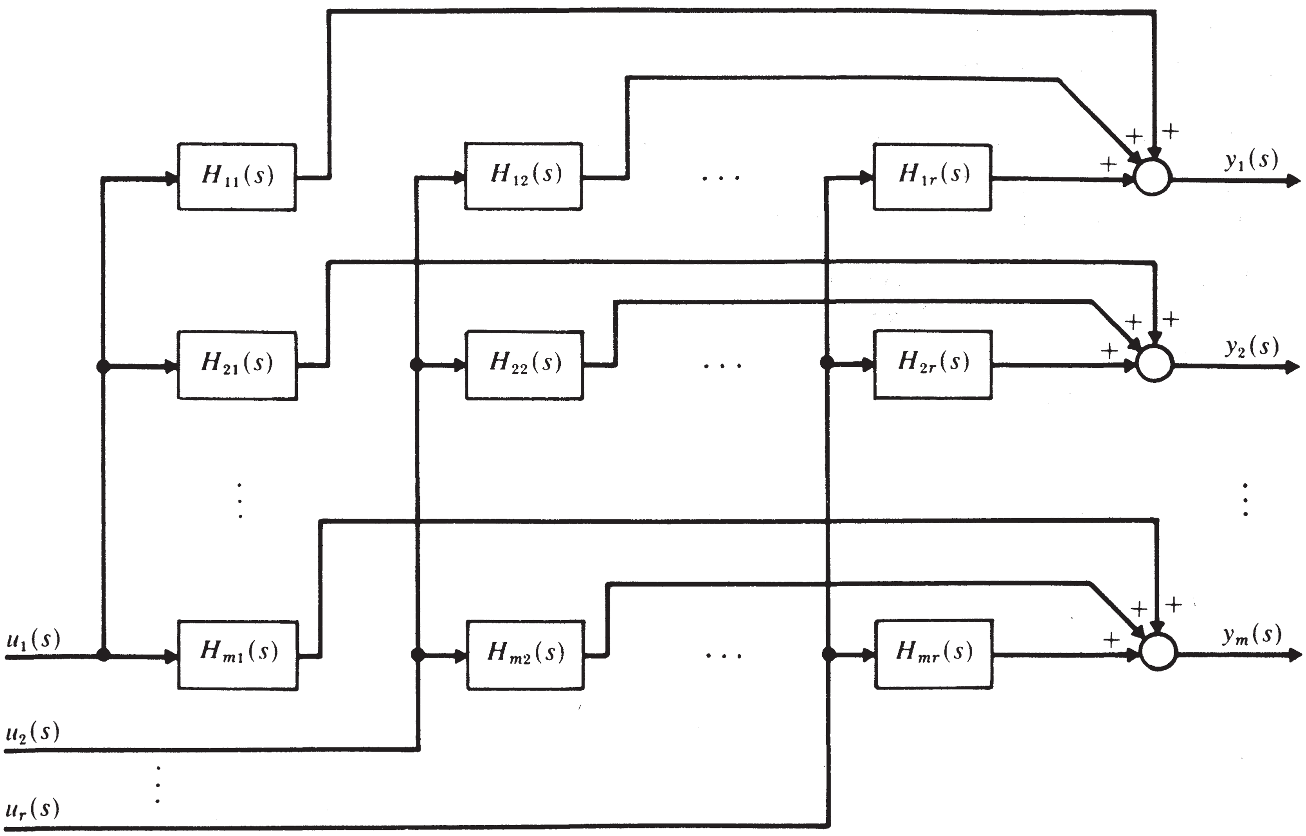



**Figure 12.1** Block diagram of $H(s)$ without internal coupling.

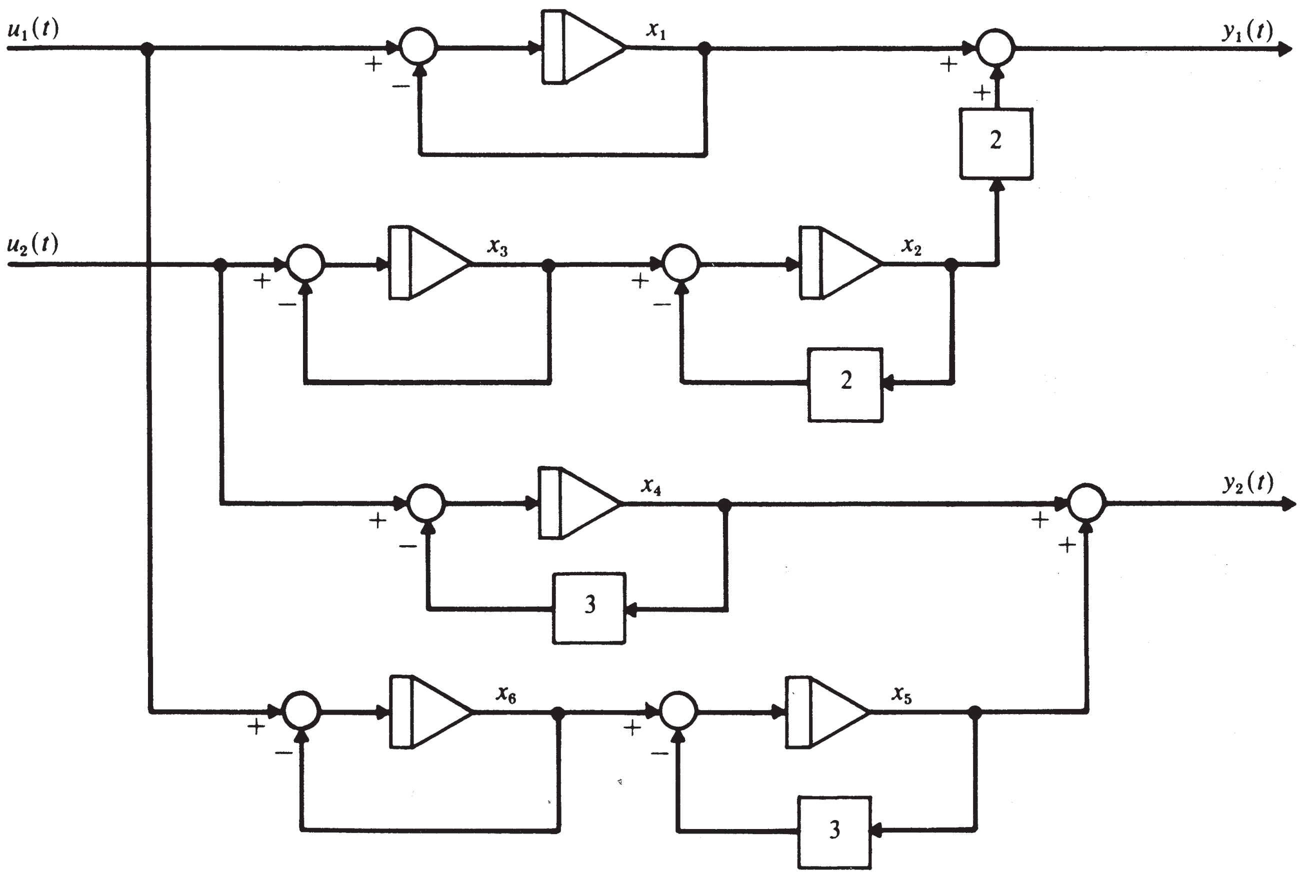



**Figure 12.2** Realization for Example 12.2.

Application of Eq. (*12.5*) shows that this sixth-order state representation does have the specified input-output transfer matrix. ■

The method just presented yields a realization for any transfer matrix. The realization of $\mathbf{H}(s)$ is not unique. If one $n$th-order realization of $\mathbf{H}(s)$ can be found, then there are an infinite number of $n$th-order realizations.

**EXAMPLE 12.3** Let $\{\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1\}$ be a realization of $\mathbf{H}(s)$, where $\mathbf{A}_1$ is an $n \times n$ matrix. Let $\mathbf{M}$ be any constant nonsingular $n \times n$ matrix and define a new state vector $\mathbf{x}_2$ by the transformation (change of basis in the state space $\Sigma$) $\mathbf{x}_1 = \mathbf{M}\mathbf{x}_2$. Now

$$\dot{\mathbf{x}}_1 = \mathbf{A}_1\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}, \; \mathbf{y} = \mathbf{C}_1\mathbf{x}_1 + \mathbf{D}_1\mathbf{u}$$

become

$$\mathbf{M}\dot{\mathbf{x}}_2 = \mathbf{A}_1\mathbf{M}\mathbf{x}_2 + \mathbf{B}_1\mathbf{u}, \; \mathbf{y} = \mathbf{C}_1\mathbf{M}\mathbf{x}_2 + \mathbf{D}_1\mathbf{u}.$$

Defining $\mathbf{A}_2 = \mathbf{M}^{-1}\mathbf{A}_1\mathbf{M}$, $\mathbf{B}_2 = \mathbf{M}^{-1}\mathbf{B}_1$, $\mathbf{C}_2 = \mathbf{C}_1\mathbf{M}$, and $\mathbf{D}_2 = \mathbf{D}_1$ gives

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2\mathbf{x}_2 + \mathbf{B}_2\mathbf{u}, \; \mathbf{y} = \mathbf{C}_2\mathbf{x}_2 + \mathbf{D}_2\mathbf{u}.$$

Then

$$\mathbf{C}_2[s\mathbf{I} - \mathbf{A}_2]^{-1}\mathbf{B}_2 + \mathbf{D}_2 = \mathbf{C}_1\mathbf{M}[s\mathbf{I} - \mathbf{M}^{-1}\mathbf{A}_1\mathbf{M}]^{-1}\mathbf{M}^{-1}\mathbf{B}_1 + \mathbf{D}_1 = \mathbf{C}_1[s\mathbf{I} - \mathbf{A}_1]^{-1}\mathbf{B}_1 + \mathbf{D}_1$$

Thus $\{\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1\}$ and $\{\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2, \mathbf{D}_2\}$ are two different realizations. Eq. (*12.7*) indicates that $\mathbf{D}$ is determined only by $\mathbf{H}(s)$. The choice of basis vectors for $\Sigma$ does not influence $\mathbf{D}$. Hence $\mathbf{D}_1 = \mathbf{D}_2$. ■

This establishes the nonuniqueness of system representations. A more interesting result is that $\dim(\Sigma)$ is not uniquely defined by $\mathbf{H}(s)$.

## 12.4 DEFINITION AND IMPLICATION OF IRREDUCIBLE REALIZATIONS

The methods of Sec. 12.3 will produce state variable models for either continuous-time or discrete-time systems from a transfer function matrix. However, this method ignores any commonalities that may exist among the various elements $H_{ij}$. Such commonalities *may* allow a sharing of integrators or delay elements by more than one element $H_{ij}$. When these opportunities are ignored, the resulting state equations will be of an unnecessarily high order. The state space is thus of unnecessarily high dimension.

**Definition 12.1.** Of all the possible realizations of $\mathbf{H}(s)$, $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is said to be an *irreducible* (or minimum) *realization* if the associated state space has the smallest possible $\dim(\Sigma)$.

***Important Fact***

A minimal realization is both completely controllable and completely observable.

In the case of a scalar transfer function, the minimum dimension required is equal to the order of the denominator of the transfer function after all common pole-zero cancellations are made. The corresponding result for transfer matrices is not so obvious and is discussed in Sec. 12.5.

Whenever a pole-zero pair is cancelled from a transfer function, the system mode associated with the cancelled pole will not be evident in the state equations. Yet, in order to achieve an irreducible realization, these cancellations must be made. What is the implication of irreducible realizations in view of this apparent loss of information about the system?

An irreducible realization is a system, of minimal dimension, which is capable of reproducing the measurable relationships between inputs and outputs. This assumes that the system is originally relaxed, i.e., the initial state vector is zero. For this reason a system and its irreducible realization are said to be *zero state equivalent.* Often, the only knowledge about a system is the information which is obtainable from measurements of inputs and outputs. Transfer functions and matrices can be experimentally determined from these measurements. The irreducible realization does not cause loss of information in this case, because nothing was known about the system's internal structure in the first place. It is not claimed that an irreducible realization is the best description of the internal structure of a system.

If the internal structure of a system is known (for example, a circuit diagram), then an irreducible realization may not be the appropriate one to use. State equations can always be written directly from the system's linear graph, as described in Chapter 3. The realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ obtained in this manner is the most complete system description, whether it is irreducible or not. Let this realization have $\dim(\Sigma) = n$. The transfer matrix $\mathbf{H}(s)$ can be found using Eq. (*12.5*). Starting with $\mathbf{H}(s)$, an irreducible realization $\{\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}\}$ can be found. If $\mathbf{A}_1$ is $n_1 \times n_1$, with $n_1 < n$, then information about $n - n_1$ modes would be lost by using the irreducible realization (see Problems 12.4 and 12.6). In Example 12.5, the irreducible realization gives no indication that the system is actually unstable. If $n_1 = n$, the system is said to be *completely characterized* by $\mathbf{H}(s)$. In general, transfer matrices, and irreducible realizations of them, describe only that subsystem which is both completely controllable and completely observable (see Problems 12.12 and 12.13). The incompleteness of a transfer matrix description is another reason for preferring state space techniques.

**EXAMPLE 12.4** The input-output equations for a system are

$$\begin{aligned} \dot{y}_1 + 2(y_1 - y_2) &= 4u_1 - u_2 \\ \dot{y}_2 + 3(y_2 - y_1) &= 4u_1 - u_2 \end{aligned} \qquad (12.8)$$

Using the simulation diagram of Figure 12.3, the state equations (*12.9*) and (*12.10*) are obtained:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (12.9)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (12.10)$$

The state vector has dimension 2. The transfer matrix can be derived directly from Eq. (*12.8*) by Laplace transforming and a matrix inversion,

$$\mathbf{H}(s) = \begin{bmatrix} 4/s & -1/s \\ 4/s & -1/s \end{bmatrix}$$

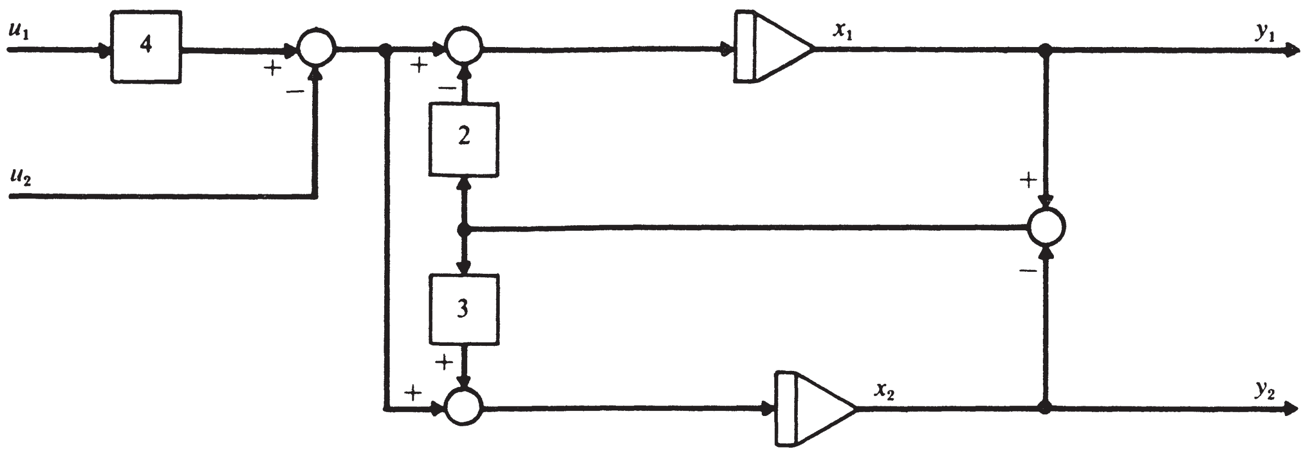


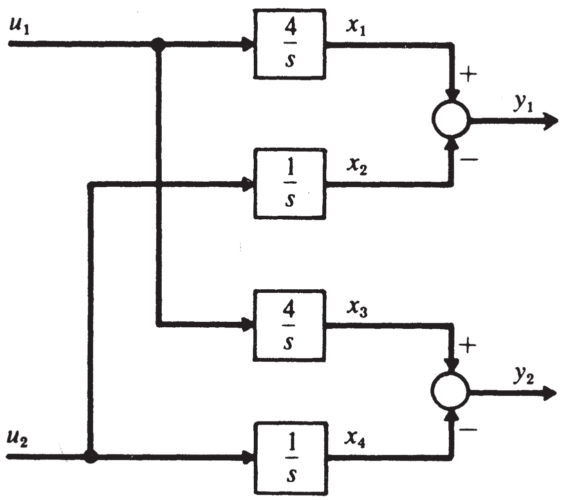

**Figure 12.3**

Assume now that $\mathbf{H}(s)$ is the only information known about the system. The method of Sec. 12.3 leads to the fourth-order system realization shown in Figure 12.4.

Two other realizations for $\mathbf{H}(s)$ are shown in Figures 12.5 and 12.6.

None of these three realizations resembles the original system. The one shown in Figure 12.5 agrees insofar as $\dim(\Sigma) = 2$, but the mode corresponding to the pole $(s + 5)$ is not present. The minimum realization, shown in Figure 12.6 has $n = 1$. The original second-order model was not of minimal order and therefore fails to have either one or both of the properties of controllability and observability. Examination shows that the original system is observable but not cotrollable. If the original system is decomposed into subsystems as in Problem 11.21, page 397, this is verified. The only mode that is completely controllable and completely observable is the one associated with the eigenvalue (pole) $s = 0$. ■

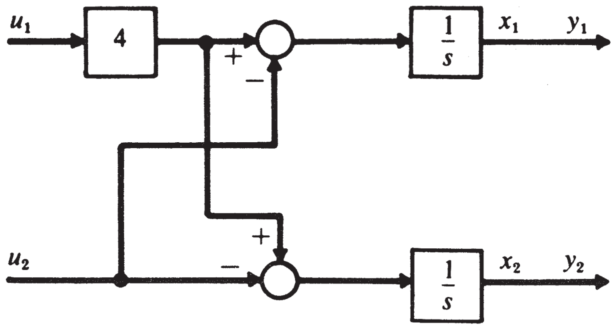
Consider a scalar transfer function, given in the form of Eq. (*12.6*). The eigenvalues of the matrix $\mathbf{A}$ are roots of $\Delta(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0$. The poles of $\mathbf{H}(s)$ will be the roots of $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = 0$, *unless* some of the factors of $\Delta(s)$ are cancelled by terms in the numerator of $\mathbf{H}(s)$. Thus the poles of a transfer function will always be eigenvalues of the system matrix $\mathbf{A}$. Eigenvalues of $\mathbf{A}$ need not always be poles of the transfer function because cancellations may occur. If a system is completely characterized by its




$$\mathbf{A} = [\mathbf{0}_{4\times4}], \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$$

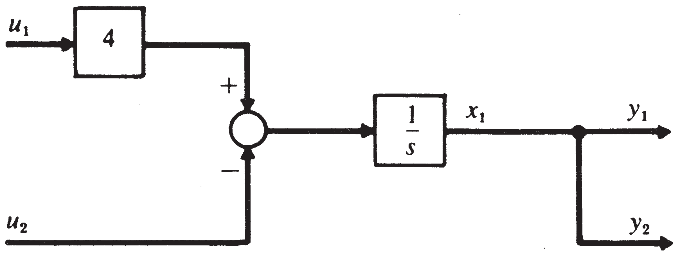
**Figure 12.4**




$$\mathbf{A} = [\mathbf{0}_{2\times 2}], \quad \mathbf{B} = \begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$$

**Figure 12.5**




$$\mathbf{A} = 0, \quad \mathbf{B} = [4 \quad -1]$$

$$\mathbf{C} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{D} = [0]$$

**Figure 12.6**

transfer function (equivalently if **A** is associated with an irreducible realization of **H**($s$)), then the poles and the eigenvalues are the same. This will be true if the system is completely controllable and completely observable. A similar relationship exists between the eigenvalues of **A** and the roots of the "denominator" of a transfer matrix. This assumes a suitable definition for a matrix denominator (see Sec. 12.6).

## 12.5 THE DETERMINATION OF IRREDUCIBLE REALIZATIONS

There are two generic approaches to finding a minimal realization for a given matrix **H**($s$). The first consists of finding a state realization by any available means and then reducing it, if necessary, in the state space domain. Methods of Chapter 3 or Sec. 12.4 may be used to find the original realization. The reduction process consists of using a suitable decomposition in order to determine the controllable and observable subsystem. This section presents Jordan form and Kalman canonical form approaches to the decomposition and reduction process.

The second generic approach consists of appropriate reduction of **H**($s$) (or **H**($z$)) first and then finding a state variable realization of the reduced transfer function. In the single-input, single-output case, reduction of **H** simply consists of canceling any common pole-zero pairs. Reduction of a matrix **H**($s$) is not so obvious. Sec. 12.6 gives a procedure which uses elementary (polynomial) operations on the matrix fraction description (MFD) in order to achieve the required reduction.

### 12.5.1. Jordan Canonical Form Approach

Any state variable model can be decomposed into four subsystems which are (1) completely controllable and completely observable, (2) completely controllable but not observable, (3) completely observable but not controllable, and (4) neither controllable nor observable (see Problems 11.18 through 11.21). The subsystem which is completely controllable and completely observable constitutes an irreducible realization of $\mathbf{H}(s)$ [1, 2].

**EXAMPLE 12.5** Suppose that a system realization has been obtained, and it has been transformed into the Jordan canonical form

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Because this Jordan form system has distinct eigenvalues, controllability criterion 1 of Sec. 11.3 applies. Since row 3 of $\mathbf{B}_n$ is all zero, mode $q_3$ is uncontrollable. The second column of $\mathbf{C}_n$ is all zero, so $q_2$ is an unobservable mode. Modes $q_2$ and $q_3$ have no effect on the input-output behavior and can be dropped. An irreducible realization is

$$\dot{q}_1 = -5q_1 + [1 \quad 0]\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_1 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

The transfer matrix is $\mathbf{H}(s) = \begin{bmatrix} 1/(s+5) & 0 \\ 0 & 1 \end{bmatrix}$ for both forms of the state equations. ∎

**EXAMPLE 12.6** A system realization is found to be

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + [\mathbf{D}]\mathbf{u}$$

Even though the coefficient matrix $\mathbf{A}$ is diagonal, controllability and observability criteria 1 of Sec. 11.3 cannot be used because of the repeated eigenvalues of $\mathbf{A}$. This realization is not controllable because rows 1 and 3 of $\mathbf{B}$ are not independent (see Problem 11.16). It is not observable because column 3 of $\mathbf{C}$ is a linear combination of columns 1 and 2 (see Problem 11.17). In this simple case direct manipulations show that two new states, which are both controllable and observable, can be formed from the original $q_1 + q_3$ and $q_2 + q_3$.

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} q_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} q_3 + \mathbf{Du} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (q_1 + q_3) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (q_2 + q_3) + \mathbf{Du}$$

Because of the linear dependence of the columns of $\mathbf{C}_n$, it is possible to define two new state variables $\tilde{q}_1 = q_1 + q_3$ and $\tilde{q}_2 = q_2 + q_3$. Then $\dot{\tilde{q}}_1 = \dot{q}_1 + \dot{q}_3 = -a(\tilde{q}_1) + 2u_1$ and $\dot{\tilde{q}}_2 = -a\tilde{q}_2 + u_1 + u_2$, or

$$\dot{\tilde{\mathbf{q}}} = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \tilde{\mathbf{q}} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{\mathbf{q}} + \mathbf{D}\mathbf{u}$$

This reduced system is completely controllable and observable, hence irreducible. The transfer matrix for it and for the original third-order system is

$$\mathbf{H}(s) = \begin{bmatrix} 1/(s+a) & 1/(s+a) \\ 2/(s+a) & 0 \end{bmatrix} + \mathbf{D}$$ ■

Rather than first finding some arbitrary realization and then using the modal matrix in a similarity transformation (see Secs. 9.4 and 9.10), it is possible to obtain a Jordan form realization directly from $\mathbf{H}(s)$ or $\mathbf{H}(z)$. This is done by expanding each $H_{ij}(s)$ element, using partial fractions. Regroupng terms gives a matrix version of the partial fraction expansion of $\mathbf{H}(s)$. A completely controllable realization can be written directly from this.

**EXAMPLE 12.7** The transfer matrix of Example 12.2 is reconsidered. Using partial fractions, it can be written as

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+1} - \dfrac{2}{s+2} \\ \dfrac{1/2}{s+1} - \dfrac{1/2}{s+3} & \dfrac{1}{s+3} \end{bmatrix} = \frac{\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}}{s+2} + \frac{\begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}}{s+3}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \quad 2] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[\frac{1}{2} \quad 0]}{s+1} + \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}[0 \quad -2]}{s+2} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}[-\frac{1}{2} \quad 1]}{s+3}$$

In this case $\lim_{s\to\infty} \mathbf{H}(s) = [\mathbf{0}] = \mathbf{D}$, so that, from Eq. (*12.5*), $\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$. Comparing this with the expanded form gives

$$\mathbf{H}(s) = \left[\begin{array}{cc|c|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} \dfrac{1}{s+1} & & & \mathbf{0} \\ & \dfrac{1}{s+1} & & \\ & & \dfrac{1}{s+2} & \\ \mathbf{0} & & & \dfrac{1}{s+3} \end{bmatrix} \left[\begin{array}{cc} 1 & 2 \\ \frac{1}{2} & 0 \\ \hline 0 & -2 \\ \hline -\frac{1}{2} & 1 \end{array}\right]$$

In this form the matrices $\mathbf{C}$, $[s\mathbf{I} - \mathbf{A}]^{-1}$, and $\mathbf{B}$ are clearly evident. Since $[s\mathbf{I} - \mathbf{A}]^{-1}$ is diagonal, $\mathbf{A}$ is also diagonal and a realization of $\mathbf{H}(s)$ is given by

$$\mathbf{A} = \text{diag}[-1, -1, -2, -3], \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 0 \\ 0 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = [\mathbf{0}]$$

This is an irreducible realization, since it is completely controllable and observable. This realization is completely controllable because the rows of $\mathbf{B}$ associated with the simple eigenvalues $-2$ and $-3$ are both nonzero and the two rows of $\mathbf{B}$ associated with the two $1 \times 1$ blocks for the eigenvalue $-1$ are independent. The controllability is ensured by the construction process. The observability is not guaranteed by the process in cases where multiple poles (terms such as $1/(s+p_i)^m$) occur. In this example the realization is observable, since the columns of $\mathbf{C}$ satisfy the independence requirements of Problem 11.17. Therefore this realization is irreducible. The required state space satisfies $\dim(\Sigma)=4$, and not 6 as in Example 12.2. ■

The method illustrated by Example 12.7 consists of

1. Writing $\mathbf{H}(s)$ in a matrix form of partial fraction expansion:

$$\mathbf{H}(s) = \mathbf{D} + \sum_{i=1}^{k} \frac{\mathbf{N}_i}{s+p_i}$$

(note that terms like $1/(s+p_i)^m$ are *not* considered here, but will be later);

2. Determining the rank $r_i$ of each of the $\mathbf{N}_i$ matrices;
3. Factoring each $\mathbf{N}_i$ into the sum of $r_i$ outer products $\mathbf{N}_i = \sum_{j=1}^{r_i} \mathbf{c}_{ij}\rangle\langle\mathbf{b}_{ij}$;

(The double subscripts on the column vectors $\mathbf{c}$ and the row vectors $\mathbf{b}^T$ are used to distinguish which pole they are associated with. This distinction is not always necessary and a single subscript then suffices.)

4. Setting

$$\mathbf{A} = \operatorname{diag}[-p_1\mathbf{I}_{r_1}, -p_2\mathbf{I}_{r_2}, \ldots, -p_k\mathbf{I}_{r_k}]$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_{11}^T \\ \vdots \\ \mathbf{b}_{1r_1}^T \\ \hline \mathbf{b}_{21}^T \\ \vdots \\ \mathbf{b}_{2r_2}^T \\ \hline \vdots \\ \hline \mathbf{b}_{k1}^T \\ \vdots \\ \mathbf{b}_{kr_k}^T \end{bmatrix}, \qquad \mathbf{C} = [\mathbf{c}_{11} \;\cdots\; \mathbf{c}_{1r_1} \mid \mathbf{c}_{21} \;\cdots\; \mathbf{c}_{2r_2} \mid \cdots \mid \mathbf{c}_{k1} \;\cdots\; \mathbf{c}_{kr_k}]$$

The dimension of $\mathbf{A}$ is $n \times n$, where $n = r_1 + r_2 + \cdots + r_k$. Since $\mathbf{A}$ is diagonal, it is in Jordan form. There is just a single $1 \times 1$ Jordan block associated with those poles $p_i$ for which $r_i = 1$. The controllability and observability of these modes is ensured by criteria 1 of Sec. 11.3, since the rows $\mathbf{b}_{i1}^T$ and columns $\mathbf{c}_{i1}$ are not zero. In general, there will be $r_i$ $1 \times 1$ Jordan blocks associated with a given pole $p_i$. These modes will be both controllable and observable because rank $\mathbf{N}_i = r_i$ implies that the $r_i$ rows $\{\mathbf{b}_{i1}^T, \mathbf{b}_{i2}^T, \ldots, \mathbf{b}_{ir_i}^T\}$ are linearly independent and so are the $r_i$ columns $\{\mathbf{c}_{i1}, \mathbf{c}_{i2}, \ldots, \mathbf{c}_{ir_i}\}$ (see Problems 11.16 and 11.17). Since every mode of this realization is both controllable and observable, it is irreducible.

**Multiple Poles.** If any one element $H_{ij}(s)$ has a multiple pole, say $1/(s + p_1)^m$, then the matrix partial fraction expansion for $\mathbf{H}(s)$ will also contain this term, plus terms to the $m - 1, m - 2, \ldots, 1$ powers. This requires a modification of the method, which is best explained by example. The essence of the modification amounts to obeying the admonition of Sec. 3.4 against using *unnecessary* integrators when using simulation diagrams. This results in realizations with nondiagonal $\mathbf{A}$ matrices.

**EXAMPLE 12.8** Find an irreducible realization of

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{(s+2)^3(s+5)} & \dfrac{1}{s+5} \\ \dfrac{1}{s+2} & 0 \end{bmatrix}$$

Using partial fraction expansion gives

$$\mathbf{H}(s) = \frac{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix}}{(s+2)^3} + \frac{\begin{bmatrix} -\frac{1}{9} & 0 \\ 0 & 0 \end{bmatrix}}{(s+2)^2} + \frac{\begin{bmatrix} \frac{1}{27} & 0 \\ 1 & 0 \end{bmatrix}}{s+2} + \frac{\begin{bmatrix} -\frac{1}{27} & 1 \\ 0 & 0 \end{bmatrix}}{s+5}$$

$$= \frac{\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}[1 \quad 0]}{(s+2)^3} + \frac{\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}[1 \quad 0]}{(s+2)^2} + \frac{\begin{bmatrix} \frac{1}{27} \\ 1 \end{bmatrix}[1 \quad 0]}{s+2} + \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}[-\frac{1}{27} \quad 1]}{s+5}$$

There is only one $\mathbf{b}$ vector associated with the three terms involving $s + 2$, namely, $\mathbf{b}_1^T = [1 \quad 0]$. This means that these three terms can be simulated from a series connection of $1/(s + 2)$ terms with a single input $\mathbf{b}_1^T\mathbf{u}$. These terms will also form a single Jordan block. Figure 12.7 gives a simulation diagram from which state equations can be written. Note that

$$x_1(s) = \frac{1}{(s+2)^3}[1 \quad 0]\mathbf{u}(s) \qquad x_2(s) = \frac{1}{(s+2)^2}[1 \quad 0]\mathbf{u}(s)$$

$$x_3(s) = \frac{1}{(s+2)}[1 \quad 0]\mathbf{u}(s)$$

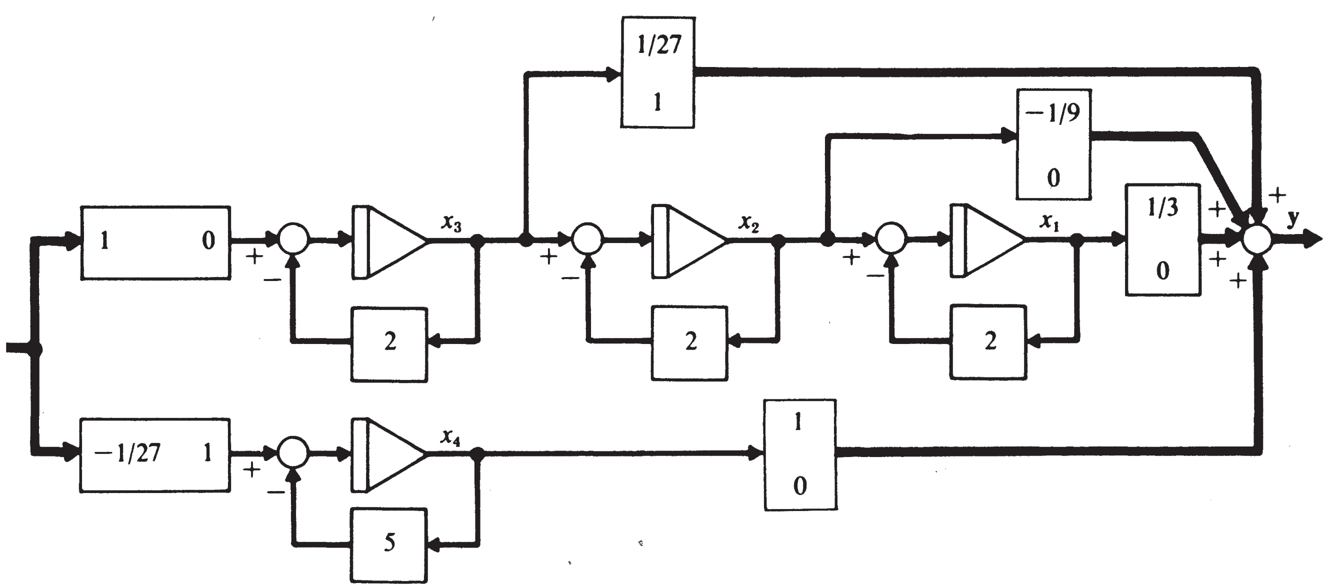



**Figure 12.7**

From Figure 12.7,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \hline \dot{x}_4\end{bmatrix} = \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & -2 & 0\\ \hline 0 & 0 & 0 & -5\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\\ \hline x_4\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ \hline -\frac{1}{27} & 1\end{bmatrix}\mathbf{u}$$

and

$$\mathbf{y} = \begin{bmatrix}\frac{1}{3}\\ 0\end{bmatrix}x_1 + \begin{bmatrix}-\frac{1}{9}\\ 0\end{bmatrix}x_2 + \begin{bmatrix}\frac{1}{27}\\ 1\end{bmatrix}x_3 + \begin{bmatrix}1\\ 0\end{bmatrix}x_4 = \left[\begin{array}{ccc|c}\frac{1}{3} & -\frac{1}{9} & \frac{1}{27} & 1\\ 0 & 0 & 1 & 0\end{array}\right]\mathbf{x}$$

This realization is both completely controllable and completely observable and is therefore irreducible. ■

For each repeated pole the previous method will yield a number of Jordan blocks equal to the number of *linearly independent* $\mathbf{b}_i$ vectors associated with that pole. The realization obtained in this manner will always be completely controllable because one of these $\mathbf{b}_i^T$ vectors will form the last row for each block in $\mathbf{B}$ associated with a Jordan block. (See Problem 11.16.) Observability cannot be ensured in advance. Depending on the example, observability may or may not result with the first realization. If it does not, then less than systematic modifications, i.e., eliminating states, will be required until observability is achieved. In general, the value of the Jordan form approach is greatly diminished in the multiple-pole case. The next two examples illustrate what can happen and show a more or less trial-and-error path to a minimal realization. Problem 12.7 presents another example, which uses a somewhat more systematic—but still tedious—approach to the reduction process. The Kalman canonical form approach of Sec. 12.5.2, by way of contrast, provides an algorithm which is easily implementable on a computer.

**EXAMPLE 12.9** Suppose

$$\mathbf{H}(s) = \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}[1\quad 0\quad 0]}{(s+p)^3} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}[0\quad 1\quad 0]}{(s+p)^3} + \frac{\begin{bmatrix}1\\0\\1\end{bmatrix}[1\quad 0\quad 0]}{(s+p)^2}$$

$$+ \frac{\begin{bmatrix}2\\5\\2\end{bmatrix}[0\quad 0\quad 1]}{(s+p)^2} + \frac{\begin{bmatrix}2\\2\\3\end{bmatrix}[-1\quad -1\quad 0]}{s+p}$$

There are three independent $\mathbf{b}_i^T$ vectors, labeled $\mathbf{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. The last $\mathbf{b}^T$ vector can be written as $\begin{bmatrix}-1\\-1\\0\end{bmatrix} = -\mathbf{b}_1 - \mathbf{b}_2$.

Defining $\mathbf{c}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix}2\\5\\2\end{bmatrix}$, and $\mathbf{c}_5 = \begin{bmatrix}2\\2\\3\end{bmatrix}$ allows $\mathbf{H}(s)$ to be written as

$$\mathbf{H}(s) = \frac{\mathbf{c}_1\rangle\langle\mathbf{b}_1}{(s+p)^3} + \frac{\mathbf{c}_3\rangle\langle\mathbf{b}_1}{(s+p)^2} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_1}{s+p} + \frac{\mathbf{c}_2\rangle\langle\mathbf{b}_2}{(s+p^3)} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_2}{s+p} + \frac{\mathbf{c}_4\rangle\langle\mathbf{b}_3}{(s+p)^2}$$

A simulation diagram for this equation is given in Figure 12.8. The three terms with input $\mathbf{b}_1^T\mathbf{u}$ are linked together in what is called a *Jordan chain.* The three associated state variables will be described by a $3 \times 3$ Jordan block. The two terms with input $\mathbf{b}_2^T\mathbf{u}$ form another Jordan chain and hence a Jordan block. Likewise, the term with input $\mathbf{b}_3^T\mathbf{u}$ leads to a third Jordan block. The dimension of a Jordan block is determined by the number of integrators required to simulate the Jordan chain.

The completely controllable realization is given by

$$\mathbf{A} = \left[\begin{array}{ccc|ccc|cc} -p & 1 & 0 & & & & & \\ 0 & -p & 1 & & & & \mathbf{0} & \\ 0 & 0 & -p & & & & & \\ \hline & & & -p & 1 & 0 & & \\ & & & 0 & -p & 1 & & \\ & \mathbf{0} & & 0 & 0 & -p & & \\ \hline & & & & & & -p & 1 \\ & & & & & & 0 & -p \end{array}\right], \quad \mathbf{B} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{b}_1^T \\ \hline \mathbf{0} \\ \mathbf{0} \\ \mathbf{b}_2^T \\ \hline \mathbf{0} \\ \mathbf{b}_3^T \end{array}\right],$$

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_3 \quad -\mathbf{c}_5 \mid \mathbf{c}_2 \quad \mathbf{0} \quad -\mathbf{c}_5 \mid \mathbf{c}_4 \quad \mathbf{0}]$$

It is *not* completely observable because $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is not a linearly independent set. (See Problem 11.17.) In fact, $\mathbf{c}_4 = 2\mathbf{c}_1 + 3\mathbf{c}_2$. Therefore, this realization can be reduced. ■

**EXAMPLE 12.10** Find an *irreducible* realization for $\mathbf{H}(s)$ of the previous example.

In order to reduce the order of the realization by one, one integrator must be eliminated from Figure 12.8 without altering the input-output characteristics, that is, $\mathbf{H}(s)$. Using $\mathbf{c}_4 = 2\mathbf{c}_1 + 3\mathbf{c}_2$ allows $\mathbf{H}(s)$ to be rewritten as

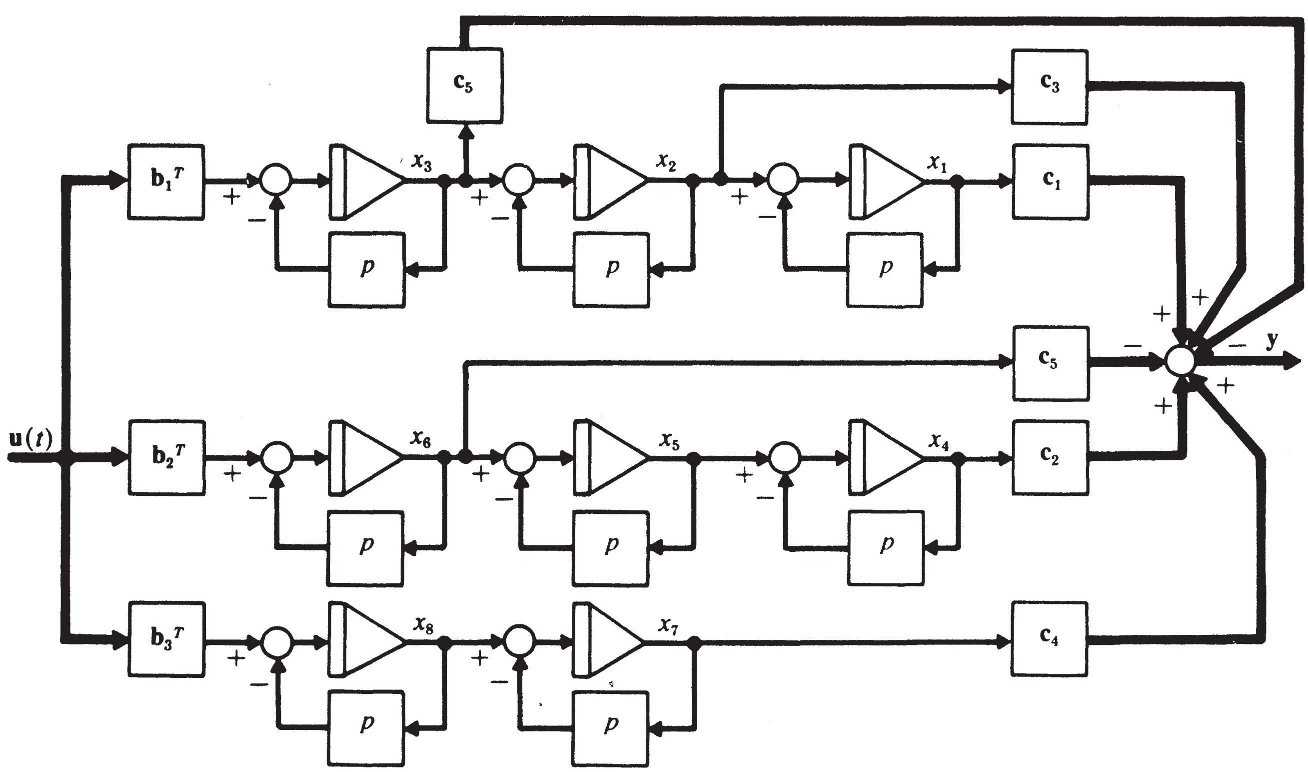



Figure 12.8

$$\mathbf{H}(s) = \frac{\mathbf{c}_1\rangle\langle[\mathbf{b}_1 + 2(s+p)\mathbf{b}_3]}{(s+p)^3} + \frac{\mathbf{c}_3\rangle\langle\mathbf{b}_1}{(s+p)^2} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_1}{s+p} + \frac{\mathbf{c}_2\rangle\langle[\mathbf{b}_2 + 3(s+p)\mathbf{b}_3]}{(s+p)^3} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_2}{s+p} \tag{12.11}$$

This form indicates that if additional inputs to the first and second Jordan blocks are used, the second-order term in the third Jordan block can be eliminated. However, a new first-order term will apparently be required in the third Jordan block to subtract out all additional unwanted outputs from blocks one and two due to their added inputs. This is made clear by rewriting Eq. (*12.11*) as

$$\mathbf{H}(s) = \frac{\mathbf{c}_1\rangle\langle[\mathbf{b}_1 + 2(s+p)\mathbf{b}_3]}{(s+p)^3} + \frac{\mathbf{c}_3\rangle\langle[\mathbf{b}_1 + 2(s+p)\mathbf{b}_3]}{(s+p)^2} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_1}{s+p} + \frac{\mathbf{c}_2\rangle\langle[\mathbf{b}_2 + 3(s+p)\mathbf{b}_3]}{(s+p)^3} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_2}{s+p} - \frac{2\mathbf{c}_3\rangle\langle\mathbf{b}_3}{s+p} \tag{12.12}$$

Figure 12.9 gives the reduced simulation diagram. The new realization is given by

$$\mathbf{A} = \left[\begin{array}{ccc|ccc|c} -p & 1 & 0 & & & & \\ 0 & -p & 1 & & \mathbf{0} & & \\ 0 & 0 & -p & & & & \\ \hline & & & -p & 1 & 0 & \\ & \mathbf{0} & & 0 & -p & 1 & \\ & & & 0 & 0 & -p & \\ \hline & & & & & & -p \end{array}\right], \qquad \mathbf{B} = \left[\begin{array}{c} \mathbf{0} \\ 2\mathbf{b}_3^T \\ \mathbf{b}_1^T \\ \hline \mathbf{0} \\ 3\mathbf{b}_3^T \\ \mathbf{b}_2^T \\ \hline \mathbf{b}_3^T \end{array}\right],$$

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_3 \quad -\mathbf{c}_5 \mid \mathbf{c}_2 \quad \mathbf{0} \quad -\mathbf{c}_5 \mid -2\mathbf{c}_3]$$

This realization is still not observable, and can be further reduced since $-2\mathbf{c}_1 + 2\mathbf{c}_2 = -2\mathbf{c}_3$. This result could have been recognized from Eq. (*12.12*) and the reduction could have been completed in one step.

Rewriting Eq. (*12.12*) gives

$$\mathbf{H}(s) = \frac{\mathbf{c}_1\rangle\langle\mathbf{b}_1 + 2(s+p)\mathbf{b}_3 - 2(s+p)^2\mathbf{b}_3]}{(s+p)^3} + \frac{\mathbf{c}_3\rangle\langle[\mathbf{b}_1 + 2(s+p)\mathbf{b}_3]}{(s+p)^2} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_1}{s+p} + \frac{\mathbf{c}_2\rangle\langle[\mathbf{b}_2 + 3(s+p)\mathbf{b}_3 + 2(s+p)^2\mathbf{b}_3]}{(s+p)^3} - \frac{\mathbf{c}_5\rangle\langle\mathbf{b}_2}{s+p}$$

The simulation diagram of Figure 12.9 can be modified to obtain an irreducible sixth-order realization:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} -p & 1 & 0 & & & \\ 0 & -p & 1 & & \mathbf{0} & \\ 0 & 0 & -p & & & \\ \hline & & & -p & 1 & 0 \\ & \mathbf{0} & & 0 & -p & 1 \\ & & & 0 & 0 & -p \end{array}\right], \qquad \mathbf{B} = \left[\begin{array}{c} -2\mathbf{b}_3^T \\ 2\mathbf{b}_3^T \\ \mathbf{b}_1^T \\ \hline 2\mathbf{b}_3^T \\ 3\mathbf{b}_3^T \\ \mathbf{b}_2^T \end{array}\right],$$

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_3 \quad -\mathbf{c}_5 \mid \mathbf{c}_2 \quad \mathbf{0} \quad -\mathbf{c}_5]$$

■

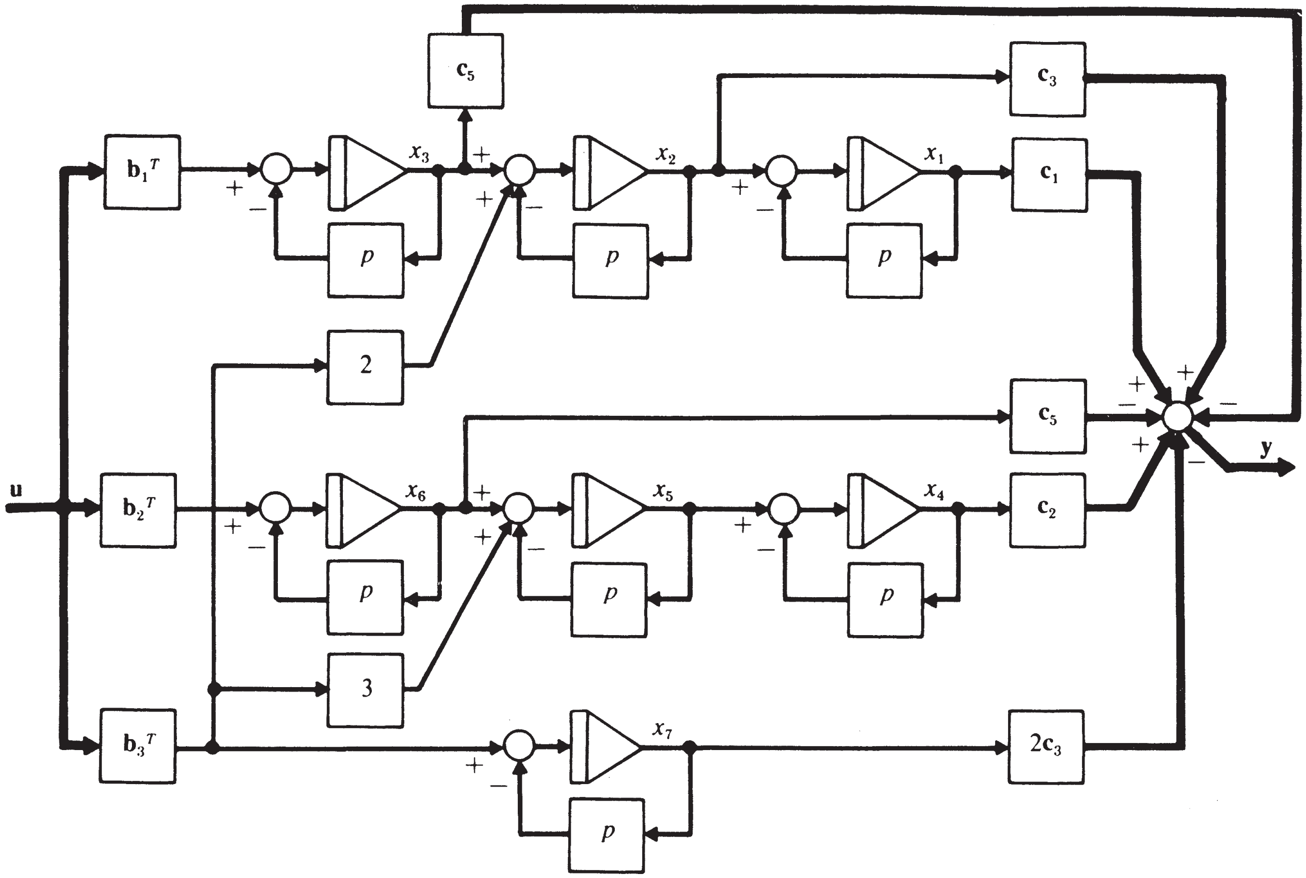



Figure 12.9

### 12.5.2 Kalman Canonical Form Approach to Minimal Realizations

In Sec. 11.7 it was shown that the dimension of the controllable subspace, here called $\Sigma_c$, for a given state variable realization is equal to the rank of the controllability matrix $\mathbf{P}$. In fact, the independent columns of $\mathbf{P}$ form a basis for $\Sigma_c$. An orthogonal decomposition of the state space, $\Sigma = \Sigma_c \oplus \Sigma_c^\perp$, was obtained, with the orthonormal basis vectors of these two subspaces forming columns of the matrices $\mathbf{T}_1$ and $\mathbf{T}_2$. The modified Gram-Schmidt process was used to obtain the **QR** decomposition of **P**, and this gave $\mathbf{T}_1$ directly. For an alternate method using singular value decomposition, see Reference 3.

Likewise, the observable subspace, call it $\Sigma_o$, was found by applying the **QR** decomposition algorithm to the observability matrix **Q**. $\Sigma_o$ has an orthonormal basis set, which forms the columns of the matrix $\mathbf{V}_1$. An alternative orthonormal decomposition of the state space, $\Sigma = \Sigma_o \oplus \Sigma_o^\perp$, was obtained by completing the set of basis vectors with $\mathbf{V}_2$.

It is possible to use these notions to decompose the state variable model into four subsystems [1]:

1. The intersection $\Sigma_c \cap \Sigma_o$ contains all the states which are both controllable and observable. This is the only part of the system that is described by the transfer function, and its model is a minimal realization.

2. The portion of $\Sigma_c$ that is spanned by columns of $\mathbf{T}_1$ which are *not* included in (1) contains states which are controllable but not observable.
3. The portion of $\Sigma_o$ that is spanned by columns of $\mathbf{V}_1$ which are *not* included in (1) contains the states which are observable but not controllable.
4. The remainder of $\Sigma$ contains those states which are neither controllable nor observable.

The full decomposition just described is unnecessary for the determination of a minimal realization. If only $\mathbf{T}_1$ is found from $\mathbf{P}$, then

$$\dot{\mathbf{w}}_1 = \mathbf{T}_1^T\mathbf{A}\mathbf{T}_1\mathbf{w}_1 + \mathbf{T}_1^T\mathbf{B}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{C}\mathbf{T}_1\mathbf{w}_1 + \mathbf{D}\mathbf{u} \tag{12.13}$$

is a realization of the controllable subsystem. Comparing this with Eq. (*11.8*) shows that the $\mathbf{w}_2$ states have been deleted. Even though the $\mathbf{w}_2$ states may be observable in the output $\mathbf{y}$, and even though the $\mathbf{w}_2$ states may *in general* be nonzero due to initial conditions, recall that the transfer function describes only the zero state response. Since $\mathbf{w}_2$ is not controllable, if it is initially zero, it stays zero. Thus in the case of zero initial conditions, $\mathbf{w}_2$ plays no role and can be dropped.

Starting with the reduced order controllable system of Eq. (*12.13*), a new observability matrix $\mathbf{Q}$ can be calculated. If this system is observable, then Eq. (*12.13*) constitutes a minimal realization. If it is not, then the **QR** decomposition process can be applied to $\mathbf{Q}$ to find $\mathbf{V}_1$, an orthonormal basis for the observable subspace of the reduced system. A similarity transformation (actually an orthogonal transformation because of the orthonormality of the basis set) on Eq. (*12.13*) gives

$$\dot{\mathbf{q}} = \mathbf{V}_1^T\mathbf{T}_1^T\mathbf{A}\mathbf{T}_1\mathbf{V}_1\mathbf{q} + \mathbf{V}_1^T\mathbf{T}_1^T\mathbf{B}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{C}\mathbf{T}_1\mathbf{V}_1\mathbf{q} + \mathbf{D}\mathbf{u} \tag{12.14}$$

Equation (*12.14*) models a system that is both controllable and observable and hence is a minimal realization. That is, Eq. (*12.14*) and the original system $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ have exactly the same transfer function.

Clearly the preceding sequence of the two orthogonal transformations could have been reversed. That is, an observable realization could have been constructed first from the original $\mathbf{Q}$ to yield a reduced system

$$\dot{\mathbf{v}}_1 = \mathbf{V}_1^T\mathbf{A}\mathbf{V}_1\mathbf{v}_1 + \mathbf{V}_1^T\mathbf{B}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{v}_1 + \mathbf{D}\mathbf{u} \tag{12.15}$$

This model describes the observable part of Eq. (*11.9*). Then a new matrix $\mathbf{P}$ could be constructed for this system. If it is full rank, then Eq. (*12.15*) constitutes a minimal realization. If it is not, then a second **QR** decomposition would give a matrix $\mathbf{T}_1$ to be used in the final orthogonal transformation, yielding

$$\dot{\mathbf{q}} = \mathbf{T}_1^T\mathbf{V}_1^T\mathbf{A}\mathbf{V}_1\mathbf{T}_1\mathbf{q} + \mathbf{T}_1^T\mathbf{V}_1^T\mathbf{B}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{T}_1\mathbf{q} + \mathbf{D}\mathbf{u} \tag{12.16}$$

In general, $\mathbf{q}$, $\mathbf{T}_1$, and $\mathbf{V}_1$ in Eq. (*12.14*) and Eq. (*12.16*) will not be the same. These constitute two alternative and equally valid forms of minimum realizations. By checking the ranks of both $\mathbf{P}$ and $\mathbf{Q}$ first, using the one with the smaller rank first, *might* possibly eliminate the need for the second orthogonal transformation.

**EXAMPLE 12.11** The second-order realization of Example 12.4 is reconsidered, using Kalman canonical forms. Since $\mathbf{C} = \mathbf{I}_2$, the observability matrix $\mathbf{Q}$ has full rank 2, so the system is observable. It is important to realize that any similarity transformation on an observable (or controllable) system will give a new system which is also observable (or controllable), so that the observability of any reduced order system derived from this example will not need to be rechecked. The controllability matrix is $\mathbf{P} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 4 & -1 & 0 & 0 \end{bmatrix}$ and has rank 1. A single basis vector spans the controllable subspace. It is selected as $\mathbf{T}_1 = [0.7071 \quad 0.7071]^T$. Then the first-order controllable subsystem is given by

$$\dot{w}_1 = \mathbf{T}_1^T \mathbf{A} \mathbf{T}_1 w_1 + \mathbf{T}_1^T \mathbf{B} u = \mathbf{0} w_1 + [5.65685 \quad -1.4142]\mathbf{u}$$

and $\mathbf{y} = \mathbf{I}_2 \mathbf{T}_1 w_1 = [0.707 \quad 0.707]^T w_1$. This is a minimal realization and agrees with the minimal realization presented earlier in Figure 12.6 (except for an inconsequential scaling up of **B** and scaling down of **C** by a factor of 1.414). ■

**EXAMPLE 12.12** Start with the sixth-order realization of Example 12.2 and find a minimal realization using the Kalman canonical form technique. Note that this same system was reduced to a fourth-order minimal realization in Example 12.7 using the Jordan form method. The 6 × 12 controllability matrix is

$$\mathbf{P} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 7 & 0 & -15 & 0 & 31 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 9 & 0 & -27 & 0 & 81 & 0 & -243 \\ 0 & 0 & 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 & 121 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

The rank of **P** is 5. The observability matrix is

$$\mathbf{Q} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & -4 & 0 & 8 & 0 & -16 & 0 & 32 & 0 & -64 & 0 \\ 0 & 0 & 2 & 0 & -6 & 0 & 14 & 0 & -30 & 0 & 62 & 0 \\ 0 & 1 & 0 & -3 & 0 & 9 & 0 & -27 & 0 & 81 & 0 & -243 \\ 0 & 1 & 0 & -3 & 0 & 9 & 0 & -27 & 0 & 81 & 0 & -243 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 & 121 \end{array}\right]$$

Its rank is 4. A set of four orthonormal six-component vectors is selected from **Q** and used to form the columns of

$$\mathbf{V}_1 = \begin{bmatrix} 0.4472136 & 0 & 0.3651484 & 0 \\ 0.8944272 & 0 & -0.1825742 & 0 \\ 0 & 0 & 0.9128709 & 0 \\ 0 & 0.7071068 & 0 & 0 \\ 0 & 0.7071068 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The fourth-order observable subsystem is found to be (rounded)

$$\dot{\mathbf{q}} = \begin{bmatrix} -1.8 & 0 & 0.9797959 & 0 \\ 0 & -3 & 0 & 0.707107 \\ 0.163299 & 0 & -1.2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0.447214 & 0 \\ 0 & 0.707107 \\ 0.365148 & 0.912871 \\ 1 & 0 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 2.23607 & 0 & 0 & 0 \\ 0 & 1.41421 & 0 & 0 \end{bmatrix}\mathbf{q}$$

The new $4 \times 8$ matrix $\mathbf{P}$ is of full rank, so that this realization is both controllable and observable and hence irreducible. Note that if the original matrix $\mathbf{P}$ had been used to construct five orthonormal basis vectors, a fifth-order controllable realization would be found, but it would not be observable (rank($\mathbf{Q}$) = 4). It would then be necessary to construct a set of four orthonormal basis vectors from the new $\mathbf{Q}$ and carry out a second orthogonal transformation. It should not be inferred that just because the original realization had a controllable subspace $\Sigma_c$ of dimension 5, which is larger than the dimension of the original $\Sigma_o$ of 4, that $\Sigma_o \subset \Sigma_c$, as was the case in this example. Problem 12.10 illustrates a case where $\dim(\Sigma) = 8$, $\dim(\Sigma_c) = 7$, and $\dim(\Sigma_o) = 6$. Selecting a sixth-order observable realization and then testing its new controllability matrix shows that rank($\mathbf{P}$) = 5 = order of the minimal realization. The following sketch helps understand this behavior.

$\longleftarrow \dim(\Sigma) = 8 \longrightarrow$
$\longleftarrow \dim(\Sigma_c) = 7 \longrightarrow$
$\longleftarrow \dim(\Sigma_c \cap \Sigma_0) = 5 \longrightarrow$
$\longleftarrow \dim(\Sigma_0) = 6 \longrightarrow$

■

A bounding inequality on the minimum order $n_{\min}$ is

$$\dim(\Sigma_c) + \dim(\Sigma_o) - n \le n_{\min} \le \min\{\dim(\Sigma_c), \dim(\Sigma_o)\}.$$

## 12.6 MINIMAL REALIZATIONS FROM MATRIX FRACTION DESCRIPTION

For a scalar transfer function, the order of the irreducible realization is the degree of the denominator after all common pole/zero pairs are canceled. The poles of the reduced denominator will be eigenvalues of the matrix $\mathbf{A}$ of the minimal realization. A similar result is true for matrix transfer functions. It hinges on what is meant by the denominator. Chen [4] defines the denominator of a rational transfer function matrix as the lowest common denominator of all minors of all orders, after all possible cancellations are made in each minor. This applies to sampled and continuous transfer functions.

**EXAMPLE 12.13** A $Z$-transform transfer function matrix is given by

$$\mathbf{H}(z) = \begin{bmatrix} 0.2z/[(z-1)(z-0.5)] & 3z(z+0.3)/[(z-1)(z-0.8)] \\ (z+0.8)/[(z-0.5)^2(z-0.7)] & z/(z-0.5) \end{bmatrix}$$

The four first-order minors are just the entries of $\mathbf{H}(z)$, and the lowest common denominator is $(z-1)(z-0.5)^2(z-0.7)(z-0.8)$. The only second-order minor is the determinant

$$H_{11}H_{22} - H_{12}H_{21} = \{0.2z^2(z-0.7)(z-0.8) - 3z(z+0.3)(z+0.8)\}$$
$$/[(z-1)(z-0.5)^2(z-0.7)(z-0.8)]$$

In this case the lowest common denominator of all minors is the same as the lowest common denominator of the entries in $\mathbf{H}$. This will not usually be the case. Here the denominator is of degree 5 and has poles at $z = 1, 0.5, 0.5, 0.7$, and $0.8$. ■

The order of the minimum or irreducible state variable realization is equal to the degree of the denominator in the sense just defined. For the example, a completely controllable and completely observable fifth-order realization of $\mathbf{H}(z)$ is possible, and its matrix **A** will have eigenvalues at 1, 0.5, 0.5, 0.7, and 0.8. The *determination* of a minimal realization by means of Jordan form or Kalman canonical form techniques was described previously. The matrix fraction description MFD of the transfer function can also be used. (See Chapter 4 and Problems 4.30 through 4.32.) The matrix equivalent of canceling all common pole-zero pairs is accomplished by using a series of polynomial-restricted elementary operations. (See Sec. 6.3.1 and 6.3.2.) Recall that a given transfer matrix can be written as either a left or right MFD,

$$\mathbf{H}(s) = \mathbf{P}_1^{-1}(s)\mathbf{N}_1(s) = \mathbf{N}_2(s)\mathbf{P}_2^{-1}(s)$$

Thus a series of elementary row operations on $\mathbf{P}_1(s)\mathbf{H}(s) = \mathbf{N}_1(s)$ or a series of elementary column operations on $\mathbf{H}(s)\mathbf{P}_2(s) = \mathbf{N}_2(s)$ will leave $\mathbf{H}(s)$ unaltered. The order of the state variable realization required for a given MFD description is the degree of the determinant of the "denominator matrix" $\mathbf{P}_1$ or $\mathbf{P}_2$. Do not confuse the denominator matrix or its determinant degree with Chen's denominator, which is always a scalar and whose degree is uniquely determined by **H.** Here the determinant degree of the denominator matrix can be changed by choice of the elementary operations that are used. A systematic sequence of elementary row operations can be carried out on $[\mathbf{P}_1 \mid \mathbf{N}_1]$ (or elementary column operations on $\left(\begin{bmatrix}\mathbf{P}_2\\ \hline \mathbf{N}_2\end{bmatrix}\right)$ until the determinant of the modified $\mathbf{P}_i$ portion takes on the a priori known minimal degree. It is a fact that the same minimal order applies, whether the left or right MFD version is being used. The systematic reduction procedure can be carried out until the Hermite (row or column) form is reached, but this is more than necessary. When the minimal order is reached, the factors $\mathbf{N}_i$ and $\mathbf{P}_i$ are said to be *coprime* (left coprime in the case $i = 1$ and right coprime in the case $i = 2$). Being coprime is the matrix equivalent of being maximally reduced—that is, having all common pole-zero factors canceled in the scalar case. For a much more thorough treatment of these topics and the related proofs, see Kailath [5]. Once the reduced form of the MFD is obtained, the state equations can be obtained using one of the methods of Chapter 3. For example, if the left MFD is being used then the corresponding coupled differential (or difference) equations are obtained from

$$\mathbf{P}_1\mathbf{Y} = \mathbf{N}_1\mathbf{U}$$

Then the nested integrator approach of Sec. 3.4 is used to obtain the matrix version of the observable canonical form.

If the right MFD is being used, the intermediate variables $g_i$ are first simulated from $\mathbf{P}_2\mathbf{g} = \mathbf{U}$ and then the outputs are constructed from $\mathbf{Y} = \mathbf{N}_2\mathbf{g}$. This is the matrix version of the controllable canonical form approach in Figure 3.9. In both cases the state variables are the outputs of integrators, or delay elements, and will be of the minimum possible number. The realizations thus obtained will be both controllable and observable and hence irreducible.

**EXAMPLE 12.14** Find a minimal realization for the system of Example 12.13.

An initial MFD can always be obtained by finding the common denominator $d(z)$ of all the first-order minors and then writing either $\mathbf{H}(z) = [\mathbf{I}d(z)]^{-1}\mathbf{N}(z)$ or $\mathbf{N}(z)[\mathbf{I}d(z)]^{-1}$. Here the left MFD will be used, and the starting form of $[\mathbf{P}_1(z) \mid \mathbf{N}_1(z)]$ is of determinant degree 10:

$$\left[\begin{matrix} (z-1)(z-0.5)^2(z-0.7)(z-0.8) & 0 \\ 0 & (z-1)(z-0.5)^2(z-0.7)(z-0.8) \end{matrix}\right.$$

$$\left.\begin{matrix} 0.2z(z-0.5)(z-0.7)(z-0.8) & 3z(z+0.3)(z-0.5)^2(z-0.7) \\ (z+0.8)(z-1)(z-0.8) & z(z-0.5)(z-0.7)(z-0.8)(z-1) \end{matrix}\right]$$

When using the so-called polynomial restricted elementary row operations, the division operation is allowed only if it leaves no remainder. In row 1 a factor $(z-0.5)(z-0.7)$ can be canceled from each term. In row 2 a factor $(z-1)(z-0.8)$ can be canceled, leaving a determinant degree of 6:

$$\left[\begin{array}{cc|cc} (z-1)(z-0.5)(z-0.8) & 0 & 0.2z(z-0.8) & 3z(z+0.3)(z-0.5) \\ 0 & (z-0.5)^2(z-0.7) & (z+0.8) & z(z-0.5)(z-0.7) \end{array}\right]$$

No more easy common factors remain, but it is known that the degree must be reduced by one more factor, and the factor to be removed must be $(z-0.5)$, since currently there are three of these and the final result must only have two. If $\alpha = 0.03/1.3$ is multiplied times row 2 and if the result is added to row 1, then every term in the altered row 1 will have at least one factor $(z-0.5)$. Doing this and canceling leaves the final form with determinant degree 5:

$$\left[\begin{array}{cc|cc} (z-1)(z-0.8) & \alpha(z-0.5)(z-0.7) & 0.2z+(\alpha-0.06) & z[3(z+0.3)+\alpha(z-0.7)] \\ 0 & (z-0.5)^2(z-0.7) & (z+0.8) & z(z-0.5)(z-0.7) \end{array}\right]$$

From this, a pair of coupled difference equations are written:

$$y_1(k+2) - 1.8y_1(k+1) + 0.8y_1(k) + \alpha y_2(k+2) - 1.2\alpha y_2(k+1) + 0.35\alpha y_2(k)$$
$$= 0.2u_1(k+1) + (\alpha - 0.06)u_1(k) + (\alpha+3)u_2(k+2) + (0.9-0.7\alpha)u_2(k+1)$$
$$y_2(k+3) - 1.7y_2(k+2) + 0.95y_2(k+1) - 0.175y_2(k)$$
$$= u_1(k+1) + 0.8u_1(k) + u_2(k+3) - 1.2u_2(k+2) + 0.35u_2(k+1)$$

By delaying the first equation twice and the second equation three times, a nested-delay form of these equations is obtained:

$$y_1(k) = -\alpha y_2(k) + (\alpha+3)u_2(k) + \mathfrak{D}\{1.8y_1(k) + 1.2\alpha y_2(k) + 0.2u_1(k)$$
$$+ (0.9-0.7\alpha)u_2(k) + \mathfrak{D}[-0.8y_1(k) - 0.35\alpha y_2(k) + (\alpha-0.06)u_1(k)]\}$$
$$y_2(k) = u_2(k) + \mathfrak{D}\{1.7y_2(k) - 1.2u_2(k) + \mathfrak{D}[-0.95y_2(k) + u_1(k) + 0.35u_2(k)$$
$$+ \mathfrak{D}\langle 0.175y_2(k) + 0.8u_1(k)\rangle]\}$$

where $\mathfrak{D}$ is a unit delay.

These can be represented as an interconnection of five delay elements. Picking the delay outputs as states leads to the minimal realization

$$\mathbf{A} = \begin{bmatrix} 1.8 & 0 & -0.6\alpha & 0 & 0 \\ -0.8 & 0 & 0.45\alpha & 0 & 0 \\ 0 & 0 & 1.7 & 1 & 0 \\ 0 & 0 & -0.95 & 0 & 1 \\ 0 & 0 & 0.175 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0.2 & (6.3+0.5\alpha) \\ (\alpha-0.06) & -(2.4+0.35\alpha) \\ 0 & 0.5 \\ 1 & -0.6 \\ 0.8 & 0.175 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad \alpha = 0.0230769$$

■

## 12.7 CONCLUDING COMMENTS

Linear constant coefficient systems can be described by either state variable techniques or by transfer functions. This chapter has explored the relationships between these alternatives. If the state equations are given, the transfer function is given uniquely by Eq. (*12.5*). The reverse process is not unique. For a given transfer function there exist many sets of corresponding state equations. Several methods of determining state equations, called realizations, have been presented, starting with Chapter 3. In this chapter a major emphasis has been placed on finding state variable realizations with the minimum possible number of states. Three different approaches to the determination of minimal realizations have been presented. Each has advantages and disadvantages, but the Kalman canonical form technique seems best matched to the developments in this book. It is also well suited to machine implementation.

Minimal realizations are both completely controllable and observable. These properties are important in the following chapters, since they guarantee the existence of solutions to various control design problems.

An alternative method for determining irreducible realizations was presented in the fundamental contributions by B. L. Ho [6, 7]. His work is also discussed in Reference 4. Raven [8] presents a method for determining *approximate* realizations when only approximations of the transfer functions are known. Additional related material may be found in References 9, 10, and 11.

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## ILLUSTRATIVE PROBLEMS

### *Transfer Functions and State Equations*

**12.1** A single-input, single-output system has the transfer function

$$H(s) = \frac{(s+5)(s+\alpha)}{(s+2)(s+3)(s+\alpha)}$$

**(a)** Cancel the common pole, zero pair and write the input-output differential equation for this system. Use a simulation diagram to select state variables.
**(b)** Repeat part a without canceling the pole, zero pair.
**(c)** Compare the controllability and observability of the two realizations obtained in (a) and (b).

**(a)** The transfer function $y(s)/u(s) = (s+5)/[(s+2)(s+3)]$ implies the differential equation $\ddot{y} + 5\dot{y} + 6y = 5u + \dot{u}$. A simulation diagram is given in Figure 12.10. Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = [1 \quad 0]\mathbf{x}$$

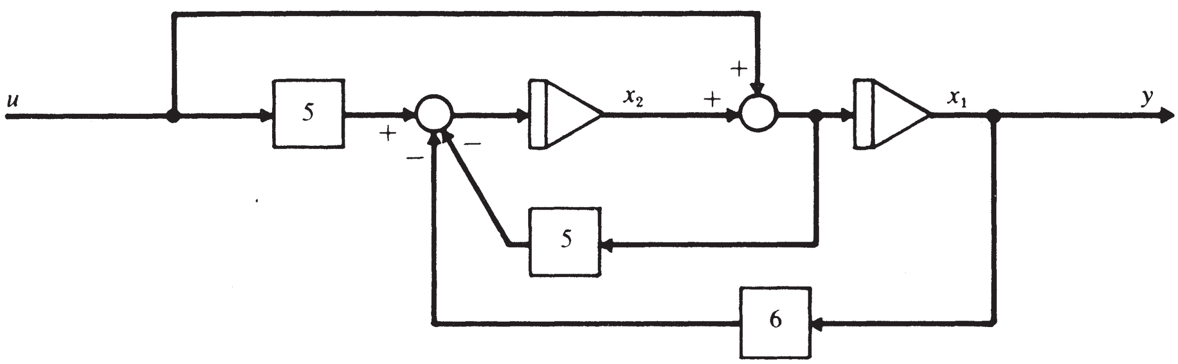



**Figure 12.10**

**(b)** The differential equation now is $\dddot{y} + (5+\alpha)\ddot{y} + (6+5\alpha)\dot{y} + 6\alpha y = \ddot{u} + (5+\alpha)\dot{u} + 5\alpha u$. From this,

$$y = \int\left\{[u-(5+\alpha)y] + \int\left[(5+\alpha)u - (6+5\alpha)y + \int(5\alpha u - 6\alpha y)\,dt\right]dt\right\}dt$$

A simulation diagram is given in Figure 12.11. Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -(5+\alpha) & 1 & 0 \\ -(6+5\alpha) & 0 & 1 \\ -6\alpha & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5+\alpha \\ 5\alpha \end{bmatrix} u, \qquad y = [1 \quad 0 \quad 0]\mathbf{x}$$

**(c)** For system a, $\mathbf{P} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & -6 \end{array}\right]$ has rank 2. $\mathbf{Q}\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right]$ has rank 2. System a is completely controllable and observable. For system b, $\mathbf{P} = \left[\begin{array}{c|c|c} 1 & 0 & -6 \\ 5+\alpha & -6 & -6\alpha \\ 5\alpha & -6\alpha & 0 \end{array}\right]$ has rank 2. $\mathbf{Q} = \left[\begin{array}{c|c|c} 1 & -5-\alpha & \alpha^2+5\alpha+19 \\ 0 & 1 & -5-\alpha \\ 0 & 0 & 1 \end{array}\right]$ has rank 3. System b is completely observable but not controllable. Other simulation diagrams can be drawn for system b, which result in a realization which is completely controllable but not observable. It is not possible to obtain a third-order realization which is both completely controllable and observable.

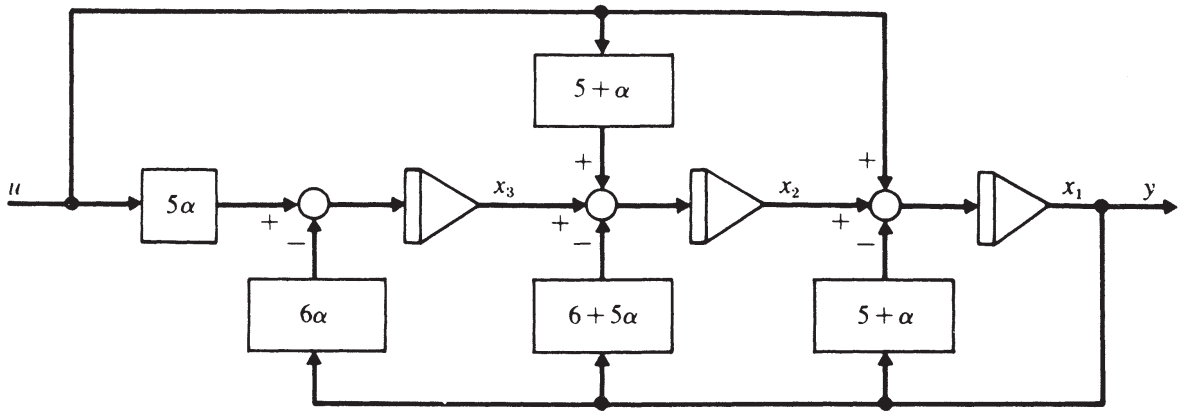



**Figure 12.11**

**12.2** Find state equations and the input-output transfer matrix for the system of Figure 12.12.

The fact that there are multiple inputs and outputs does not change the procedure for selecting state variables from the linear graph, described in Sec. 3.4.5. The tree to be used is shown in heavy lines. Use the capacitor voltage $x_1$ and inductor current $x_2$ as state variables. Then $\dot{x}_1 = x_2/C + u_1/C$, $\dot{x}_2 = -x_1/L - R_2x_2/L + u_2/L$, and $y_1 = x_1, y_2 = x_1 + R_2x_2$. Hence

$$\mathbf{A} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R_2/L \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1/C & 0 \\ 0 & 1/L \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & R_2 \end{bmatrix}, \quad D = 0$$

Then

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \frac{\begin{bmatrix} (s + R_2/L)/C & 1/LC \\ s/C & (R_2s + 1/C)/L \end{bmatrix}}{s^2 + R_2s/L + 1/LC}$$

**12.3** A system with three inputs and two outputs is described by

$$\ddot{y}_1 + 6\dot{y}_1 - \dot{y}_2 + 10y_1 - 6y_2 = \frac{\dot{u}_1}{2} - 2u_1 + \frac{3\dot{u}_2}{2} + 4u_2 - 5\dot{u}_3$$

$$\ddot{y}_2 + 8\dot{y}_2 - 2\dot{y}_1 + 12y_2 - 4y_1 = \dot{u}_1 + 2u_1 + \dot{u}_2 + 2u_2 + \dot{u}_3 + 2u_3$$

Find the transfer matrix $\mathbf{H}(s)$.

A simulation diagram could be drawn, utilizing four integrators. Then Eq. (*12.5*) could be used to obtain $\mathbf{H}(s)$. Alternatively, the Laplace transforms of the original equations are used

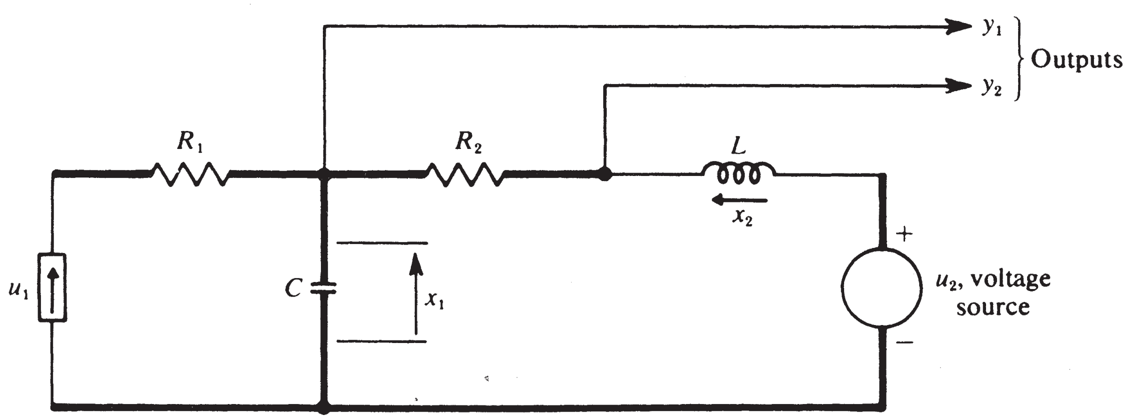



**Figure 12.12**

here, giving

$$\mathbf{y}(s) = \begin{bmatrix} s^2+6s+10 & -s-6 \\ -2s-4 & s^2+8s+12 \end{bmatrix}^{-1} \begin{bmatrix} s/2-2 & 3s/2+4 & -5 \\ s+2 & s+2 & s+2 \end{bmatrix} \mathbf{u}(s)$$

Therefore,

$$\mathbf{H}(s) = \frac{\begin{bmatrix} s/2-1 & 3s/2+5 & -4 \\ s+1 & s+3 & s \end{bmatrix}}{(s+2)(s+4)}$$

**12.4** Determine the input-output transfer matrix for a system described by

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

Using Eq. (*12.5*), the transfer matrix is $\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$. Substituting for **A, B, C,** and **D** and carrying out the multiplication gives

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{s^2+2s+3}{(s+1)^2} & \dfrac{s^2+6s+3}{(s+1)^2} \\ \dfrac{1}{s+1} & \dfrac{-1}{s+1} \\ \dfrac{-s+2}{(s+1)^2} & \dfrac{7s+4}{(s+1)^2} \end{bmatrix}$$

**12.5** Use the linearized state variable model of the aircraft in Problem 11.9. Find the transfer function matrix which relates the aircraft aileron and rudder deflections to the state variables $p$, $r$, $\beta$, and $\phi$. Draw a block diagram of the relationships contained in the state equations and verify the entries in the transfer function matrix. Use the block diagram to explain in physical terms the mathematical results on controllability (Problem 11.9) and observability (Problem 11.10).

Since the transfer function to *all* the states is desired, set $\mathbf{C} = \mathbf{I}$. Then

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \begin{bmatrix} s+10 & 0 & 10 & 0 \\ 0 & s+0.7 & -9 & 0 \\ 0 & 1 & s+0.7 & 0 \\ 1 & 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{20}{s+10} & \dfrac{2.8(s-0.776)(s+2.176)}{(s+10)(s^2+1.4s+9.49)} \\ 0 & \dfrac{-3.13(s+0.7)}{s^2+1.4s+9.49} \\ 0 & \dfrac{3.13}{s^2+1.4s+9.49} \\ \dfrac{20}{s(s+10)} & \dfrac{2.8(s-0.776)(s+2.176)}{s(s+10)(s^2+1.4s+9.49)} \end{bmatrix}$$

The block diagram of Figure 12.13 is drawn directly from the state equations. Using elementary block diagram reduction technques, or Mason's gain formula for a one-step reduction, the transfer functions $H_{ij}$ from each input $j$ to each of the four outputs $i$ can be found. The eight results are of course the entries in the previous matrix.

Figure 12.13 makes it clear why the system is uncontrollable with just $\delta_a$. There is no signal path from this input to either $r$ or $\beta$. The other input does feed into all four state variables, either

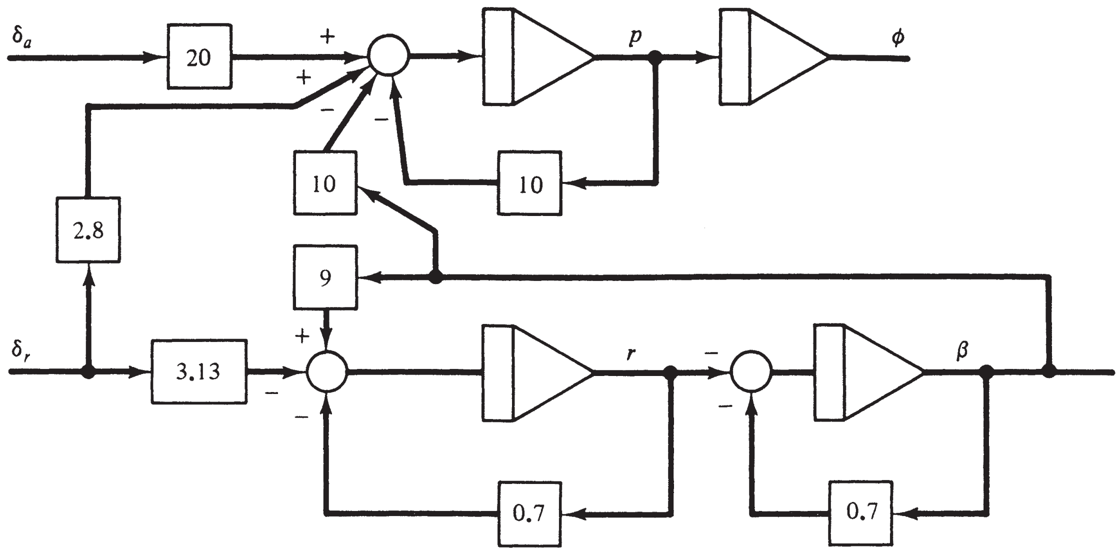



**Figure 12.13**

directly or indirectly. The observability results of Problem 11.10 are equally obvious. If $\phi$ is not an output, it can never be determined, since there is no path by which $\phi$ affects any other state or output. Conversely, if $\phi$ can be monitored, it is intuitive that its rate, i.e., $p$, can be deduced. Likewise, the rate of change of $p$ can be deduced, and it contains information about $\beta$. This in turn is strongly related to $r$.

### *Jordan Form Irreducible Realizations*

**12.6** Find an irreducible realization of $\mathbf{H}(s)$ from Problem 12.4.
Expanding each $H_{ij}$ element yields

$$\mathbf{H}(s) = \begin{bmatrix} 1 + \dfrac{2}{(s+1)^2} & 1 - \dfrac{2}{(s+1)^2} + \dfrac{4}{s+1} \\ \dfrac{1}{s+1} & -\dfrac{1}{s+1} \\ \dfrac{3}{(s+1)^2} - \dfrac{1}{s+1} & \dfrac{-3}{(s+1)^2} + \dfrac{7}{s+1} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 3 & -3 \end{bmatrix}}{(s+1)^2} + \frac{\begin{bmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 7 \end{bmatrix}}{s+1} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This form indicates that $\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. The remaining two matrices are of rank 1 and 2, respectively. Writing them as outer products gives

$$\mathbf{H}(s) = \frac{\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} [1 \quad -1]}{(s+1)^2} + \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} [1 \quad -1]}{s+1} + \frac{\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} [0 \quad 1]}{s+1} + \mathbf{D}$$

The vectors $\mathbf{b}_1^T = [1 \quad -1]$, $\mathbf{b}_3^T = [0 \quad 1]$ are linearly independent, and could be used to form the last rows associated with two Jordan blocks $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ \hdashline 0 & 1 \end{bmatrix}$. However, the corresponding output matrix $\mathbf{C} = \left[\begin{array}{cc:c} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 3 & -1 & 6 \end{array}\right]$ has its first and third columns linearly dependent. This would lead to a realization which is completely controllable, but not observable and hence reducible. Since $\mathbf{c}_3 = 2\mathbf{c}_1$, the first and third terms can be combined:

$$\mathbf{H}(s) = \frac{\mathbf{c}_1\rangle\langle[\mathbf{b}_1 + 2\mathbf{b}_3(s+1)]}{(s+1)^2} + \frac{\mathbf{c}_2\rangle\langle\mathbf{b}_1}{s+1} + \mathbf{D}$$

The corresponding simulation diagram is given in Figure 12.14. The irreducible realization is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This realization is of lesser dimension than the actual system from which the transfer function was obtained. The original system was not completely controllable.

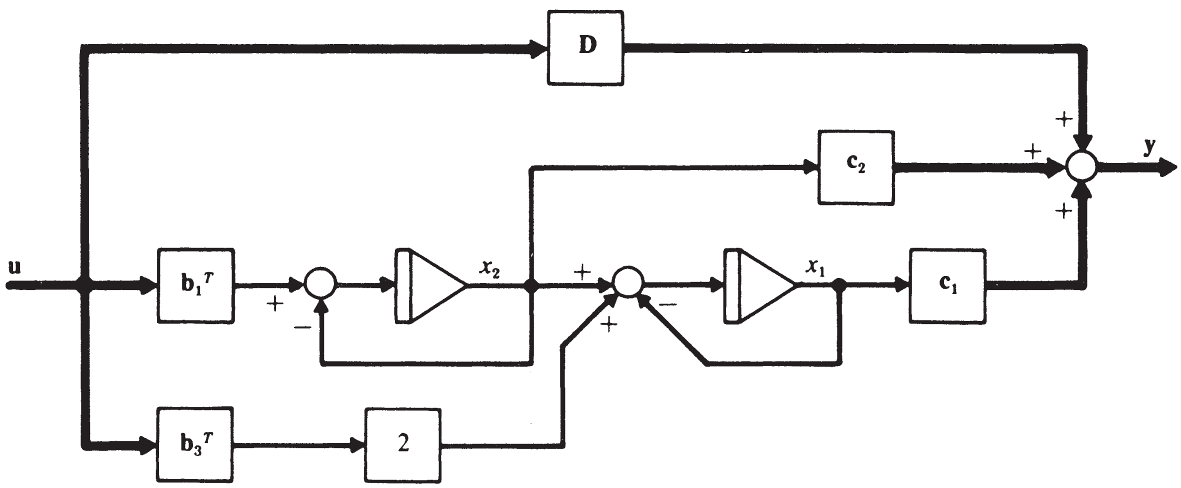



**Figure 12.14**

**12.7** Find a minimal order state variable realization for the discrete system described by

$$\mathbf{T}(z) = \begin{bmatrix} \dfrac{0.2z}{(z-1)(z-0.5)} & \dfrac{3z(z+0.3)}{(z-1)(z-0.8)} \\ \dfrac{(z+0.8)}{(z-0.5)^2(z-0.7)} & \dfrac{z}{(z-0.5)} \end{bmatrix}$$

A partial fraction expanded form for $\mathbf{T}(z)$ is

$$\mathbf{T}(z) = \frac{1}{(z-0.5)}\begin{bmatrix} -0.2 & 0 \\ -37.5 & 0.5 \end{bmatrix} + \frac{1}{(z-0.5)^2}\begin{bmatrix} 0 & 0 \\ -6.5 & 0 \end{bmatrix}$$

$$+ \frac{1}{(z-1)}\begin{bmatrix} 0.4 & 19.5 \\ 0 & 0 \end{bmatrix} + \frac{1}{(z-0.8)}\begin{bmatrix} 0 & -13.2 \\ 0 & 0 \end{bmatrix}$$

$$+ \frac{1}{(z-0.7)}\begin{bmatrix} 0 & 0 \\ 37.5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

All of the 2 × 2 expansion matrices are of rank one except the first, which has rank 2. Therefore, a seventh-order state variable model could be written down immediately as

$$\mathbf{A} = \text{diag}\left[0.5, 0.5, \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}, 1, 0.8, 0.7\right]$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 0 \\ \hline 0.4 & 19.5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{C} = \left[\begin{array}{cc|cc|ccc} -0.2 & 0 & 0 & 0 & 1 & -13.2 & 0 \\ -37.5 & 0.5 & 0 & -6.5 & 0 & 0 & 37.5 \end{array}\right] \qquad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

Note that a 2 × 2 Jordan block is included for the $(z - 0.5)^2$ factor. The first state in this segment has a direct connection to the input but not the output. The reverse is true for the second state in this subsegment. These two states are related, as shown in Figure 12.15. It can be shown (easily with the right computational aids, otherwise more laboriously) that the above seventh-order system is neither controllable nor observable. The ranks of the controllability and observability matrices are both 5. This indicates that two unneeded states can be deleted. The two states shown in Figure 12.15, plus the other two uncoupled states associated with $z = 0.5$, must be reduced to just two states. A systematic way of accomplishing this is to assume the segment shown in Figure 12.16 and select the eight scalar parameters $a, \dots, h$ so that this segment realization yields the same output as the original fourth-order segment. That is,

$$\mathbf{w} = \left\{ \frac{\begin{bmatrix} e \\ f \end{bmatrix}[a \quad b] + \begin{bmatrix} g \\ h \end{bmatrix}[c \quad d]}{(z - 0.5)} + \frac{\begin{bmatrix} g \\ h \end{bmatrix}[a \quad b]}{(z - 0.5)^2} \right\} \mathbf{u}$$

$$= \left\{ \frac{\begin{bmatrix} -0.2 & 0 \\ -37.5 & 0.5 \end{bmatrix}}{(z - 0.5)} + \frac{\begin{bmatrix} 0 & 0 \\ -6.5 & 0 \end{bmatrix}}{(z - 0.5)^2} \right\} \mathbf{u}$$

This requires that $ae + cg = -0.2, be + dg = 0, af + ch = -37.5, bf + dh = 0.5, ag = 0, bg = 0, ah = -6.5, bh = 0$. Of the many possibilities, pick $b = c = g = 0$ and $a = 1$. Then solve for $e = -0.2$, $h = -6.5$, $d = -0.076923$, and $f = -37.5$. The 2 × 2 segment associated with $z = 0.5$

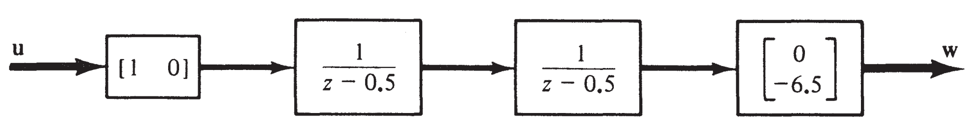



**Figure 12.15**

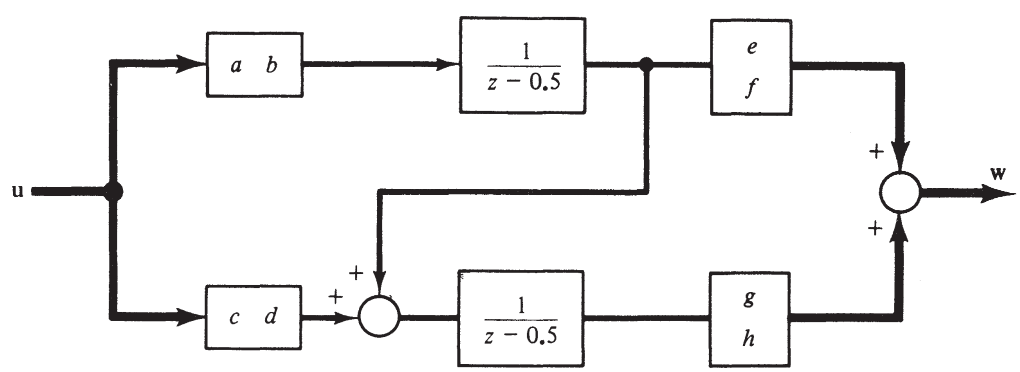



**Figure 12.16**

thus has the state equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 & -0.076923 \\ 1 & 0 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{w}(k) = \begin{bmatrix} 0 & -0.2 \\ -6.5 & -37.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

By substituting this second-order segment for the original fourth-order segment, the final fifth-order system realization, which is controllable with index 3 and observable with index 3, is

$$\mathbf{A} = \begin{bmatrix} 0.5 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -0.07693 \\ 1 & 0 \\ 0.4 & 19.5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -0.2 & 1 & -13.2 & 0 \\ -6.5 & -37.5 & 0 & 0 & 37.5 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

**12.8** Find an irreducible realization for the transfer matrix

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{3}{(s^2+2s+10)(s+5)} & \dfrac{2s}{s+5} \\ \dfrac{s+1}{(s^2+2s+10)(s+5)} & \dfrac{s+1}{s+5} \end{bmatrix}$$

Partial fraction expansion gives

$$H_{11}(s) = \frac{\alpha}{s+1+3j} + \frac{\bar{\alpha}}{s+1-3j} + \frac{3/25}{s+5}, \quad H_{12}(s) = 2 - \frac{10}{s+5}$$

$$H_{21}(s) = \frac{-j\alpha}{s+1+3j} + \frac{-\overline{j\alpha}}{s+1-3j} - \frac{4/25}{s+5}, \quad H_{22}(s) = 1 - \frac{4}{s+5}$$

where $\alpha = -3/50 + 2j/25$. Let $p_1 = 1 + 3j$. Then

$$\mathbf{H}(s) = \frac{\begin{bmatrix} \alpha & 0 \\ -j\alpha & 0 \end{bmatrix}}{s+p_1} + \frac{\begin{bmatrix} \bar{\alpha} & 0 \\ -\overline{j\alpha} & 0 \end{bmatrix}}{s+\bar{p}_1} + \frac{\begin{bmatrix} \frac{3}{25} & -10 \\ -\frac{4}{25} & -4 \end{bmatrix}}{s+5} + \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} \alpha \\ -j\alpha \end{bmatrix}[1 \quad 0]}{s+p_1} + \frac{\begin{bmatrix} \bar{\alpha} \\ -\overline{j\alpha} \end{bmatrix}[1 \quad 0]}{s+\bar{p}_1} + \frac{\begin{bmatrix} \frac{3}{25} \\ -\frac{4}{25} \end{bmatrix}[1 \quad 0]}{s+5} + \frac{\begin{bmatrix} 10 \\ 4 \end{bmatrix}[0 \quad -1]}{s+5} + \mathbf{D}$$

This transfer matrix can be realized with four $1 \times 1$ Jordan blocks. Since $\mathbf{c}_3$ and $\mathbf{c}_4$ associated with the pole at $-5$ are linearly independent, an irreducible realization is given by

$$\mathbf{A} = \text{diag}[-p_1, -\bar{p}_1, -5, -5], \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \alpha & \bar{\alpha} & \frac{2}{25} & 10 \\ -j\alpha & -\overline{j\alpha} & -\frac{4}{25} & 4 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Because of the complex numbers in **A** and **C**, this realization cannot be directly mechanized. The next problem illustrates a transformation which eliminates the complex numbers.

**12.9** Find a suitable transformation (change of basis in $\Sigma$) which transforms the realization of Problem 12.8 into one with all real coefficients.

The original state vector is $\mathbf{x}$. A new state vector $\mathbf{x}'$ is sought, satisfying $\mathbf{x} = \mathbf{T}\mathbf{x}'$. Then

$$\dot{\mathbf{x}}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{x}' + \mathbf{T}^{-1}\mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{T}\mathbf{x}' + \mathbf{D}\mathbf{u}$$

and it is required that $\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\mathbf{B}' = \mathbf{T}^{-1}\mathbf{B}$, and $\mathbf{C}' = \mathbf{C}\mathbf{T}$ all be real.

A tentative transformation is $\mathbf{T} = \left[\begin{array}{cc|c} 1 & j & \mathbf{0} \\ 1 & -j & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{I}_2 \end{array}\right]$. Then

$$\mathbf{C}' = \begin{bmatrix} \alpha+\bar{\alpha} & j\alpha - j\bar{\alpha} & \frac{3}{25} & 10 \\ -j\alpha + \overline{j\alpha} & \alpha+\bar{\alpha} & -\frac{4}{25} & 4 \end{bmatrix} \quad \text{or} \quad \mathbf{C}' = \begin{bmatrix} -\frac{3}{25} & -\frac{4}{25} & \frac{3}{25} & 10 \\ \frac{4}{25} & -\frac{3}{25} & -\frac{4}{25} & 4 \end{bmatrix}$$

The selected transformation does yield a real $\mathbf{C}'$. To evaluate $\mathbf{A}'$ and $\mathbf{B}'$,

$$\mathbf{T}^{-1} = \left[\begin{array}{cc|c} 1/2 & 1/2 & \mathbf{0} \\ -j/2 & j/2 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{I}_2 \end{array}\right]$$

is required. Matrix multiplication then gives

$$\mathbf{A}' = \left[\begin{array}{cc|cc} -1 & 3 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ \hline 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{array}\right] \quad \text{and} \quad \mathbf{B}' = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & -1 \end{array}\right]$$

The $\mathbf{D}$ matrix is unaffected by this change of basis, so a real, irreducible realization is $\{\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}\}$.

### *Irreducible Realization Using Kalman Canonical Form*

**12.10** Find another minimal realization for the transfer function of Examples 12.13 and 12.14 and Problem 12.7 by first finding an eighth-order realization with no internal coupling (as in Sec. 3.5) and then reducing it using the Kalman canonical decomposition method.

By using a controllable canonical form realization for each of the four scalar transfer function elements, two second-order subsystems, one third-order, and one first-order are obtained. The total eighth-order realization has

$$\mathbf{A} = \left[\begin{array}{cc|ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.175 & -0.95 & 1.7 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.8 & 1.8 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{array}\right]$$

$$\mathbf{B} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \\ \hline 0 & 1 \end{array}\right] \quad \mathbf{C}^T = \left[\begin{array}{cc} 0 & 0 \\ 0.2 & 0 \\ \hline 0 & 0.8 \\ 0 & 1 \\ 0 & 0 \\ \hline -2.4 & 0 \\ 6.3 & 0 \\ \hline 0 & 0.5 \end{array}\right] \quad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

Computer evaluation of the $8 \times 16$ controllability matrix $\mathbf{P}$ shows that it has rank 7, so the eighth-order system is not controllable. The $8 \times 16$ observability matrix is found to have rank 6. A set of six orthonormal basis vectors are selected from $\mathbf{Q}$ and used to find a sixth-order observable realization having

$$\mathbf{A} = \begin{bmatrix} 1.50534 & 0 & 0.226920 & 0 & 0 & 0 \\ 0 & 0.48942 & 0 & 0.81318 & 0 & 0 \\ -1.57135 & 0 & 0.29339 & 0 & 0.027686 & 0 \\ 0 & -0.78092 & 0 & 1.22054 & 0 & 1.61228 \\ -0.50807 & 0 & -0.00469 & 0 & 0.50127 & 0 \\ 0 & 0.24190 & 0 & 0.16458 & 0 & -0.009961 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.02965 & 0.93407 \\ 0 & 0.36370 \\ -0.00070 & -0.35521 \\ 0.89450 & 0.004733 \\ -0.70741 & 0.00335 \\ 0.43832 & 0.17404 \end{bmatrix} \quad \mathbf{C}^T = \begin{bmatrix} 6.7446 & 0 \\ 0 & 1.37477 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

The new $6 \times 12$ controllability matrix has rank 5, so the reduced system is observable but not controllable. A new matrix $\mathbf{T}_1$ is selected from columns of $\mathbf{P}$ and used in an orthonormal transformation to give the final fifth-order minimal realization

$$\mathbf{A} = \begin{bmatrix} 1.28861 & -0.018929 & -0.33158 & -0.02789 & -0.023257 \\ 0.17151 & 1.60667 & 0.03270 & -0.46532 & -0.78351 \\ 0.96580 & -0.22085 & 0.25438 & 0.07253 & 0.04190 \\ 0 & 1.20235 & -0.48397 & 0.09238 & -0.45202 \\ 0 & 0 & 0.68778 & 0.13848 & 0.25796 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1.2221 & 0.08681 \\ 0 & 1.07412 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C}^T = \begin{bmatrix} 0.16365 & 0 \\ 5.85202 & 0.46550 \\ -1.00342 & 0.76456 \\ -1.28731 & -0.40300 \\ -2.92306 & 0.86074 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

There are an infinite number of other valid fifth-order realizations. To check the validity of other answers, use Eq. (*12.5*) to see if the original $\mathbf{H}(s)$ is reconstructed.

**12.11** Use the method of Sec. 12.5.2 to find a minimal realization for the state variable model of Problem 12.4.

The observability matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 & 3 & -2 & 0 & -3 & 2 & 0 & 3 \\ -1 & 1 & -1 & 3 & -1 & 4 & -5 & 1 & -7 \\ 3 & 0 & 6 & -3 & 0 & -6 & 3 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \text{rank}(\mathbf{Q}) = 3$$

The controllability matrix is given by $\mathbf{P} = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 2 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$. Using **QR** decomposition, this becomes

$$\begin{bmatrix} 0.81650 & 0.57735 \\ 0.40825 & -0.57735 \\ -0.40825 & 0.57735 \end{bmatrix} \begin{bmatrix} 2.4495 & -0.8165 & -1.6330 & 0 & 0.81650 & 0.81650 \\ 0 & 1.1547 & 0.57735 & -1.7321 & -1.1547 & 2.3094 \end{bmatrix}$$

From this it is obvious that rank($\mathbf{P}$) = 2 and that an orthonormal basis for the two-dimensional controllable subspace is given by the first two columns above. Using these for $\mathbf{T}_1$ in Eq. (*12.13*) gives the minimal realization

$$\mathbf{A} = \begin{bmatrix} -0.6667 & -0.4714 \\ 0.2357 & -1.3333 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2.4495 & -0.8165 \\ 0 & 1.1547 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 3.4641 & 1.41421 \\ 0.40825 & -0.57735 & 0.707107 \\ -0.40825 & 0.57735 & 3.53553 \end{bmatrix}$$

Of course, **D** remains unchanged.

**12.12** Find a minimal realization for the state variable model which has

$$\mathbf{A} = \begin{bmatrix} -\frac{8}{3} & -1 & -\frac{5}{3} \\ -\frac{2}{3} & -2 & -\frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{7}{3} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The controllability matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 & -4 \\ 0 & -1 & 0 & 2 & 0 & -4 \\ -1 & 1 & 1 & -2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 0.707107 & 0 \\ 0 & 1 \\ -0.707106 & 0 \end{bmatrix} \mathbf{R} = \mathbf{T}_1 \mathbf{R}$$

where **R** is a $2 \times 6$ upper triangular matrix. Thus **P** has rank 2, and the system is not controllable. The observability matrix **Q** is of rank three, so the original system is observable. Using $\mathbf{T}_1$ leads to the Kalman controllable canonical form of Eq. (*12.13*),

$$\dot{\mathbf{w}} = \left[\begin{array}{cc|c} -1 & 1.41421 & -0.333333 \\ 0 & -2 & 0.942809 \\ \hline 0 & 0 & -4 \end{array}\right] \mathbf{w} + \left[\begin{array}{cc} 1.41421 & -1.41421 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cc|c} 0.707106 & 1 & 2.12132 \\ 0 & -1 & 1.41421 \end{array}\right] \mathbf{w}$$

The minimal realization is thus the second-order system with

$$\mathbf{A} = \begin{bmatrix} -1 & 1.41421 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1.41421 & -1.41421 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0.707106 & 1 \\ 0 & -1 \end{bmatrix}$$

Both this system and the original system can be verified to have the same transfer function matrix $\mathbf{H}(s) = \text{Diag}[1/(s+1) \quad 1/(s+2)]$.

### *Irreducible Realizations Using MFD*

**12.13** Use the methods of Sec. 12.6 to find a minimal realization for [5]

$$\mathbf{H}(s) = \frac{\begin{bmatrix} s+6 & s+3 \\ 4 & s+3 \end{bmatrix}}{(s+2)(s+3)}$$

The order of the minimal realization is first found. The common denominator of all first-order minors is $(s+2)(s+3)$. The only second-order minor is the determinant

$$(s+6)/[(s+3)(s+2)^2] - 4/[(s+3)(s+2)^2] = (s+2)/[(s+3)(s+2)^2]$$

After canceling the common factor, this also has $(s+2)(s+3)$ as the denominator, so the minimal order is 2. The left MFD form is used, starting with $\mathbf{D}(s) = [(s+2)(s+3)\mathbf{I}]$ and

$\mathbf{N}(s) = \begin{bmatrix} s+6 & s+3 \\ 4 & s+3 \end{bmatrix}$. Therefore, $\mathbf{D}(s)\mathbf{H}(s) = \mathbf{N}(s)$ can be reduced by using elementary row operations shown next. Subtract row 2 from row 1.

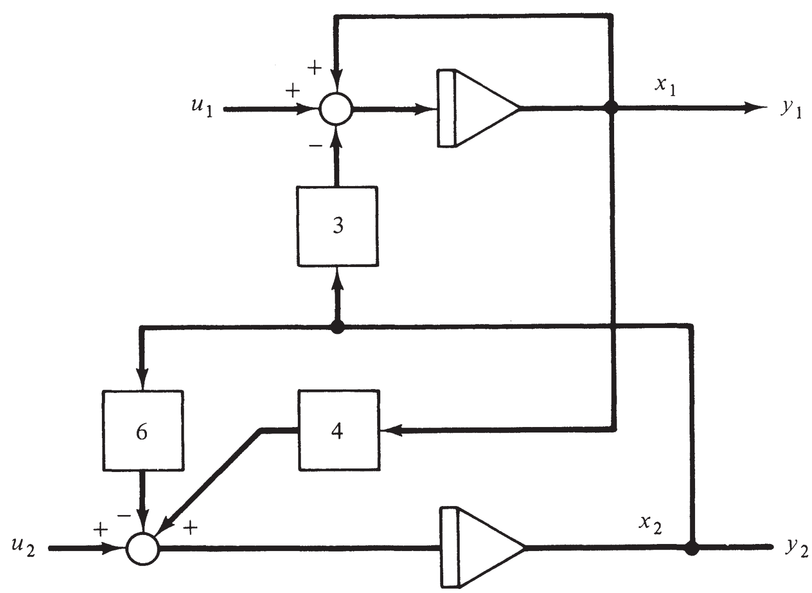
$$\left[\begin{array}{cc|cc} s^2+5s+6 & 0 & s+6 & s+3 \\ 0 & s^2+5s+6 & 4 & s+3 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} s^2+5s+6 & -(s^2+5s+6) & s+2 & 0 \\ 0 & s^2+5s+6 & 4 & s+3 \end{array}\right]$$

Now a factor of $s+2$ can be canceled from each term in row 1. Subtract 4 times row 1 from row 2 to get

$$\left[\begin{array}{cc|cc} s+3 & -(s+3) & 1 & 0 \\ -4(s+3) & s^2+9s+16 & 0 & s+3 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} s+3 & -(s+3) & 1 & 0 \\ -4 & s+6 & 0 & 1 \end{array}\right]$$

The reduction process could stop here (the determinant of the reduced $\mathbf{D}(s)$ has the minimal degree 2), but the state realization is slightly easier if row 2 is first added to row 1. This leaves $\begin{bmatrix} s-1 & s \\ -4 & s+6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. From this, the simulation diagram is drawn. Then the minimal-order state equations are immediately written.

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{I}\mathbf{x}$$




**12.14** The transfer function for a system with two inputs and three outputs is

$$\mathbf{H}(s) = \begin{bmatrix} 2/(s+1)^2 & (4s+2)/(s+1)^2 \\ 1/(s+1) & -1/(s+1) \\ (-s+2)/(s+1)^2 & (7s+4)/(s+1)^2 \end{bmatrix}$$

Find a minimal realization using elementary operations on a right MFD. Note that this is the same $\mathbf{H}(s)$ as in Problem 12.4 but with $\mathbf{D}$ subtracted out.

The minimal order is first determined. The six first-order minors are just the six entries of $\mathbf{H}$, and the lowest common denominator of these is $(s+1)^2$. There are three second-order minors:

$$H_{11}H_{22} - H_{12}H_{21} = -2(s+3)(s+1)/(s+1)^3$$

$$H_{11}H_{32} - H_{12}H_{31} = (8s+6)(s+1)^2/(s+1)^4$$

$$H_{21}H_{32} - H_{22}H_{32} = 6(s+1)/(s+1)^3$$

After canceling common factors, the lowest common denominator of these nine terms is $(s + 1)^2$. Therefore, this system can be realized by a second-order model. From a right MFD we have

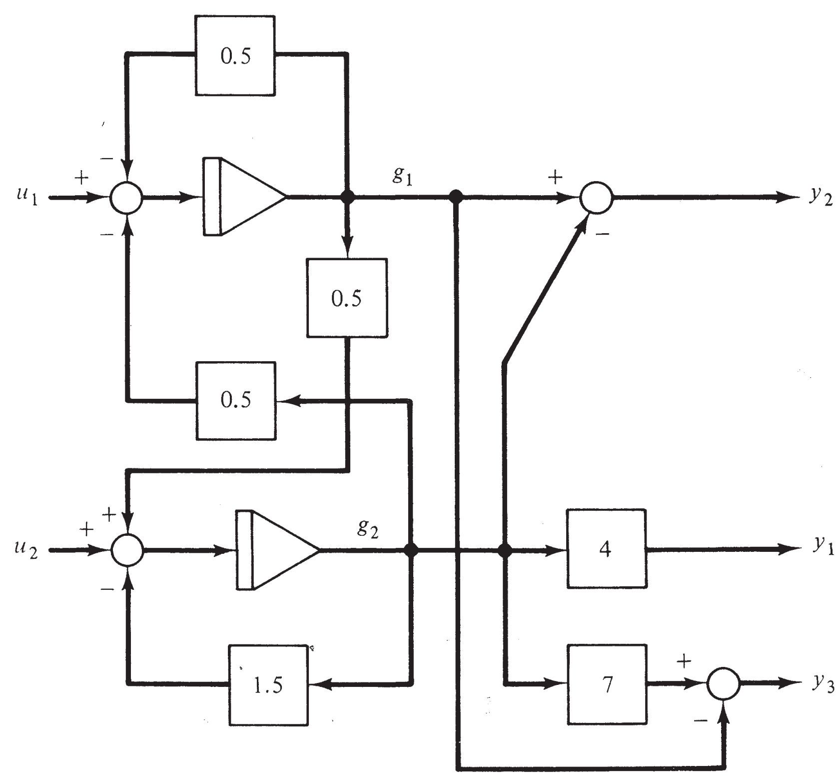
$$\begin{bmatrix} \mathbf{N}(s) \\ \hline \mathbf{P}(s) \end{bmatrix} = \begin{bmatrix} 2 & 4s+2 \\ s+1 & -(s+1) \\ -s+2 & 7s+4 \\ \hline (s+1)^2 & 0 \\ 0 & (s+1)^2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 \\ s+1 & 0 \\ -s+2 & 6 \\ \hline (s+1)^2 & (s+1) \\ 0 & (s+1) \end{bmatrix}$$

Column 1 was added to column 2 and then a factor of $(s + 1)$ was canceled from column 2. This leaves a determinant degree of 3. Subtracting 0.5 times column 2 from column 1 leaves a factor of $(s + 1)$ in every nonzero member of column 1. Canceling gives a form in which $\mathbf{P}(s)$ has the desired determinant degree of 2, as follows. However, one more operation, subtracting column 1 from column 2, reduces the degree of $P_{12}$, and this will simplify the construction of a simulation diagram.

$$\begin{bmatrix} \mathbf{N}(s) \\ \hline \mathbf{P}(s) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ s+1 & 0 \\ -s-1 & 6 \\ \hline (s+1)(s+0.5) & (s+1) \\ -0.5(s+1) & (s+1) \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ -1 & 6 \\ \hline (s+0.5) & (s+1) \\ -0.5 & (s+1) \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 7 \\ \hline s+0.5 & 0.5 \\ -0.5 & s+1.5 \end{bmatrix}$$

Thus $\mathbf{H}(s) = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} s+0.5 & 0.5 \\ -0.5 & s+1.5 \end{bmatrix}^{-1}$. Then $\mathbf{Pg} = \mathbf{u}$ and $\mathbf{Ng} = \mathbf{y}$ can be simulated as shown. The second-order state model is

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -1.5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 7 \end{bmatrix} \mathbf{x}$$



**12.15** In Problems 4.30 and 4.31 the transfer function $\mathbf{H}(s) = \begin{bmatrix} 1/(s+1) & 2/[(s+1)(s+2)] \\ 1/[(s+1)(s+3)] & 1/(s+3) \end{bmatrix}$ was expressed in left and right MFD form, respectively. In both cases elementary operations were used to reduce this to a form with a determinant degree 4.

**(a)** Use Chen's definition of the denominator of **H** to show that the minimal realization will be of fourth order.

**(b)** Use the left MFD to find a fourth-order realization and verify that its matrix **A** has eigenvalues which agree with the poles of Chen's denominator.

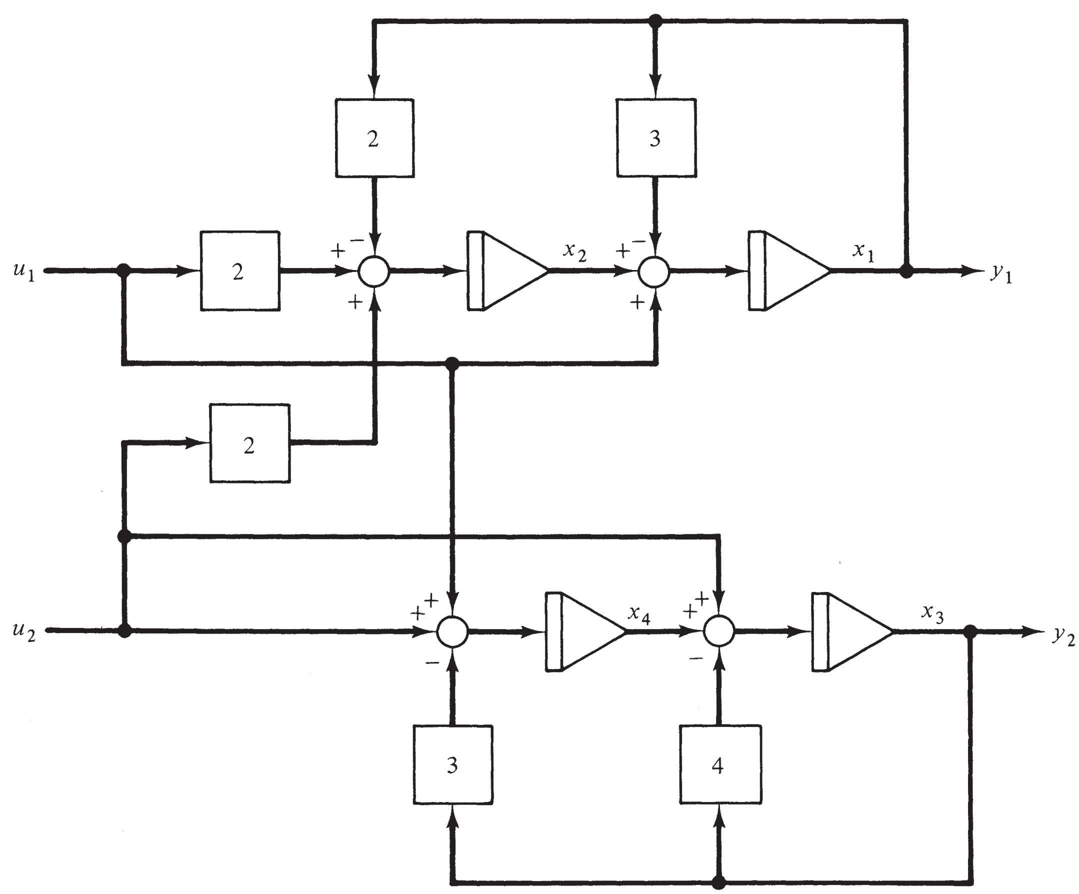
**(a)** The common denominator of the four first-order minors is $(s+1)(s+2)(s+3)$. This was used in Problem 4.30 as an initial MFD with determinant degree 6, due to the $\mathbf{I}_2$ factor. The second-order minor of **H** is the determinant; its denominator is $(s+1)^2(s+2)(s+3)$, so this is the common denominator of all minors. The degree is four, so the minimal realization will be fourth order. Its eigenvalues will be at $-1$, $-1$, $-2$, and $-3$.

**(b)** From the final result in Problem 4.30,

$$\begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+3) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} s+2 & 2 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

A nested integrator form of the simulation diagram is as follows. From the diagram the state equations are

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}$$



and

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}$$

Because the form of the denominator polynomial $\mathbf{P}(s)$ was block diagonal, so is $\mathbf{A}$. Thus it is easy to verify that its eigenvalues are at $-1$, $-1$, $-2$, and $-3$.

**12.16** Use Chen's definition of the denominator of $\mathbf{H}(s)$ to verify that the minimal order realization of the following transfer function is $n = 3$.

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+2)} \\ \dfrac{1}{(s+1)(s+3)} & \dfrac{1}{s+3} \end{bmatrix}$$. The first-order minors are just the $H_{ij}$ elements, and the lowest common denominator of these four terms is $(s+1)(s+2)(s+3)$. There is just one second-order minor, namely,

$$|\mathbf{H}(s)| = \frac{1}{(s+1)(s+3)} - \frac{2}{(s+1)(s+2)(s+3)} = \frac{s}{(s+1)(s+2)(s+3)}$$

The lowest common denominator of all first- and second-order minors is $(s+1) \times (s+2)(s+3)$. It has the order $n = 3$. Furthermore, the poles at $s = -1$, $-2$, and $-3$ will be the eigenvalues of the $3 \times 3$ irreducible $\mathbf{A}$ matrix.

### *Relation Between Irreducibility and Controllability, Observability*

**12.17** Show that an irreducible realization of $\mathbf{H}(s)$ must be completely controllable and observable. Show that a completely controllable and observable realization cannot be reduced. Assume that $\mathbf{D} = [\mathbf{0}]$.

Every $m \times r$ transfer matrix can be expanded into an infinite series $\mathbf{H}(s) = \mathbf{H}_1/s + \mathbf{H}_2/s^2 + \mathbf{H}_3/s^3 + \cdots$, where $\mathbf{H}_i$ are $m \times r$ constant matrices. Every realization satisfies $\mathbf{H}(s) = \mathbf{C}\boldsymbol{\Phi}(s)\mathbf{B}$, and $\boldsymbol{\Phi}(s) = \mathcal{L}\{e^{\mathbf{A}t}\} = \mathbf{I}/s + \mathbf{A}/s^2 + \mathbf{A}^2/s^3 + \cdots$. Therefore, for every realization, $\mathbf{H}_i = \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}$.

Let $n$ be the smallest integer such that $\mathbf{H}_j$ can be written as a combination of $\mathbf{H}_1$ through $\mathbf{H}_n$, for $j > n$. Then $\mathbf{H}(s) = \beta_1(s)\mathbf{H}_1 + \beta_2(s)\mathbf{H}_2 + \cdots + \beta_n(s)\mathbf{H}_n$. If $\dim(\Sigma) = n$ so that $\mathbf{A}$ is $n \times n$, the Cayley-Hamilton theorem gives $\boldsymbol{\Phi}(s) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{n-1}\mathbf{A}^{n-1}$. $\mathbf{A}$ cannot be less than $n \times n$, otherwise $\mathbf{A}^{n-1}$ could be expressed as a linear combination of lower powers of $\mathbf{A}$. This implies that $\mathbf{H}_n$ can be expressed as a combination of $\mathbf{H}_1$ through $\mathbf{H}_{n-1}$ and contradicts the manner in which $n$ was chosen. Thus minimum $\dim(\Sigma) = n$.

Suppose $\mathbf{A}$ is $p \times p$, with $p > n$. Then

$$\mathbf{H}(s) = \beta_1\mathbf{CB} + \beta_2\mathbf{CAB} + \cdots + \beta_n\mathbf{CA}^{n-1}\mathbf{B}$$

must equal

$$\mathbf{H}(s) = \mathbf{C}[\alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{p-1}\mathbf{A}^{p-1}]\mathbf{B}$$

This requires that for $i \geq n$, either $\mathbf{A}^i\mathbf{B}$ is a linear combination of $\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}$, or $\mathbf{CA}^i$ is a linear combination of $\mathbf{C}, \mathbf{CA}, \ldots, \mathbf{CA}^{n-1}$. The first possibility means the system is uncontrollable. The second means the system is unobservable. A realization can have $\dim(\Sigma) > n$ only if it is either not controllable or not observable. Conversely, if $\mathbf{A}$ is $n \times n$ and both controllable and observable, then $\mathbf{H}(s)$ will contain terms up to and including $\mathbf{H}_n$, and thus cannot be reduced.

**12.18** Give a geometrical interpretation of the relationship between irreducible realizations of transfer matrices and controllability and observability.

Assume that $\mathbf{D} = [\mathbf{0}]$ since $\mathbf{D}$ is not influenced by the dimension of the state space $\Sigma$. Then

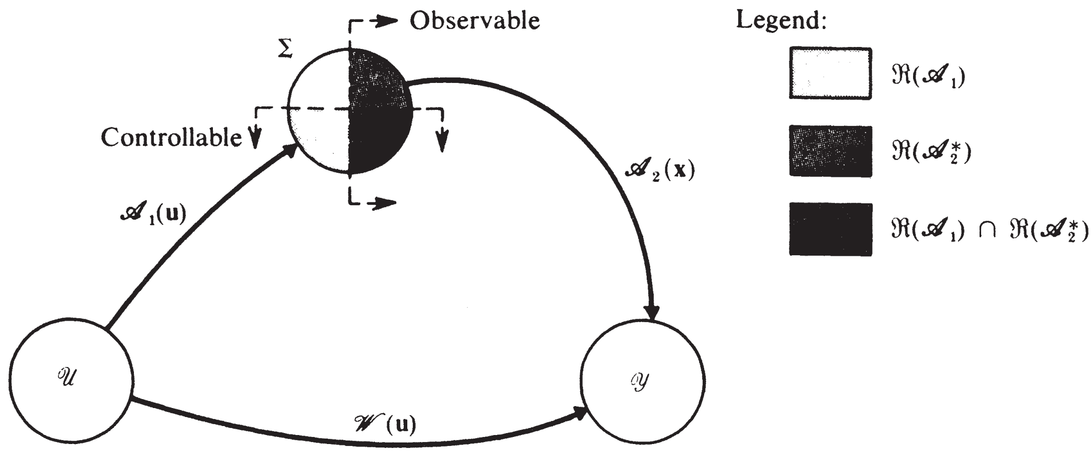



**Figure 12.17**

$\mathbf{H}(s) = \mathbf{C}\boldsymbol{\Phi}(s)\mathbf{B}$. In the time domain the mapping from the input space $\mathcal{U}$ to the output space $\mathcal{Y}$ is given by

$$\mathbf{y}(t) = \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Define $\mathcal{A}_1 : \mathcal{U} \to \Sigma$ by $\mathbf{x} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$ and $\mathcal{A}_2 : \Sigma \to \mathcal{Y}$ by $\mathbf{y} = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}$. Then $\mathbf{y} = \mathcal{A}_2(\mathcal{A}_1(\mathbf{u}))$. The state space can be decomposed into $\Sigma = \mathcal{R}(\mathcal{A}_1) \oplus \mathcal{N}(\mathcal{A}_1^*)$ or alternatively into $\Sigma = \mathcal{R}(\mathcal{A}_2^*) \oplus \mathcal{N}(\mathcal{A}_2)$. Thus $\dim(\Sigma) = \dim(\mathcal{R}(\mathcal{A}_1) + \dim(\mathcal{N}(\mathcal{A}_1^*) = \dim(\mathcal{R}(\mathcal{A}_2^*)) + \dim(\mathcal{N}(\mathcal{A}_2))$. No matter which input $\mathbf{u}(t)$ is used, $\mathcal{A}_1(\mathbf{u}) \in \mathcal{R}(\mathcal{A}_1)$, so the component of $\mathbf{x} \in \mathcal{N}(\mathcal{A}_1^*)$ will be zero. Since $\mathcal{A}_2(\mathbf{0}) = \mathbf{0}$, no part of the output $\mathbf{y}$ due to any input $\mathbf{u}$ will depend upon states in $\mathcal{N}(\mathcal{A}_1^*)$. Thus $\dim(\Sigma)$ can be reduced without affecting the input-output relationship by requiring $\dim(\mathcal{N}(\mathcal{A}_1^*)) = 0$, that is, $\mathcal{N}(\mathcal{A}_1^*) = \{\mathbf{0}\}$. This is the condition for complete controllability. Likewise, any state $\mathbf{x} \in \mathcal{N}(\mathcal{A}_2)$ contributes nothing to the output $\mathbf{y}$.

Making $\dim(\mathcal{N}(\mathcal{A}_2) = 0$, that is, $\mathcal{N}(\mathcal{A}_2) = \{\mathbf{0}\}$, reduces the dimension of $\Sigma$ without affecting the input-output relationship. But $\mathcal{N}(\mathcal{A}_2) = \{\mathbf{0}\}$ is the condition for complete observability. If the realization is completely controllable and observable, the $\dim(\Sigma) = \dim(\mathcal{R}(\mathcal{A}_1)) = \dim(\mathcal{R}(\mathcal{A}_2^*)) = n$, the dimension of the irreducible realization. A suggestive sketch is given in Figure 12.17.

## PROBLEMS

**12.19** Find the state equations and the input-output transfer function for the circuit of Figure 12.18

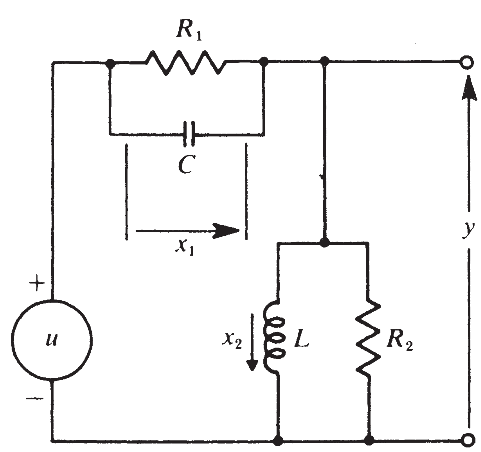



**Figure 12.18**

**12.20** Find the state equations for the circuit of Figure 12.19.

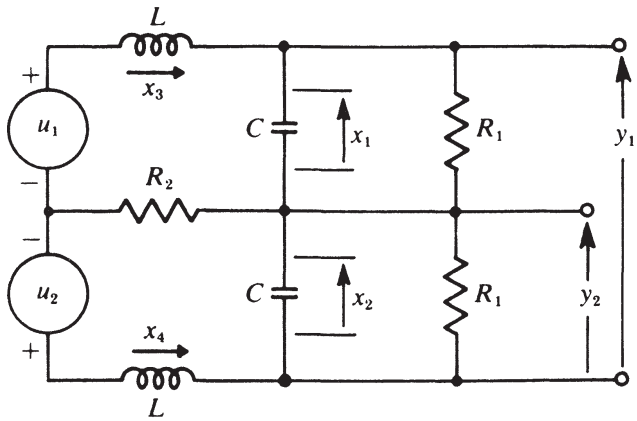



**Figure 12.19**

**12.21** Find the transfer matrix for the system given in Problem 11.21, page 397.

**12.22** Interchange the current source and voltage source of Figure 12.12, and repeat Problem 12.2.

**12.23** Find an irreducible realization for $\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{(s+1)} & \dfrac{1}{(s+1)(s+2)} & \dfrac{s}{(s+1)} \\ 0 & \dfrac{1}{(s+1)(s+3)} & \dfrac{(s+1)}{(s+3)} \end{bmatrix}$.

**12.24** Which of the following system realizations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are irreducible?

**(a)** $\mathbf{A} = \begin{bmatrix} -6 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$

**(b)** Same **A** as in (a), $\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$.

**(c)** Same **A** as in (a), same **B** as in (b), same **C** as in (a), $\mathbf{D} = [\mathbf{0}]$.

**12.25** Find an irreducible realization of $\mathbf{H}(s) = \begin{bmatrix} \dfrac{s}{s+1} & \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{s+3} \\ \dfrac{-1}{s+1} & \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{s} \end{bmatrix}$. See page 220 of Reference 4.

**12.26** Find an irreducible realization of $\mathbf{H}(s) = \begin{bmatrix} \dfrac{-s}{(s+1)^2} & \dfrac{1}{s+1} \\ \dfrac{2s+1}{s(s+1)} & \dfrac{1}{s+1} \end{bmatrix}$.

**12.27** Find an irreducible realization of

$$\mathbf{H}(s) = \frac{1}{s^4}\begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix}$$

See pages 244 and 249 of Reference 4.

**12.28** A discrete-time system's matrices are

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.8 \\ 0 & -0.8 & 0.5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0.5 & -1 & 0 \\ 0.5 & 1 & 1 \end{bmatrix}, \qquad \mathbf{D} = [0]$$

Find the input-output transfer function matrix $\mathbf{T}(z)$.

**12.29** Find a minimal realization for the following system. Be sure that the final system matrices **A, B, C,** and **D** are all real so that they can be synthesized with real hardware.

$$\mathbf{T}(z) = \begin{bmatrix} \dfrac{z}{(z^2 - z + 0.5)(z - 1)(z - 0.3679)} \\ \dfrac{(z + 0.2)}{(z - 1)(z - 0.3679)} \end{bmatrix}$$

*Hint:* Use partial fraction expansion on all real poles, but leave the complex conjugate pair in the form of a second-order segment.

**12.30** A system has the $Z$-transform transfer function

$$\mathbf{H}(z) = \begin{bmatrix} 0.2z/[(z-1)(z-0.5)] & 3z(z+0.3)/[(z-1)(z-0.5)] \\ (z+0.8)/[(z-0.5)^2(z-0.7)] & z/(z-0.5) \end{bmatrix}$$

Use a left MFD, reduce it, and then find a minimal state variable realization. Note that H($z$) is identical to that of Example 12.14 except for one denominator factor.

**12.31** Consider the same system as in Problem 12.30. Construct an eighth-order state variable realization with no internal coupling. Then use the Kalman canonical form method to obtain a fifth-order (minimal) realization.

**12.32** Rework Problem 12.8 using a right MFD approach. Subtract out the constant **D** matrix and then work with $\mathbf{H}'(s) = \mathbf{H}(s) - \mathbf{D}$.

**12.33** Rework Problem 12.8 by first finding a reducible realization with no internal coupling and then reduce to minimal order using the Kalman canonical form procedure.

**12.34** Find a minimal realization for

$$\mathbf{H}(s) = \begin{bmatrix} (s+2)/[(s+1)^2(s+3)] & s/[(s+1)(s+5)] \\ 2/(s+1) & (s+1)/[(s+2)(s+5)] \end{bmatrix}$$

using a left MFD.

**12.35** Use a left MFD representation for the system in Example 12.8. Reduce it and thus find another minimal realization.

# 13

# Design of Linear Feedback Control Systems

## 13.1 INTRODUCTION

A discussion of open-loop versus closed-loop control was presented in Chapter 1. Important advantages of feedback were discussed at that time. Chapter 2 presented a review of the analysis and design of single-input, single-output feedback systems using classical control techniques. It will be recalled that a fundamental method of classical design consists of forcing the dominant closed-loop poles to be suitably located in the $s$-plane or the $Z$-plane. Just what constitutes a suitable location depends upon the design specifications regarding relative stability, response times, accuracy, and so on.

This chapter considers the design of feedback compensators for linear, constant coefficient multivariable systems. One of the fundamental design objectives is, again, the achievement of suitable pole locations in order to ensure satisfactory transient response. This problem is analyzed, first under the assumption that *all* state variables can be used in forming feedback signals. Output feedback, i.e., incomplete state feedback, is also considered.

An additional design objective, which cannot arise in single-input, single-output systems, is the achievement of a decoupled or noninteracting system. This means that each input component affects just one output component, or possibly some prescribed subset of output components.

## 13.2 STATE FEEDBACK AND OUTPUT FEEDBACK

The state equations were introduced in Chapter 3 and analyzed extensively in the intervening chapters. It is assumed here that the open-loop system, often called the plant, is described in state variable form. Most of what is to be discussed in this chapter applies equally well to continuous-time systems,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{13.1}$$

or to discrete-time systems,

$$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)$$
$$\mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \tag{13.2}$$

The system matrices $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ have different meanings in the two cases, and of course the locations of good poles will differ between the $s$-plane and the $Z$-plane. However, if $s_1$ is a good pole location in the $s$-plane, then its image $z_1 = \exp(s_1 T)$ will inherit the same good features in the $Z$-plane. In any event, the methods and procedures to be developed will look the same for both types of systems in terms of the four system matrices.

It is assumed that plant—and hence these four matrices—are specified and cannot be altered by the designer to improve performance. It will be consistently assumed that the state vector $\mathbf{x}$ is $n \times 1$, the input vector $\mathbf{u}$ is $r \times 1$ and the output or measurement vector $\mathbf{y}$ is $m \times 1$. Thus $\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $n \times r$, $\mathbf{C}$ is $m \times n$, and $\mathbf{D}$ is $m \times r$. If the system performance is to be altered, it must be accomplished by some form of signal manipulation outside of the given open-loop system. Two commonly used possibilities are shown in Figures 13.1 and 13.2 for continuous-time systems. These arrangements are referred to as state variable feedback and output feedback, respectively.

The feedback gain matrices $\mathbf{K}$ and $\mathbf{K}'$ are $r \times n$ and $r \times m$, respectively, and are assumed constant. The external inputs $\mathbf{v}$ and $\mathbf{v}'$ are assumed to be $p \times 1$ vectors for generality, although usually $\mathbf{v}$ will have the same number of components as $\mathbf{u}$, so that $r = p$. The feed-forward matrices $\mathbf{F}$ and $\mathbf{F}'$ in the most general case could be of dimension $r \times p$ but are most often square $r \times r$ constant matrices.

It could be justifiably argued that state variable feedback is only of academic interest because, by definition, the outputs are the only signals which are accessible. State variable feedback seems to violate our dictum about using only signals external

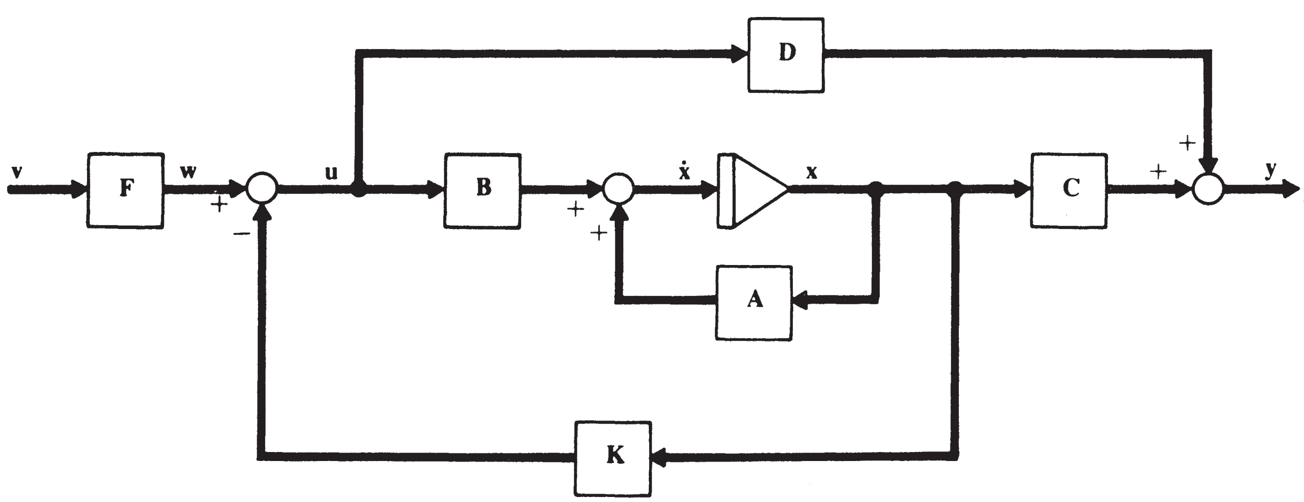



**Figure 13.1** State variable feedback system.

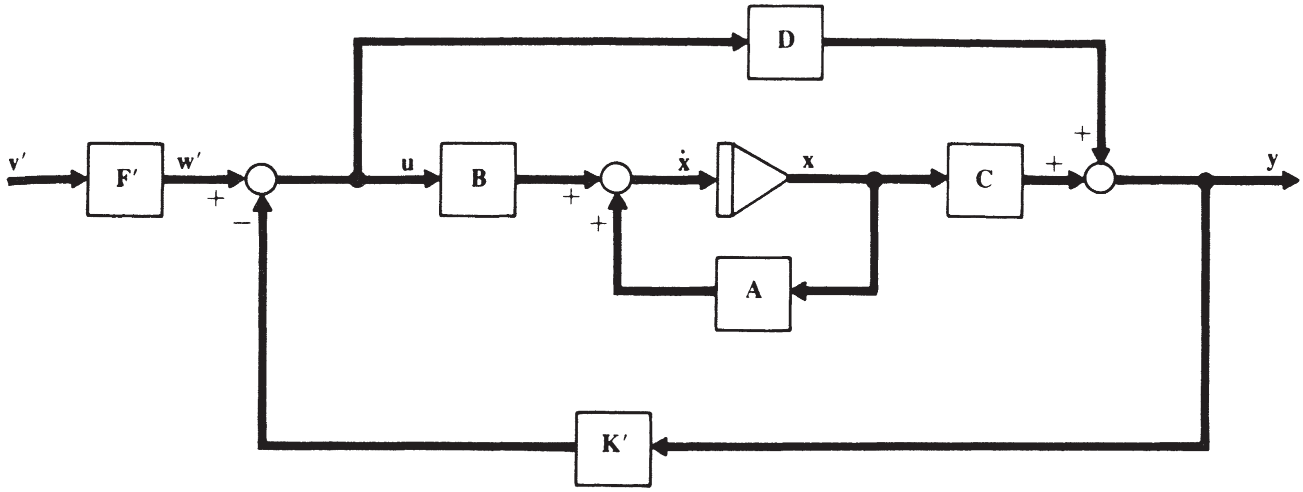



**Figure 13.2** Output feedback system.

to the given open-loop system. In spite of this objection, state feedback is considered for the following reasons:

1. The state **x** contains all pertinent information about the system. It is of interest to determine what can be accomplished by using feedback in this ideal limiting case.
2. There are instances for which the state variables are all measurable, i.e., outputs. This will be the case if $\mathbf{C} = \mathbf{I}_n$ and $\mathbf{D} = [\mathbf{0}]$.
3. Several optimal control laws (see Chapter 14) take the form of a state feedback control. Anticipating this result, it is worthwhile to have an understanding of the effects of state feedback.
4. There are effective means available for estimating or reconstructing the state variables from the available inputs and outputs (see Sec. 13.6).

The equations which describe the state feedback problem are Eq. (*13.1*) or (*13.2*) plus the relation

$$\mathbf{u}(t) = \mathbf{F}\mathbf{v}(t) - \mathbf{K}\mathbf{x}(t) \quad \text{or} \quad \mathbf{u}(k) = \mathbf{F}\mathbf{v}(k) - \mathbf{K}\mathbf{x}(k) \tag{13.3}$$

Combining gives

$$\dot{\mathbf{x}} = [\mathbf{A} - \mathbf{B}\mathbf{K}]\mathbf{x} + [\mathbf{B}\mathbf{F}]\mathbf{v} \quad \text{or} \quad \mathbf{x}(k+1) = [\mathbf{A} - \mathbf{B}\mathbf{K}]\mathbf{x}(k) + [\mathbf{B}\mathbf{F}]\mathbf{v}(k) \tag{13.4}$$

and

$$\mathbf{y} = [\mathbf{C} - \mathbf{D}\mathbf{K}]\mathbf{x} + [\mathbf{D}\mathbf{F}]\mathbf{v} \quad \text{or} \quad \mathbf{y}(k) = [\mathbf{C} - \mathbf{D}\mathbf{K}]\mathbf{x}(k) + [\mathbf{D}\mathbf{F}]\mathbf{v}(k) \tag{13.5}$$

Equations (*13.4*) and (*13.5*) are of the same form as Eqs. (*13.1*) and (*13.2*). Considering $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ as fixed system elements, the question is, "What changes in overall system characteristics can be achieved by choice of **K** and **F**?" *Stability* of the state feedback system depends on the eigenvalues of $[\mathbf{A} - \mathbf{B}\mathbf{K}]$. *Controllability* depends on the pair $\{[\mathbf{A} - \mathbf{B}\mathbf{K}], \mathbf{B}\mathbf{F}\}$. *Observability* depends on the pair $\{[\mathbf{A} - \mathbf{B}\mathbf{K}], [\mathbf{C} - \mathbf{D}\mathbf{K}]\}$. The effect of feedback on these properties is investigated in Sec. 13.3.

The output feedback system is described by Eq. (*13.1*) or (*13.2*) plus $\mathbf{u}(t) =$

$\mathbf{F'v'} - \mathbf{K'y}$. Therefore, for continuous-time systems

$$\mathbf{y}(t) = \mathbf{Cx} + \mathbf{DF'v'} - \mathbf{DK'y}$$

or

$$\mathbf{y}(t) = [\mathbf{I}_m + \mathbf{DK'}]^{-1}\{\mathbf{Cx} + \mathbf{DF'v'}\} \tag{13.6}$$

Using this gives $\dot{\mathbf{x}} = \{\mathbf{A} - \mathbf{BK'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{C}\}\mathbf{x} + \mathbf{B}\{\mathbf{F'} - \mathbf{K'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{DF'}\}\mathbf{v'}$. The matrix inversion identity $\mathbf{I}_r - \mathbf{K'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{D} \equiv [\mathbf{I}_r + \mathbf{K'D}]^{-1}$ simplifies this to

$$\dot{\mathbf{x}} = \{\mathbf{A} - \mathbf{BK'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{C}\}\mathbf{x} + \mathbf{B}[\mathbf{I}_r + \mathbf{K'D}]^{-1}\mathbf{F'v'} \tag{13.7}$$

Again, Eqs. (*13.7*) and (*13.6*) are of the same form as Eq. (*13.1*). The properties of stability, controllability, and observability are now determined by $\{\mathbf{A} - \mathbf{BK'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{C}\}$, $\mathbf{B}[\mathbf{I}_r + \mathbf{K'D}]^{-1}\mathbf{F'}$, and $[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{C}$. Exactly the same kind of results applies to the discrete-time system.

## 13.3 THE EFFECT OF FEEDBACK ON SYSTEM PROPERTIES

The closed-loop systems, obtained by using either state feedback or output feedback, are described by four new system matrices. The reason for adding feedback is to improve the system characteristics in some sense. The effect of feedback on the properties of controllability, observability, and stability should be understood.

### Controllability

Let $\mathbf{P}$ be the controllability matrix of Chapter 11. The open-loop system has $\mathbf{P} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$. With state feedback the controllability matrix becomes

$$\tilde{\mathbf{P}} = [\mathbf{BF} \mid (\mathbf{A} - \mathbf{BK})\mathbf{BF} \mid (\mathbf{A} - \mathbf{BK})^2\mathbf{BF} \mid \cdots \mid (\mathbf{A} - \mathbf{BK})^{n-1}\mathbf{BF}]$$

If the feed-forward matrix $\mathbf{F}$ satisfies rank$(\mathbf{F}) = r$, then $\mathbf{F}$ will not affect the rank of $\tilde{\mathbf{P}}$. Physically, this means that there are as many independent input components $\mathbf{v}$ after adding feedback as there were in the input $\mathbf{u}$ without feedback. Assuming this is true, and letting $\mathbf{F} = \mathbf{I}_r$ for convenience, a series of elementary column operations can be used to reduce $\tilde{\mathbf{P}}$ to $\mathbf{P}$. For example, the columns of $\mathbf{BKB}$ are linear combinations of the columns of $\mathbf{B}$. Elementary operations can reduce these, as well as all other extra terms in $\tilde{\mathbf{P}}$, to $\mathbf{0}$. Therefore, rank $(\tilde{\mathbf{P}})$ = rank$(\mathbf{P})$ for any gain matrix $\mathbf{K}$. Thus state feedback does not alter the controllability of the open-loop system.

Since $\tilde{\mathbf{P}}$ and $\mathbf{P}$ have the same rank for *any* $\mathbf{K}$, including the special case $\mathbf{K} = \mathbf{K'}[\mathbf{I}_m + \mathbf{DK'}]^{-1}\mathbf{C}$, system controllability is also unaltered when output feedback is used. This assumes that $\mathbf{I}_r + \mathbf{K'D}$ is nonsingular and rank $(\mathbf{F'}) = r$.

### Observability

The open-loop observability matrix is $\mathbf{Q} = [\overline{\mathbf{C}}^T \mid \overline{\mathbf{A}}^T\overline{\mathbf{C}}^T \mid \cdots \mid (\overline{\mathbf{A}}^{n-1})^T\overline{\mathbf{C}}^T]$. From Eq. (*13.5*), it is obvious that when state feedback is used, observability is lost if $\mathbf{C} = \mathbf{DK}$. This set of simultaneous linear equations has a solution $\mathbf{K}$ if rank $[\mathbf{D}]$ = rank $[\mathbf{D} \mid \mathbf{C}]$.

This is just one illustration of the general result: state feedback *can* cause a loss of observability. When output feedback is used, observability of the open- and closed-loop systems is the same, as shown next.

Assume, without loss of generality, that $\mathbf{D} = [\mathbf{0}]$. Then the output feedback observability matrix is

$$\tilde{\mathbf{Q}} = [\overline{\mathbf{C}}^T \mid (\overline{\mathbf{A}} - \overline{\mathbf{BK'C}})^T \overline{\mathbf{C}}^T \mid \cdots \mid (\overline{\mathbf{A}} - \overline{\mathbf{BK'C}})^{n-1^T} \overline{\mathbf{C}}^T]$$

A series of elementary column operations, precisely like those used on $\tilde{\mathbf{P}}$, can be used to reduce $\tilde{\mathbf{Q}}$ to $\mathbf{Q}$. This proves that $\text{rank}(\tilde{\mathbf{Q}}) = \text{rank}(\mathbf{Q})$. Observability is preserved when output feedback is used, regardless of $\mathbf{K}'$. The reason why system observability is invariant for output feedback and not for state feedback is the presence of the matrix $\mathbf{C}$ in the $\mathbf{A} - \mathbf{BK'C}$ terms. That is, columns of $\overline{\mathbf{C}}^T \overline{\mathbf{K}}'^T \overline{\mathbf{B}}^T \overline{\mathbf{C}}^T$ can be shown to be linearly related to columns of $\overline{\mathbf{C}}^T$, whereas columns of $\overline{\mathbf{K}}^T \overline{\mathbf{B}}^T \overline{\mathbf{C}}^T$ need not be.

### *Stability*

Stability of linear, constant systems depends entirely on the location of the eigenvalues in the complex plane. Both state and output feedback yield closed-loop eigenvalues which differ from the open-loop eigenvalues. This means that the stability of the closed-loop system is not necessarily the same as that of the open-loop system. The degree of freedom we have in specifying closed-loop eigenvalue locations by choice of $\mathbf{K}$ or $\mathbf{K}'$ is the crux of the pole-assignment problem. This topic is the central theme of this chapter.

### *Poles and Zeros*

Poles and zeros are primarily transfer function concepts. The poles are the values of the complex variable $s$ (or $z$ in discrete-time cases) for which one or more elements of the transfer function matrix have unbounded magnitude. In the single-input, single-output case, the transfer function is a scalar function, a ratio of polynomials. Those values of $s$ (or $z$) for which the denominator is zero are poles. In the case of matrix transfer functions, nothing surprising happens to the definition of poles. This is because if the common denominator vanishes, all terms in the transfer function matrix increase without bound unless the offending denominator root happens to cancel. From Chapter 12 it is seen that the transfer function denominator is the characteristic determinant. Thus the poles are the same as the system eigenvalues except in the unusual case of cancellation. Actually, the poles will always be eigenvalues, but all eigenvalues may not appear as poles because of cancellation. Loosely speaking, the poles and the eigenvalues are the same. This chapter is predominantly devoted to methods of designing feedback systems which give "good" closed loop poles.

In the single-input, single-output case, zeros are those values of $s$ (or $z$) for which the numerator of the transfer function is zero. With such a value of $s$ (or $z$), nonzero inputs $u$ will cause zero output from the system. When multiple-input, multiple-output systems are considered, the generalization of the concept of zero is somewhat more involved. The easiest case to generalize is the one where the numbers of inputs and outputs are equal. The transfer function matrix is thus square. The question of transmission zeros reduces to the question of whether a nonzero input vector $\mathbf{u}$ can cause a zero output $\mathbf{y}$. Since $\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$, it is known from Chapters 4 and 6 that nonzero inputs $\mathbf{u}(s)$ will allow for zero outputs if $\mathbf{H}(s)$ is singular. The zeros in this square transfer function matrix case are the values of $s$ (or $z$) for which the determinant of the transfer function is zero. This definition leads to situations where a transfer function matrix has *no* finite zeros even though the numerators of the individual matrix elements are functions of $s$ and can thus go to zero individually. Individual terms vanishing is not sufficient for causing the total output vector $\mathbf{y}$ to vanish.

It is shown in Problem 13.20 that state feedback does not alter the transfer function zeros. If $s_1$ is a zero of the open-loop system, it is still a zero of the closed-loop system.

For nonsquare transfer functions, zeros are defined as those particular values of $s$ or $z$ which cause the rank of the transfer function matrix to drop below its usual value. Further discussion and alternate definitions may be found in the references.

## 13.4 POLE ASSIGNMENT USING STATE FEEDBACK [1]

Equation (*13.4*) indicates that the eigenvalues of the closed-loop state feedback system are roots of

$$\Delta'(\lambda) \triangleq |\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0 \tag{13.8}$$

It has been proven [2, 3] that if (and only if) the open-loop system $(\mathbf{A}, \mathbf{B})$ is completely controllable, then any set of desired closed-loop eigenvalues $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ can be achieved using a constant state feedback matrix $\mathbf{K}$. In order to synthesize the system with real hardware, all elements of $\mathbf{K}$ must be real. This will be the case if, for each complex $\lambda_i \in \Gamma$, $\bar{\lambda}_i$ is also assigned to $\Gamma$.

One direct and simple method of finding the values of the unknown gain matrix $\mathbf{K}$ which gives specified eigenvalues is to expand Eq. (*13.8*) and equate it to the desired characteristic polynomial. Then equating like powers of $\lambda$ gives expressions or, in simple cases, the values directly for the elements of $\mathbf{K}$.

**EXAMPLE 13.1** Let $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and assume that closed-loop eigenvalues $\lambda = -3$ and $-4$ are desired. This means that the desired characteristic polynomial is

$$(\lambda + 3)(\lambda + 4) = \lambda^2 + 7\lambda + 12$$

Expanding the determinant yields

$$\begin{vmatrix} \lambda & -2 \\ K_1 & \lambda - 3 + K_2 \end{vmatrix} = \lambda^2 + (K_2 - 3)\lambda + 2K_1$$

Equating the constant terms gives $2K_1 = 12$. Equating the first-order terms gives $K_2 - 3 = 7$, so that $\mathbf{K} = [6 \quad 10]$. ■

The explicit method just demonstrated works well for low-order problems. It generalizes easily to single-input, $n$th-order systems whose state equations are in the controllable canonical form, introduced in Chapter 3:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

For this special form of the matrices **A** and **B,** the determinant of Eq. (*13.8*) becomes

$$\begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & -1 & \cdots & 0 \\ \vdots & & & & & -1 \\ K_1 + a_0 & K_2 + a_1 & K_3 + a_2 & \cdots & & \{\lambda + K_n + a_{n-1}\} \end{vmatrix}$$

$$= \lambda^n + (K_n + a_{n-1})\lambda^{n-1} + \cdots + (K_2 + a_1)\lambda + (K_1 + a_0)$$

Equating like powers of this expression and the desired characteristic polynomial gives one equation for each unknown gain component, namely,

$$K_i = c_{i-1} - a_{i-1} \qquad (13.9)$$

where $c_i$ is the coefficient of $\lambda^i$ in the desired closed-loop characteristic polynomial. This expanded form makes it obvious that the more the closed-loop polynomial differs from the open-loop polynomial, the larger are the required feedback gains. Note that this procedure applies to the controllable canonical form. The system of Example 13.1 is *not* in this canonical form, so Eq. (*13.9*) cannot be applied directly. However, for any single-input controllable system there exists a nonsingular transformation **T,** which maps the state vector **x** into a new state vector **x**′ in such a way as to give the **x**′ state equations in controllable canonical form. For multiple inputs the simple notion of the canonical forms become more complicated. There is not universal agreement on what the controllable canonical form is in the multiple input case. In this paragraph we consider only the single-input case. Let $\mathbf{x}' = \mathbf{T}\mathbf{x}$, and then the corresponding matrices $\mathbf{A}'$ and $\mathbf{B}'$ for these new states are $\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ and $\mathbf{B}' = \mathbf{T}\mathbf{B}$. Equation (*13.9*) can be used to find the feedback gains $\mathbf{K}'$ that are appropriate for the new state vector. Then equating the feedback signals $\mathbf{K}'\mathbf{x}' = \mathbf{K}\mathbf{x}$ and using $\mathbf{x}' = \mathbf{T}\mathbf{x}$ gives $\mathbf{K} = \mathbf{K}'\mathbf{T}$.

**EXAMPLE 13.2** The system of Example 13.1 is transformed to controllable canonical form by using $\mathbf{T} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, giving $\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{B}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. From the last row of $\mathbf{A}'$, $a_0 = 0$ and $a_1 = -3$. From Eq. (*13.9*), $\mathbf{K}_1' = 12$ and $\mathbf{K}_2' = 7 + 3 = 10$. The gain **K** to be used with the original state variables is, therefore, $\mathbf{K} = \mathbf{K}'\mathbf{T} = [6 \quad 10]$. This is the same unique answer found in Example 13.1. ■

When there are multiple inputs or when the state equations are not in controllable canonical form, the preceding simple methods may not provide the best approach to the determination of feedback gain matrices. An approach which works for any order, any number of inputs, and any arbitrary form of the state equations is now developed. The problem is to determine a gain matrix $K$ such that Eq. (*13.8*) is satisfied for each of $n$ specified values $\lambda_i \in \Gamma$.

If Eq. (*13.8*) is true, then there exists at least one nonzero vector $\boldsymbol{\psi}_i$ such that

$$(\lambda_i \mathbf{I} - \mathbf{A} + \mathbf{BK})\boldsymbol{\psi}_i = \mathbf{0} \tag{13.10}$$

Rearranged, this says that

$$(\mathbf{A} - \mathbf{BK})\boldsymbol{\psi}_i = \lambda_i \boldsymbol{\psi}_i \tag{13.11}$$

This makes it clear that $\boldsymbol{\psi}_i$ is an eigenvector of the closed-loop system matrix $(\mathbf{A} - \mathbf{BK})$ associated with the closed-loop eigenvalue (pole) $\lambda_i$. Rewriting Eq. (*13.10*) in yet another way gives

$$(\lambda_i \mathbf{I} - \mathbf{A})\boldsymbol{\psi}_i = -\mathbf{BK}\boldsymbol{\psi}_i$$

or

$$[(\lambda_i \mathbf{I} - \mathbf{A}) \mid \mathbf{B}]\begin{bmatrix} \boldsymbol{\psi}_i \\ \mathbf{K}\boldsymbol{\psi}_i \end{bmatrix} = [0] \tag{13.12}$$

At this point the vector $\boldsymbol{\psi}_i$ and the matrix $\mathbf{K}$ are both unknown. Therefore, nothing is lost, and notational simplicity is gained by defining the $(n + r) \times 1$ unknown vector as

$$\boldsymbol{\xi}_i = \begin{bmatrix} \boldsymbol{\psi}_i \\ \hline \mathbf{K}\boldsymbol{\psi}_i \end{bmatrix} \tag{13.13}$$

The determination of $\mathbf{K}$ consists of two general steps. First, a sufficient number of solution vectors $\boldsymbol{\xi}_i$ is found. Then the internal structure among the components of these vectors, as expressed by Eq. (*13.13*), is used to find $\mathbf{K}$.

If the open-loop system described by the pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable, it is known from Sec. 11.5 that the $n \times (n + r)$ coefficient matrix in the homogeneous equation (*13.12*) has full rank $n$ for any value of $\lambda_i$. The solution of homogeneous equations has been discussed in Chapters 6 and 7. From this work it is known that there will thus be $r$ independent solution vectors $\boldsymbol{\xi}_i$ for each $\lambda_i$.

**EXAMPLE 13.3** Let the system of Eq. (*13.1*) have $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the solution vectors $\boldsymbol{\xi}_i$ for $\lambda_i = -3$ and $\lambda_i = -4$.

The controllability of this system is easily verified. Since $r = 1$, in this case there will be only one independent $\boldsymbol{\xi}_i$ for each $\lambda_i$. With $\lambda_1 = -3$, Eq. (*13.12*) gives

$$\begin{bmatrix} -3 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix}\boldsymbol{\xi}_1 = \mathbf{0}$$

Arbitrarily selecting $\xi_2 = 1$ gives $\xi_1 = -\frac{2}{3}$ and $\xi_3 = 6$. Recalling from Eq. (*13.13*) how $\boldsymbol{\xi}_i$ was defined, it becomes obvious that

$$\boldsymbol{\xi}_1 = [-\tfrac{2}{3} \quad 1 \quad 6]^T \quad \text{is equivalent to} \quad \mathbf{K}\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} = 6 \tag{13.14}$$

Similarly, with $\lambda_2 = -4$ the homogeneous equation (*13.12*) has a solution

$$\boldsymbol{\xi}_2 = [-\tfrac{1}{2} \quad 1 \quad 7]^T, \quad \text{which implies} \quad \mathbf{K}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = 7 \tag{13.15}$$

Taken together, these two equations are

$$\mathbf{K}\begin{bmatrix} -\frac{2}{3} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} = [6 \quad 7]$$

From Chapter 7 recall that eigenvectors for two different eigenvalues must be independent. This can be observed explicitly here. The $2 \times 2$ matrix that multiplies **K** has independent columns and hence is nonsingular. An inversion thus allows the unknown **K** to be found.

$$\mathbf{K} = \frac{[6 \quad 7]\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -\frac{2}{3} \end{bmatrix}}{(-\frac{1}{6})} = [6 \quad 10]$$

Note that although the $\boldsymbol{\xi}_i$ vectors are not unique, all other solutions for each $\lambda_i$ will be multiples of those given earlier. The arbitrary nonzero constant multipliers will cancel from Eqs. (*13.14*) and (*13.15*), thus giving a *unique* solution for **K.** This is *not* the case when there is more than one input, i.e., $r > 1$. This issue is explored and the preceding method of finding the feedback matrix **K** is generalized next. ■

Let the maximal set of $r$ linearly independent solution vectors of Eq. (*13.12*) for a given eigenvalue form the columns of an $(n + r) \times r$ matrix $\mathbf{U}(\lambda_i)$. These entire $(n + r)$ component columns constitute a basis for the null space of $[(\lambda_i \mathbf{I} - \mathbf{A} \mid \mathbf{B}]$. From Eq. (*13.13*), the top $n$ components of each column form a closed-loop eigenvector and the remaining bottom $r$ components are that same vector multiplied by an as yet unknown gain matrix **K.** The matrix **U** is partitioned accordingly as

$$\mathbf{U}(\lambda_i) = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \cdots & \boldsymbol{\psi}_r \\ \mathbf{f}_1 & \mathbf{f}_2 & \cdots & \mathbf{f}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}(\lambda_i) \\ \hline \mathcal{F}(\lambda_i) \end{bmatrix}$$

where the substitution $\mathbf{f}_i = \mathbf{K}\boldsymbol{\psi}_i$ has been made. Collectively, all these relations for the $n$ eigenvalues can be written

$$\mathbf{K}[\boldsymbol{\Psi}(\lambda_1) \quad \boldsymbol{\Psi}(\lambda_2) \quad \cdots \quad \boldsymbol{\Psi}(\lambda_n)] = [\mathcal{F}(\lambda_1) \quad \mathcal{F}(\lambda_2) \quad \cdots \quad \mathcal{F}(\lambda_n)] \tag{13.16}$$

Equation (*13.16*) cannot be solved directly for **K** because if $r > 1$, it is overdetermined and thus represents inconsistent equations. However, if the system is controllable, a nonsingular $n \times n$ matrix of $\boldsymbol{\psi}_j(\lambda_i)$'s can be found by selecting $n$ linearly independent columns from both sides of Eq. (*13.16*) and deleting the remaining columns. In this selection process, one column must be selected for each specified eigenvalue $\lambda_i$. Let the selected $n$ columns from the left-hand side form a matrix **G** and the corresponding columns from the right side form a matrix $\mathcal{G}$. If this is done, the result $\mathbf{KG} = \mathcal{G}$ can then be solved for the feedback gain matrix **K,**

$$\mathbf{K} = \mathcal{G}\mathbf{G}^{-1} \tag{13.17}$$

The freedom in arbitrarily selecting a subset of $n$ columns, subject only to the requirement that one column is chosen for each desired $\lambda_i$ and that $\mathbf{G}^{-1}$ exists, is what leads to the multiplicity of possible feedback gain matrices in the multiple-input case. Actually, any linear combination of columns can be selected from the $r$ columns found for each $\lambda_i$. That is, any vector belonging to the null space can be used.

**EXAMPLE 13.4** Consider the same matrix **A** as in the previous problem; however, now there are two inputs, with $\mathbf{B} = \mathbf{I}_2$. Find a feedback gain matrix which yields closed-loop eigenvalues at $\lambda = -3$ and $-5$.

With $\lambda_1 = -3$, Eq. (*13.12*) gives

$$\begin{bmatrix} -3 & -2 & 1 & 0 \\ 0 & -6 & 0 & 1 \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Letting $\xi_1 = \alpha$ and $\xi_2 = \beta$ leads to a solution $\boldsymbol{\xi}_1 = [\alpha \quad \beta \quad 3\alpha + 2\beta \quad 6\beta]^T$. It is possible to find two and only two independent vectors by specifying various values for $\alpha$ and $\beta$. What has been found so far can be written as

$$\mathbf{K}\begin{bmatrix} \alpha & \cdot \\ \beta & \end{bmatrix} = \begin{bmatrix} 3\alpha + 2\beta & \cdot \\ 6\beta & \end{bmatrix}$$

where the extra space has been left intentionally as a reminder that another equation will be found and filled in for $\lambda_2 = -5$. With $\lambda_2 = -5$, the same procedure gives

$$\begin{bmatrix} -5 & -2 & 1 & 0 \\ 0 & -8 & 0 & 1 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0}$$

Letting $\xi_1 = \gamma$ and $\xi_2 = \delta$ gives $\boldsymbol{\xi}_2 = [\gamma \quad \delta \quad 5\gamma + 2\delta \quad 8\delta]^T$, which means that

$$\mathbf{K}\begin{bmatrix} \cdot & \gamma \\ & \delta \end{bmatrix} = \begin{bmatrix} \cdot & 5\gamma + 2\delta \\ & 8\delta \end{bmatrix}$$

Taken together the equations for determining **K** are

$$\mathbf{K}\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} 3\alpha + 2\beta & 5\gamma + 2\delta \\ 6\beta & 8\delta \end{bmatrix} \tag{13.18}$$

Any values of $\alpha$, $\beta$, $\gamma$, and $\delta$ will give a valid gain matrix as long as the required inverse exists. The preceding degree of generality is not normally required and will be dispensed with in most of the examples to follow. To simplify calculations but also to give an orthogonal set of closed-loop eigenvectors, we select $\alpha = \delta = 1$ and $\beta = \gamma = 0$. Then, by inspection, the solution of Eq. (*13.18*) is $\mathbf{K} = \begin{bmatrix} 3 & 2 \\ 0 & 8 \end{bmatrix}$. ■

Since multiple-input systems allow an infinite number of choices for the feedback gain matrix, the designer must make choices. This freedom of choice can be exercised in various ways, which are discussed throughout this and the next chapter. One related factor can be simply stated now. Knowing that $\boldsymbol{\psi}_i$ are closed-loop eigenvectors may help guide the choice of which columns to retain. Actually, any linear combination of columns from a given partition of $\boldsymbol{\Psi}$ can be used as long as the same linear combination of $\mathbf{f}_i$ columns is used. It is often worthwhile to try to make **G** be as nearly orthogonal as possible because this improves its invertibility robustness and tends to give less interaction between modes of the closed-loop system. This orthogonality usually improves system sensitivity to parameter variations.

One final complication must be settled before a completely general method of pole assignment can be claimed. Suppose it is desired that $\lambda_i$ be a $p$th-order root of the characteristic equation. If $r$, the dimension of the null space or, equivalently, the rank of $\boldsymbol{\Psi}$, is greater than or equal to $p$, then the preceding procedure will suffice. If $r < p$, then it is not possible to specify $p$ eigenvectors. It is still possible to achieve the desired $p$ eigenvalues, however. The excess vectors for such a repeated eigenvalue will be generalized eigenvectors, as demonstrated next. Let $\boldsymbol{\psi}_i$ be an eigenvector which has already been found using this procedure. Let $\boldsymbol{\psi}_g$ denote an associated generalized eigenvector. Recall from Chapter 7 that it satisfies

$$(\mathbf{A} - \mathbf{BK})\boldsymbol{\psi}_g = \lambda\boldsymbol{\psi}_g + \boldsymbol{\psi}_i$$

This can be rearranged to read

$$(\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK})\boldsymbol{\psi}_g = -\boldsymbol{\psi}_i$$

or

$$[(\lambda\mathbf{I} - \mathbf{A}) \mid \mathbf{B}]\begin{bmatrix} \boldsymbol{\psi}_g \\ \mathbf{K}\boldsymbol{\psi}_g \end{bmatrix} = -\boldsymbol{\psi}_i \tag{13.19}$$

Equation (*13.19*) replaces Eq. (*13.12*) for the first required generalized eigenvector. If more than one generalized eigenvector is required, the second and subsequent choices may satisfy Eq. (*13.19*) but with a different eigenvector on the right-hand side, *or* the additional generalized eigenvectors may be chained to the previous ones. Which alternative prevails is dictated only by the requirement that a full set of *independent* vectors be found so that the matrix **G** is invertible. (See Illustrative Problems 13.4 and 13.5.) Generalized eigenvectors are brought into the pole-placement discussion solely because we know that they provide a needed set of independent vectors. As was shown in Chapter 7, there are several ways of finding generalized eigenvectors.

**EXAMPLE 13.5** Consider the system of Example 13.1. Find a feedback gain matrix that gives both closed-loop eigenvalues $\lambda = -1$. In this case Eq. (*13.12*) specializes to

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & -4 & 1 \end{bmatrix}\boldsymbol{\xi} = \mathbf{0}$$

All nontrivial solutions are proportional to

$$\boldsymbol{\xi} = [-2 \quad 1 \quad 4]^T$$

Obviously only one eigenvector $\boldsymbol{\psi} = [-2 \quad 1]$ can be selected from this one-dimensional space, so a generalized eigenvector is needed. It is found from Eq. (*13.19*), that is,

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & -4 & 1 \end{bmatrix}\boldsymbol{\xi}_g = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Selecting $\xi_2 = 1$ gives $\xi_1 = -4$ and $\xi_3 = 3$, so that a solution is $\boldsymbol{\xi}_g = [-4 \quad 1 \quad 3]^T$. Therefore, the two equations for finding **K** are

$$\mathbf{K}\begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix} = [4 \quad 3]$$

Matrix inversion or its equivalent gives $\mathbf{K} = [0.5 \quad 5]$. ■

### *Summary of the Pole Placement Algorithm (Eigenvalue-Eigenvector Assignment)*

Given **A, B,** and the desired set of eigenvalues, carry out the following steps.

**I.** For each $\lambda_i$:

**1.** Form $[(\lambda_i\mathbf{I} - \mathbf{A} \mid \mathbf{B}]$.

**2.** Find the null space basis set **U** by finding all independent solutions of Eq. (*13.12*).

**3.** Partition **U,** using the top $n$ rows as the $n \times r$ matrix $\mathbf{\Psi}(\lambda_i)$.

**4.** Use the remaining $r$ rows of **U** as the $r \times r$ matrix $\mathcal{F}(\lambda_i)$.

**II.** Form the composite matrices $\mathbf{\Omega} = [\mathbf{\Psi}(\lambda_1) \quad \mathbf{\Psi}(\lambda_2) \quad \cdots \quad \mathbf{\Psi}(\lambda_n)]$; $n \times nr$, rank $n$ if controllable and $\mathbf{\Lambda} = [\mathcal{F}(\lambda_1) \quad \mathcal{F}(\lambda_2) \quad \cdots \quad \mathcal{F}(\lambda_n)]$; $r \times nr$. Equation (*13.16*) can now be compactly written as $\mathbf{K\Omega} = \mathbf{\Lambda}$.

**III.** Select $n$ linearly independent columns of $\mathbf{\Omega}$ to form the $n \times n$ matrix **G.** One column (or any linear combinations of columns) must be selected from each $\mathbf{\Psi}(\lambda_i)$ partition. The selected columns will be closed-loop eigenvectors. As a preliminary screening out of linearly dependent columns, the inner products should be checked. This can be done by forming $\mathbf{\Omega}^T\mathbf{\Omega}$. The $i, j$ element, normalized by the square root of the $i, i$ and $j, j$ elements, is the cosine of the angle between columns $i$ and $j$. If this generalized cosine is 1 for any pair of selected columns, the columns are linearly dependent and the selection must be modified accordingly.

**IV.** Use the same column numbers selected in step III to form the $r \times n$ matrix $\mathcal{J}$ from $\mathbf{\Lambda}$.

**V.** Solve $\mathbf{KG} = \mathcal{J}$ for the $r \times n$ gain matrix **K.** It may be more convenient with some software packages to solve $\mathbf{G}^T\mathbf{K}^T = \mathcal{J}^T$ instead. Note that passing the pairwise inner product test in step III is necessary but not sufficient for the existence of $\mathbf{G}^{-1}$. Therefore, it may be necessary to return to step III and modify the eigenvector column selections. Also note that two columns being nearly collinear will lead to a nearly singular matrix to be inverted. This situation can be expected to give a poor, or non-robust, solution. It can be avoided by selecting columns which are as nearly orthogonal as possible.

**VI.** General Comments:

**1.** The process of selecting orthogonal columns could be automated, perhaps by using a modified Gram-Schmidt orthogonalization process or a singular value decomposition. However, allowing the designer the discretion of interactively making the selection allows judgments to be made regarding the desirability of certain closed-loop eigenvectors.

**2.** If a certain eigenvalue is specified to be a $p$th-order repeated root, with $p > r$, then $p - r$ generalized eigenvectors will need to be found, using Eq. (*13.19*) in place of Eq. (*13.12*) in step I.2.

**3.** It is never necessary to invert a complex matrix in Eq. (*13.17*), even when complex-valued eigenvalues and eigenvectors are selected. The result in Problem 4.23 allows the use of purely real arithmetic, as demonstrated in Problem 13.3.

**4.** The single-input problem ($r = 1$) can be solved by simpler methods that deal directly with the coefficients of the characteristic polynomial. A program which uses the approach of Example 13.2 is STVARFDBK [4].

**EXAMPLE 13.6** A model [5] of the lateral dynamics of an F-8 aircraft at a particular set of flight condition was given in Problem 11.9. A discrete approximation for this system is obtained using the method of Problem 9.10, with $T = 0.2$ and retaining terms through thirtieth order.

The Approximate Discrete Transition Matrix

$$\begin{bmatrix} 1.3533533\text{E}-01 & 9.8391518\text{E}-02 & -7.2121400\text{E}-01 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & 7.1751231\text{E}-01 & 1.4726298\text{E}+00 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & -1.6362552\text{E}-01 & 7.1751231\text{E}-01 & 0.0000000\text{E}+00 \\ 8.6466469\text{E}-02 & 7.8584068\text{E}-03 & -1.0389241\text{E}-01 & 1.0000000\text{E}+00 \end{bmatrix}$$

The Approximate Input Matrix **B**

$$\begin{bmatrix} 1.7293293\text{E}+00 & 2.1750930\text{E}-01 \\ 0.0000000\text{E}+00 & -5.5092329\text{E}-01 \\ 0.0000000\text{E}+00 & 5.5393357\text{E}-02 \\ 2.2706707\text{E}-01 & 3.0423844\text{E}-02 \end{bmatrix}$$

Design a state feedback controller which will provide closed-loop $Z$-plane poles corresponding to $s$-plane poles at $s = -2, -5, -8$, and $-10$. Using $z = e^{Ts}$, this means that the desired $Z$-plane poles are

$$\lambda_i = 0.6703,\ 0.3679,\ 0.2019, \text{ and } 0.1353$$

Solving Eq. (*13.12*) for each $\lambda_i$ in turn gives

$$\mathbf{U}(\lambda_1) = \left[\begin{array}{cc} -1.0000000\text{E}+00 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & 4.4247019\text{E}-01 \\ 0.0000000\text{E}+00 & 3.5991812\text{E}-01 \\ 4.7545883\text{E}-01 & 1.0787997\text{E}-02 \\ \hline 3.0937374\text{E}-01 & 8.4780902\text{E}-04 \\ 0.0000000\text{E}-00 & -1.0000000\text{E}+00 \end{array}\right]$$

$$\mathbf{U}(\lambda_2) = \left[\begin{array}{cc} -1.0000000\text{E}+00 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & 7.5502163\text{E}-01 \\ 0.0000000\text{E}+00 & 1.9485569\text{E}-01 \\ 1.8517032\text{E}-01 & 5.8909208\text{E}-03 \\ \hline 1.3450530\text{E}-01 & 8.7471344\text{E}-02 \\ 0.0000000\text{E}+00 & -1.0000000\text{E}+00 \end{array}\right]$$

$$\mathbf{U}(\lambda_3) = \left[\begin{array}{cc} -1.0000000\text{E}+00 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & 7.2149646\text{E}-01 \\ 0.0000000\text{E}+00 & 1.2148339\text{E}-01 \\ 1.1934122\text{E}-01 & 3.6517035\text{E}-03 \\ \hline 3.8512699\text{E}-02 & 1.1616344\text{E}-01 \\ 0.0000000\text{E}+00 & -1.0000000\text{E}+00 \end{array}\right]$$

$$\mathbf{U}(\lambda_4) = \left[\begin{array}{cc} -1.0000000\text{E}+00 & 0.0000000\text{E}+00 \\ 0.0000000\text{E}+00 & 6.9379878\text{E}-01 \\ 0.0000000\text{E}+00 & 9.9803299\text{E}-02 \\ 1.0003469\text{E}-01 & 2.9885261\text{E}-03 \\ \hline 0.0000000\text{E}+00 & 1.2362902\text{E}-01 \\ 0.0000000\text{E}+00 & -1.0000000\text{E}+00 \end{array}\right]$$

The portions above the partition lines are $\boldsymbol{\Psi}(\lambda_i)$ and the lower portions are $\mathcal{F}(\lambda_i)$. In order to find a suitable gain, some combination of columns (1 or 2) and (3 or 4) and (5 or 6) and (7 or 8) must be selected. For example, if columns 1, 4, 6, and 7 are used, Eqs. (*13.16*) and (*13.18*) give

The Feedback Gain Matrix

$$\mathbf{K} = \begin{bmatrix} 8.2435042\text{E}-02 & 2.4582298\text{E}-01 & -5.2851838\text{E}-01 & 8.2406455\text{E}-01 \\ 0.0000000\text{E}+00 & -1.5015275\text{E}+00 & 6.8607545\text{E}-01 & 0.0000000\text{E}+00 \end{bmatrix}$$

As a check of the result, the closed-loop system matrix $\mathbf{A} - \mathbf{BK}$ is formed. The eigenvalues of this closed-loop system matrix are given below, and verify that the desired pole locations have been achieved.

The resulting closed-loop eigenvalues are

| Real Part | Imaginary Part |
|---|---|
| 1.3529983E − 01 | 0.0000000E + 00 |
| 2.0190017E − 01 | 0.0000000E + 00 |
| 3.6790001E − 01 | 0.0000000E + 00 |
| 6.7029989E − 01 | 0.0000000E + 00 |

If columns 2, 4, 5, and 7 are selected, the following alternative gain matrix is obtained. Notice that the two sets of results have totally different **K** matrices and closed-loop eigenvectors, although the closed-loop eigenvalues will be the same.

The Feedback Gain Matrix

$$\mathbf{K} = \begin{bmatrix} 1.9954935\text{E}-01 & 1.6860582\text{E}-01 & -2.6471341\text{E}-01 & 1.9948014\text{E}+00 \\ 0.0000000\text{E}+00 & -8.8968951\text{E}-01 & -1.6846579\text{E}+00 & 0.0000000\text{E}+00 \end{bmatrix} \quad \blacksquare$$

Before concluding the discussion of state variable feedback, assume that the gain matrix **K** is factored into $\mathbf{K} = \mathbf{F}_1\mathbf{K}_1$, and that $\mathbf{K}_1$ is left in the feedback path and $\mathbf{F}_1$ is placed in the forward path, as shown in Figure 13.3. There is no change in the characteristic equation and hence the poles. The numerator of the closed-loop transfer function can differ greatly, however. If $\mathbf{F}_1$ is just a scalar, then only the gains of the transfer function elements change. If $\mathbf{F}_1$ is a full $r \times r$ matrix, the transfer function can take on a greatly modified appearance. Even though the functions of $s$ (or $z$) that appear in the numerator elements may look totally different, the zeros of the transfer function remain unchanged. As in Sec. 13.3, it is assumed here that $\mathbf{H}(s)$ or $\mathbf{H}(z)$ is square, so that zeros are easily defined as the values of $s$ or $z$ that give a zero determinant of **H**.

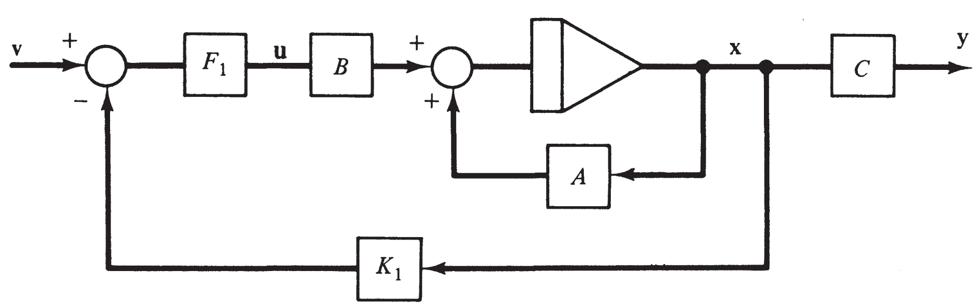



**Figure 13.3**

**EXAMPLE 13.7** Show that if

$$\mathbf{K} = \begin{bmatrix} 2 & 0 \\ -4.5 & 9 \end{bmatrix}$$

is used in a state feedback controller for the second-order system of Example 12.6, the closed-loop poles are at $-5$ and $-10$. Then consider two optional implementations,

$$\mathbf{K} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5.5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3.25 & -4.5 \\ -1.25 & 4.5 \end{bmatrix}$$

and find the closed-loop transfer functions.

If the control law is $\mathbf{u} = -\mathbf{Kx} + \mathbf{F}_1\mathbf{v}$, the expression for the closed-loop transfer function is

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{BK}]^{-1}\mathbf{BF}_1 \quad \text{and} \quad \mathbf{A} - \mathbf{BK} = \begin{bmatrix} -5 & 0 \\ 2.5 & -10 \end{bmatrix}$$

For the three values of $\mathbf{F}_1$, namely $\mathbf{I}$, $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, the resultant transfer functions are

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+5} & \dfrac{1}{s+10} \\[2mm] \dfrac{2}{s+5} & 0 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{3(s+\frac{25}{3})}{(s+5)(s+10)} & \dfrac{1}{s+10} \\[2mm] \dfrac{4}{s+5} & 0 \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} \dfrac{5}{(s+5)(s+10)} & \dfrac{2(s+\frac{15}{2})}{(s+5)(s+10)} \\[2mm] \dfrac{2}{s+5} & \dfrac{2}{s+5} \end{bmatrix}$$

The determinants of these three transfer functions are

$$-2/[(s+5)(s+10)], \qquad -4/[(s+5)(s+10)], \quad \text{and} \quad -4(s+5)/[(s+5)^2(s+10)]$$

respectively. In each case there is no finite value of $s$ that gives a zero value to the determinant. Hence there are no zeros. ■

## 13.5 PARTIAL POLE ASSIGNMENT USING STATIC OUTPUT FEEDBACK [6]

The feedback signal considered in this section is formed by premultiplying the output $\mathbf{y}$ by a constant $r \times m$ gain matrix called $\mathbf{K}'$. If $\mathbf{D}$ is not zero, a modified output vector $\mathbf{y}' = \mathbf{y} - \mathbf{Du}$ could be formed. Then a modified gain matrix $\mathbf{K}_*$ could be used with $\mathbf{y}'$. It is not difficult to show that $\mathbf{K}_*$ and $\mathbf{K}'$ are related by

$$\mathbf{K}_* = \mathbf{K}'[\mathbf{I} + \mathbf{DK}']^{-1} \quad \text{or} \quad \mathbf{K}' = [\mathbf{I} - \mathbf{K}_*\mathbf{D}]^{-1}\mathbf{K}_* \tag{13.20}$$

If $\mathbf{D} = \mathbf{0}$, these two gains are the same. For nonzero $\mathbf{D}$ it is easiest first to determine $\mathbf{K}_*$ (which is equivalent to assuming $\mathbf{D} = \mathbf{0}$) and then calculate $\mathbf{K}'$ using Eq. (*13.20*). This is the approach to be followed next. By introducing $\mathbf{K}_*$ into Eq. (*13.7*), it is apparent that when static output feedback is used, the closed-loop eigenvalues are the roots of

$$|\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK}_*\mathbf{C}| = 0 \tag{13.21}$$

Following the same arguments as in Sec. 13.4, this implies the existence of one or more nonzero vectors $\boldsymbol{\psi}$ that satisfy

$$[\lambda \mathbf{I} - \mathbf{A} \mid \mathbf{B}]\begin{bmatrix} \boldsymbol{\psi} \\ \mathbf{K}_* \mathbf{C}\boldsymbol{\psi} \end{bmatrix} = \mathbf{0} \tag{13.22}$$

Except for the presence of **C**, this is the same as Eq. (*13.12*). A vector

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\psi} \\ \mathbf{K}_* \mathbf{C}\boldsymbol{\psi} \end{bmatrix}$$

is defined. All independent nontrivial solutions for $\boldsymbol{\xi}$ must be found for each $\lambda_i$. Then $\mathbf{U}(\lambda_i)$, $\boldsymbol{\Psi}(\lambda_i)$, and $\boldsymbol{\Omega}$ are formed as in the previous algorithm, without change. The upper $n$ components of each column $\boldsymbol{\xi}_i$, now in $\mathbf{U}(\lambda_i)$, are closed-loop eigenvectors. Likewise, the lower $r$ components form the bottom partition of $\mathbf{U}(\lambda_i)$. These are used to form $\mathcal{F}'(\lambda_i)$. The meaning of these lower-partition columns is now different because of the presence of **C**. Specifically, $\mathbf{f}_i = \mathbf{K}_* \mathbf{C}\boldsymbol{\psi}_i$. These are used to form $\boldsymbol{\Lambda}'$ essentially as before. One extra operation is now required because of the presence of the matrix **C**. Define $\boldsymbol{\Omega}' = \mathbf{C}\boldsymbol{\Omega}$. Then the counterpart to Eq. (*13.16*) is $\mathbf{K}_* \boldsymbol{\Omega}' = \boldsymbol{\Lambda}'$. Now $\boldsymbol{\Omega}'$ is an $m \times nr$ matrix whose rank is $m$ if **C** is full rank. Therefore, at most $m$ linearly independent columns of $\boldsymbol{\Omega}'$ can be selected in step III of the algorithm to form $\mathbf{G}'$. The corresponding columns of $\boldsymbol{\Lambda}'$ are used to form $\mathcal{J}'$. It is clear that at most $m$ poles can be arbitrarily placed by using static output feedback. The solution for $\mathbf{K}_*$ is found as in step V of Sec. 13.4: $\mathbf{K}_* = \mathcal{J}(\mathbf{G}')^{-1}$.

Finally, if **D** is not zero, the feedback signal is

$$\mathbf{u} = -\mathbf{K}_*(\mathbf{y} - \mathbf{D}\mathbf{u}) + \mathbf{F}'\mathbf{v}'$$

Combining the two **u** terms leads to

$$\mathbf{u} = -[\mathbf{I} - \mathbf{K}_* \mathbf{D}]^{-1}\mathbf{K}_* \mathbf{y} + [\mathbf{I} - \mathbf{K}_* \mathbf{D}]^{-1}\mathbf{F}'\mathbf{v}'$$

The output feedback gain matrix, as stated in Eq. (*13.20*), is

$$\mathbf{K}' = [\mathbf{I} - \mathbf{K}_* \mathbf{D}]^{-1}\mathbf{K}_*$$

Note that the multiplier of the external input $\mathbf{v}'$ is also modified by the presence of a nonzero **D**. The term common to both the **v** and the **y** multipliers can be placed in the forward loop, as shown in Figure 13.4*a*. Figure 13.4*b* shows the same system using the $\mathbf{y}'$ signal fed back through the gain $\mathbf{K}_*$. Matrix block diagram manipulations such as those in Problems 4.2 through 4.4 can be used to reduce Figure 13.4*b* to Figure 13.4*a*, thus giving an alternate method of establishing Eq. (*13.20*).

**EXAMPLE 13.8** Specify an output feedback matrix $\mathbf{K}'$ so that the controllable and observable system

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1], \quad \mathbf{D} = [0]$$

will have $\lambda_1 = -5$ as a closed-loop eigenvalue.

Setting $\lambda = -5$ in Eq. (*13.22*) gives

$$\begin{bmatrix} -5 & -1 & 1 & 0 \\ 3 & -9 & 0 & 1 \end{bmatrix}\boldsymbol{\xi} = \mathbf{0}$$

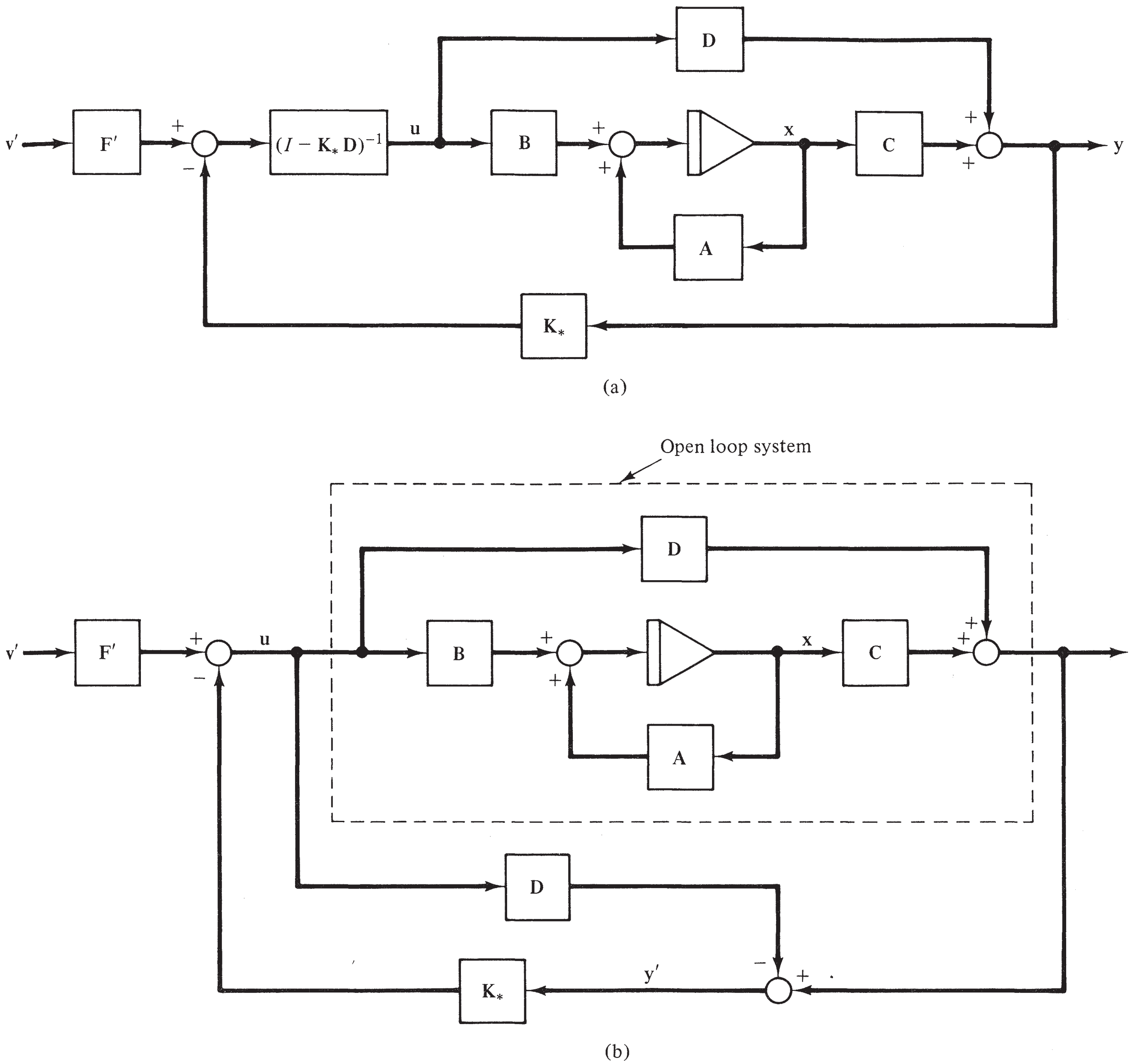



**Figure 13.4**

Letting $\xi_1 = \alpha$ and $\xi_2 = \beta$ be arbitrary constants allows the last two components to be found as $\xi_3 = 5\alpha + \beta$ and $\xi_4 = -3\alpha + \beta$. This means that

$$\mathbf{K}_* \mathbf{C}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}5\alpha + \beta\\ -3\alpha + \beta\end{bmatrix} \tag{13.23}$$

Two such independent equations can be found. For example, $\alpha = 1, \beta = 0$ gives one and $\alpha = 0$, $\beta = 1$ gives another. From these we may be tempted to write

$$\mathbf{K}_* \mathbf{C} = \begin{bmatrix}5 & 1\\ -3 & 1\end{bmatrix}$$

but this is not particularly helpful, since **C** cannot be inverted to find $\mathbf{K}_*$. What must be done is to return to Eq. (*13.23*) and explicitly combine the known **C** with the now-known values of $\alpha$ and $\beta$.

That is, $\mathbf{K}_*\{\alpha+\beta\} = \begin{bmatrix} 5\alpha+\beta \\ -3\alpha+\beta \end{bmatrix}$. Now the *scalar* $\alpha+\beta$ can be divided out to give

$$\mathbf{K}_* = \frac{\begin{bmatrix} 5\alpha+\beta \\ -3\alpha+\beta \end{bmatrix}}{\alpha+\beta}$$

An infinite number of valid choices for $\alpha$ and $\beta$ are possible, such as $\alpha=\frac{1}{4}, \beta=\frac{3}{4}$, giving $\mathbf{K}_* = \mathbf{K}' = [2 \quad 0]^T$ or $\alpha=\frac{1}{2}, \beta=\frac{1}{2}$, giving $\mathbf{K}_* = \mathbf{K}' = [\frac{7}{2} \quad -1]^T$. We leave $\alpha$ and $\beta$ general and proceed to check the closed-loop eigenvalues. The question arises whether the choice of $\alpha$ and $\beta$ can be used to affect the eigenvalues. The matrix

$$\mathbf{A} - \mathbf{BK}_*\mathbf{C} = \frac{\begin{bmatrix} -(5\alpha+\beta) & -4\alpha \\ -4\beta & -(\alpha+5\beta) \end{bmatrix}}{\alpha+\beta}$$

can be shown to have eigenvalues at $-5$ (the one requested) and at $-1$ (the one over which we have no choice) regardless of the $\alpha, \beta$ values. The original propostion that only $m$ eigenvalues can be aribtrarily assigned is reaffirmed. ■

Static output feedback is not able to reposition all $n$ eigenvalues, as just demonstrated. Even worse, those eigenvalues that are not reassignable may be unstable and hence unacceptable.

**EXAMPLE 13.9** Let $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Use static output feedback to achieve eigenvalues at $\lambda = -5$ and $-6$.

Note that the open-loop system is unstable due to the eigenvalue at $\lambda = 4$. With $\lambda_1 = -5$, Eq. (*13.22*) becomes

$$\begin{bmatrix} -3 & -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & -9 & 1 & 0 \end{bmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$$

from which we find solutions of the form $\boldsymbol{\xi}_1 = [\alpha \quad -3\alpha \quad \beta \quad 9\beta \quad -9\alpha]^T$. Similarly, with $\lambda_2 = -6$, Eq. (*13.22*) reduces to

$$\begin{bmatrix} -4 & -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 1 \\ 0 & 0 & -10 & 1 & 0 \end{bmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

from which $\boldsymbol{\xi}_2 = [\delta \quad -4\delta \quad \gamma \quad 10\gamma \quad -16\delta]^T$. Once again, the first three elements of the $\boldsymbol{\xi}$ vectors are closed-loop eigenvectors $\boldsymbol{\psi}$. The fourth and fifth components are the results of multiplying $\boldsymbol{\psi}$ by $\mathbf{K}_*\mathbf{C}$. Therefore, explicitly multiplying out the $\mathbf{C}\boldsymbol{\psi}$ products gives

$$\mathbf{K}_* \begin{bmatrix} \beta & \gamma \\ \alpha & \delta \end{bmatrix} = \begin{bmatrix} 9\beta & 10\gamma \\ -9\alpha & -16\delta \end{bmatrix}$$

One convenient choice, which avoids a matrix inversion, is $\alpha = \gamma = 0$ and $\beta = \delta = 1$. Then $\mathbf{K}_* = \begin{bmatrix} 9 & 0 \\ 0 & -16 \end{bmatrix}$, from which $\mathbf{K}' = \begin{bmatrix} -\frac{9}{8} & 0 \\ 0 & -16 \end{bmatrix}$. This solution does give closed-loop eigenvalues at

$\lambda = -5$ and $-6$ as requested, but the third eigenvalue is at $\lambda = +2$, and hence the closed-loop system is still unstable. Note that this system is stabilizable (see Sec. 11.8) when full-state feedback can be used. ■

## 13.6 OBSERVERS—RECONSTRUCTING THE STATE FROM AVAILABLE OUTPUTS

In many systems all components of the state vector are not directly available as output signals. For example, a radar may be tracking the position (states) of a vehicle and it is desired to know the velocities, accelerations, or other states. Sometimes a knowledge of the state values is desired as an end in itself, simply for performance evaluation. In many system-control problems the reason for wanting knowledge of the states is for forming feedback signals. As was shown earlier, if all states can be used for feedback, complete control over all the eigenvalues is possible, assuming the system is controllable. Several approaches to the problem of unmeasured states can be considered. First, it might be possible to add additional sensors to provide the measurements. This is generally the most expensive option. Second, some sort of ad hoc differentiation of measured states may provide an estimate of unmeasured states. This option may not give sufficiently accurate performance, especially in the case of noisy data. The third option is to use full knowledge of the mathematical models of the system in a systematic way in an attempt to estimate or reconstruct the states. This third approach is developed now. The resulting estimator algorithm is called an *observer* [7]. There are continuous-time observers and discrete-time observers. *Full state observers,* sometimes called *identity observers,* produce estimates of all state components, even those that are measured directly. This redundancy can be removed with a *reduced state observer,* which estimates only the states that are not measured and uses raw measurement data for those that are measured. When measurements are noisy, the beneficial smoothing effects which are provided by the full state observer may be more important than the elimination of redundancy.

### 13.6.1 Continuous-Time Full State Observers

The continuous-time system described by Eq. (*13.1*) is considered first. For simplicity it is now assumed that $\mathbf{D} = [0]$. If this is not true, then an equivalent output $\mathbf{y}' = \mathbf{y} - \mathbf{Du}$ can be used in place of $\mathbf{y}$ in what follows, as was demonstrated in the previous section. It is desired to obtain a good estimate of the state $\mathbf{x}(t)$, given a knowledge of the output $\mathbf{y}(t)$, the input $\mathbf{u}(t)$, and the system matrices $\mathbf{A, B, C, D}$. This problem is referred to as the *state reconstruction problem.* If $\mathbf{C}$ is square and nonsingular, then $\mathbf{x}(t) = \mathbf{C}^{-1}\mathbf{y}(t)$. In general, this trivial result will not be applicable.

Another dynamic system, called an *observer* [7], is to be constructed. Its input will depend on $\mathbf{y}$ and $\mathbf{u}$ and its state (output) should be a good approximation to $\mathbf{x}(t)$. The form of the observer is selected as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{Ly} + \mathbf{z} \tag{13.24}$$

where $\hat{\mathbf{x}}$ is the $n \times 1$ vector approximation to $\mathbf{x}$.‡ $\mathbf{A}_c$ and $\mathbf{L}$ are $n \times n$ and $n \times m$ matrices and $\mathbf{z}$ is an $n \times 1$ vector, to be determined. $\mathbf{A}_c$, $\mathbf{L}$, and $\mathbf{z}$ will be selected next in such a way that the postulated observer system meets the goal of giving good estimates for the state variables. Defining $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ and using Eq. (*13.1*) gives

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} - \mathbf{A}_c\hat{\mathbf{x}} - \mathbf{L}\mathbf{y} + \mathbf{B}\mathbf{u} - \mathbf{z} \tag{13.25}$$

By selecting $\mathbf{z} = \mathbf{B}\mathbf{u}$ and using $\mathbf{y} = \mathbf{C}\mathbf{x}$, this reduces to

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{x} - \mathbf{A}_c\hat{\mathbf{x}}$$

Selecting $\mathbf{A}_c = \mathbf{A} - \mathbf{L}\mathbf{C}$ gives

$$\dot{\mathbf{e}} = \mathbf{A}_c\mathbf{e} \tag{13.26}$$

If the eigenvalues of $\mathbf{A}_c$ all have negative real parts, then an asymptotically stable error equation results. This indicates that $\mathbf{e}(t) \to \mathbf{0}$, or $\hat{\mathbf{x}}(t) \to \mathbf{x}(t)$ as $t \to \infty$. The term $\mathbf{L}$ of the observer is still unspecified. If the original system (*13.1*) is completely observable, then it is always possible to find an $\mathbf{L}$ which will yield any set of desired eigenvalues for $\mathbf{A}_c$. Thus it is possible to control the rate at which $\hat{\mathbf{x}} \to \mathbf{x}$.

Compare the $n \times n$ matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ of Eq. (*13.26*) with the $n \times n$ matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ of Eq. (*13.4*). Both have a known $n \times n$ system matrix $\mathbf{A}$, modified by a term that is partially selectable by the designer. The order of the known terms $\mathbf{B}$ and $\mathbf{C}$ and the selectable gains $\mathbf{K}$ and $\mathbf{L}$ are reversed, so the comparison is not exact. In earlier sections methods of determining $\mathbf{K}$ were developed which give $\mathbf{A} - \mathbf{B}\mathbf{K}$ a specified set of eigenvalues. A square matrix and its transpose have the same determinant and hence the same eigenvalues. Therefore, if $\mathbf{L}$ can be selected to force the eigenvalues of

$$\mathbf{A}_c^T = \mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T \tag{13.27}$$

to have specified values, then $\mathbf{A}_c$ will have those same desirable values. The methods presented in Sec. 13.4 for pole placement apply without change to the observer problem, which is its dual. Replace $\mathbf{A}$ by $\mathbf{A}^T$ and $\mathbf{B}$ by $\mathbf{C}^T$, and find $\mathbf{L}^T$ (instead of $\mathbf{K}$) by any of the previous pole placement algorithms. The requirement in Sec. 13.4 that the system $\{\mathbf{A}, \mathbf{B}\}$ be controllable now becomes an observability requirement for $\{\mathbf{A}, \mathbf{C}\}$. To see this, note that a substitution of $\mathbf{A}^T$ and $\mathbf{C}^T$ for $\mathbf{A}$ and $\mathbf{B}$ in Eq. (*11.5*) for the controllability test automatically gives the observability test of Eq. (*11.6*) for $\{\mathbf{A}, \mathbf{C}\}$. The full-state observer design problem is the dual of the pole placement problem using full state feedback. The discussion and algorithm of Sec. 13.4 are totally applicable. For reference, the fundamental Eq. (*13.12*) is repeated here with notational changes appropriate for the observer problem.

$$[\lambda\mathbf{I} - \mathbf{A}^T \mid \mathbf{C}^T]\boldsymbol{\xi} = \mathbf{0} \tag{13.28}$$

Equation (*13.28*) is used to determine the observer gain matrix $\mathbf{L}$. As before, there will be times when generalized eigenvectors will be required in order to obtain multiple observer poles. Figure 13.5 shows the observer mechanization. Part (*a*) is directly from

‡ The circumflex ^ is used in this chapter to indicate an estimate for a quantity, e.g., $\hat{\mathbf{x}}$ is an estimate of $\mathbf{x}$. This should not be confused with the notation for a unit vector introduced in Chapter 5.

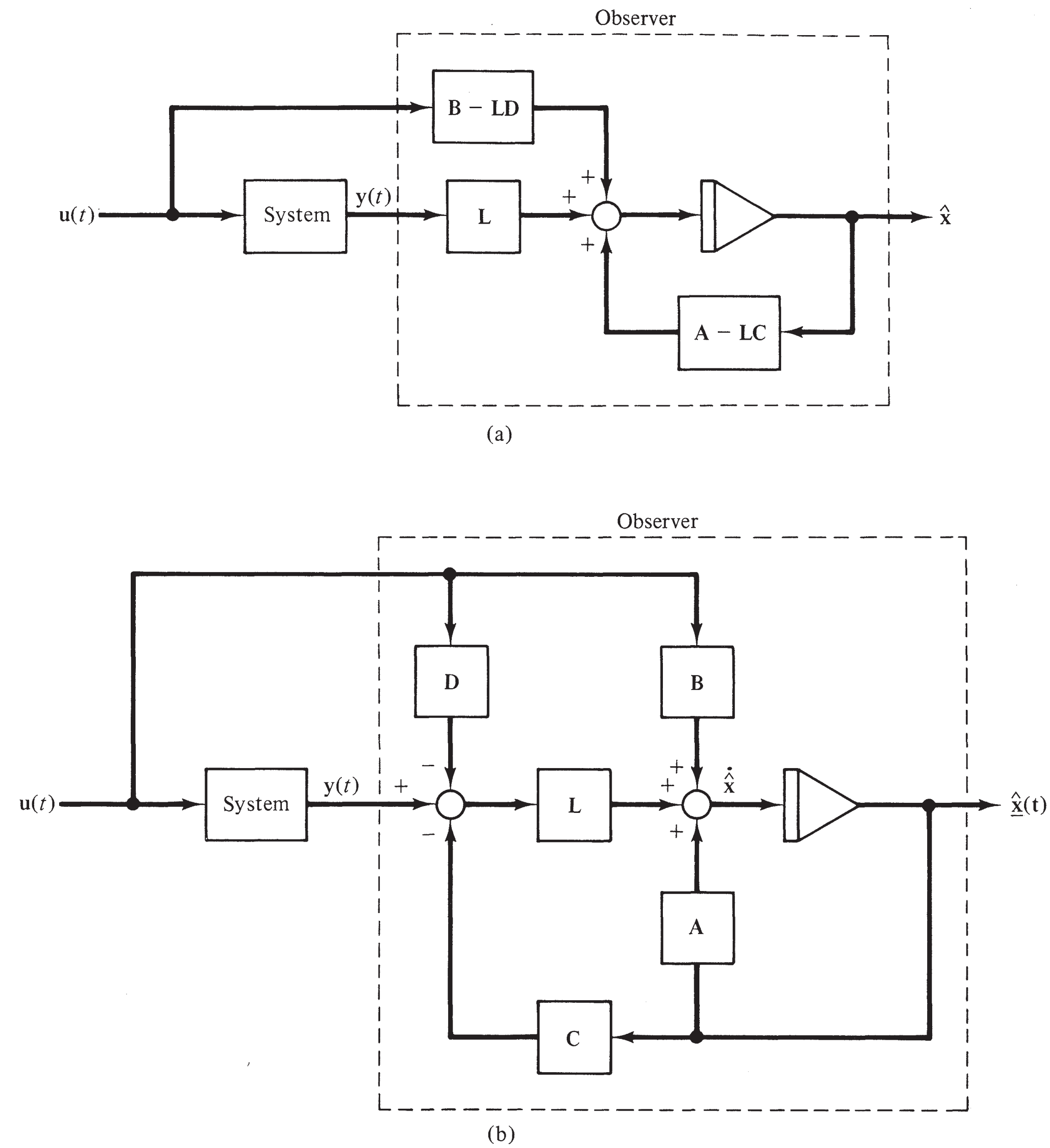



**Figure 13.5**

Eq. (*13.24*) and part (*b*) is obtained by matrix block diagram manipulations. This form better shows that a key factor in determining $\dot{\hat{\mathbf{x}}}$ is the difference between the actual measurement **y** and the "expected measurement" $\mathbf{C}\hat{\mathbf{x}}$ weighted by the observer gain **L.**

**EXAMPLE 13.10** Design an observer for the system of Example 13.1 if only $x_1$ can be measured. Put both observer eigenvalues at $\lambda = -8$. This will cause the error $\mathbf{e}(t)$ to approach zero rapidly regardless of any error in the initial estimate $\hat{\mathbf{x}}$.

Using

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = [1 \quad 0]$$

in $[-8\mathbf{I} - \mathbf{A}^T \mid \mathbf{C}^T]\boldsymbol{\xi} = \mathbf{0}$ gives component equations $8\xi_1 = \xi_3$ and $2\xi_1 = -11\xi_2$. Arbitrarily setting

$\xi_1 = 1$ gives $\xi_2 = -\frac{2}{11}$ and $\xi_3 = 8$. The first two components of $\boldsymbol{\xi}$ form the eigenvector $\boldsymbol{\psi}$. A generalized eigenvector is needed to find a second independent column. Thus

$$[-8\mathbf{I} - \mathbf{A}^T \mid \mathbf{C}^T]\boldsymbol{\xi}_g = -\boldsymbol{\psi} = \begin{bmatrix} -1 \\ \frac{2}{11} \end{bmatrix}$$

is solved to find $\boldsymbol{\xi}_g = [1 \quad -\frac{24}{121} \quad 7]^T$. The columns $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_g$ form the matrix $\mathbf{U}(-8)$. Partitioning it gives

$$\mathbf{L}^T\begin{bmatrix} 1 & 1 \\ -\frac{2}{11} & -\frac{24}{121} \end{bmatrix} = [8 \quad 7]$$

and the solution is $\mathbf{L}^T = [19 \quad 60.5]$. As a check, the matrix $\mathbf{A}_c = \begin{bmatrix} -19 & 2 \\ -60.5 & 3 \end{bmatrix}$ is found to have $(\lambda + 8)^2$ as its characteristic polynomial. The complete observer description is

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{y} + \mathbf{z}$$

or

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} -19 & 2 \\ -60.5 & 3 \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 19 \\ 60.5 \end{bmatrix}y + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Figure 13.6 gives the observer cascaded with the original system. ■

### *13.6.2 Discrete-Time Full-State Observers*

For the discrete-time system of Eq. (*13.2*) one possible form for the observer is postulated as

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}_c\hat{\mathbf{x}}(k) + \mathbf{L}\mathbf{y}(k) + \mathbf{z}(k) \qquad (13.29)$$

Notice that this implies that the estimate being calculated at time-step $k + 1$ is based upon measured data one sample old, namely, $\mathbf{y}(k)$. This time delay may be forced upon the system designer to allow time for the computational lag of the algorithm. If the sample time between $k$ and $k + 1$ is large, this delay may be undesirable. An alternate form of the observer, which replaces $\mathbf{y}(k)$ by $\mathbf{y}(k + 1)$, will be given later.

The preceding observer's estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ will satisfy the homogeneous difference equation

$$\mathbf{e}(k+1) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(k) = \mathbf{A}_c\mathbf{e}(k) \qquad (13.30)$$

provided that $\mathbf{z}(k)$ is selected to cancel all the $\mathbf{u}$ terms that come into the error equation through $\mathbf{Ly}$. This requirement is met if $\mathbf{z} = (\mathbf{B} - \mathbf{LD})\mathbf{u}(k)$. Just as in the continuous-time case, the error $\mathbf{e}(k)$ will decay toward zero if $\mathbf{A}_c$ is stable. This means that all its eigenvalues must be inside the unit circle. The decay rate depends on the location of these eigenvalues. If the original system of Eq. (*13.2*) is observable, it is possible to force the $n$ eigenvalues to any desired locations by proper choice of $\mathbf{L}$. The mechanics of doing this are exactly the same as in the continuous-time case.

Figure 13.7*a* shows the observer as described in Eq. (*13.29*). Figure 13.7*b* is a rearrangement which shows how the gain-weighted difference between the actual $\mathbf{y}$ and a predicted $\mathbf{y}$, given by $\mathbf{C}\hat{\mathbf{x}} + \mathbf{Du}$, drives the observer.

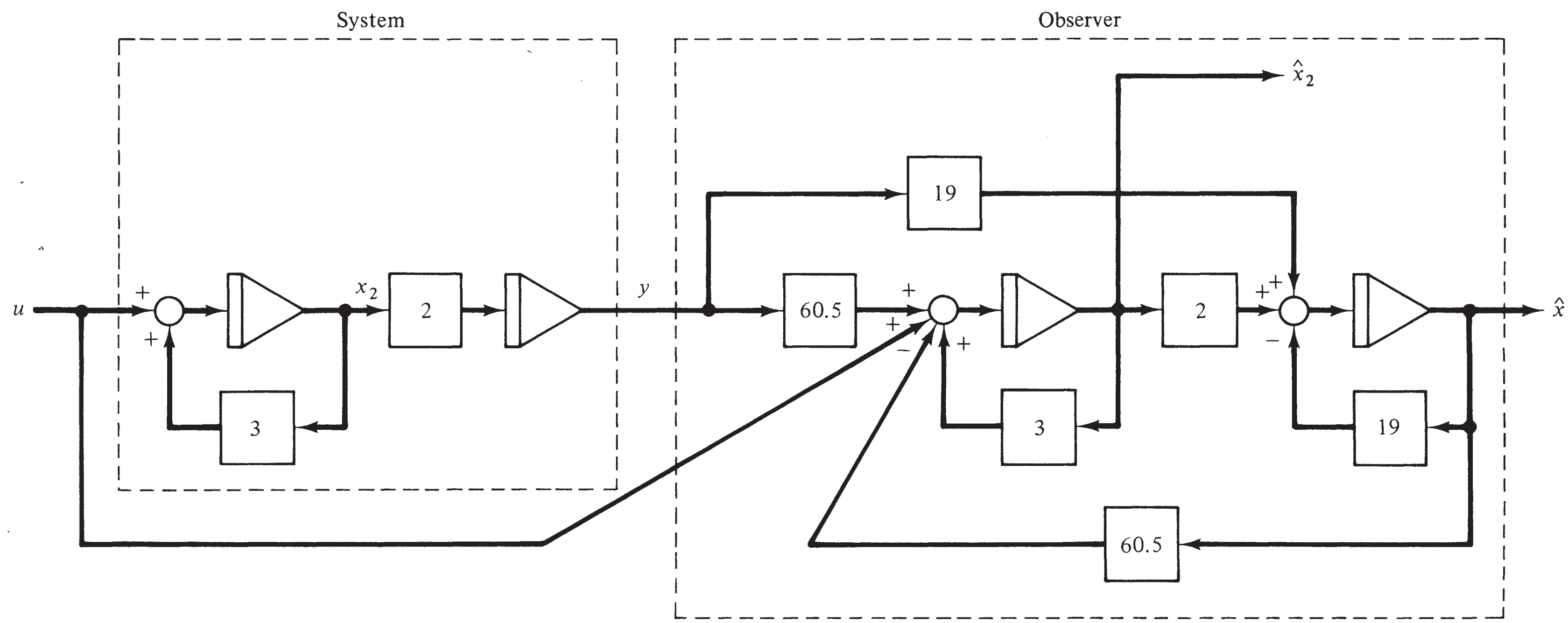



Figure 13.6

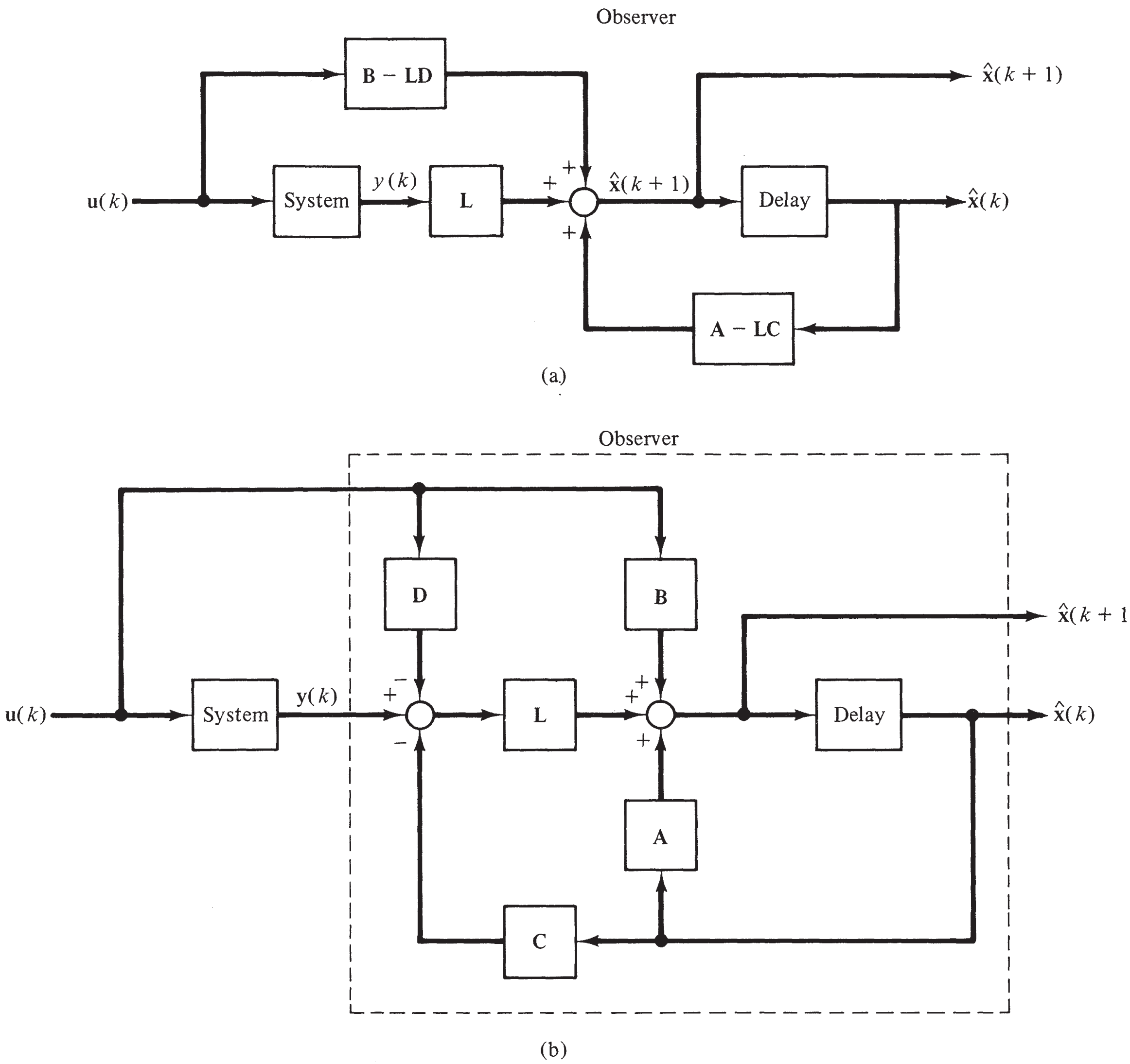



**Figure 13.7**

**EXAMPLE 13.11** Design a full-state observer with eigenvalues at $z = 0.1 \pm 0.2j$ for the discrete system with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0.2 & 0.5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{D} = [0]$, and $\mathbf{C} = [1 \quad 0]$.

For this case Eq. (*13.28*) becomes, with $z = 0.1 + 0.2j$,

$$\begin{bmatrix} 0.1+0.2j & -0.2 & 1 \\ -1 & -0.4+0.2j & 0 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0}$$

Selecting $\xi_2 = 1$ gives $\xi_1 = -0.4 + 0.2j$ and $\xi_3 = 0.28 + 0.06j$. Using the conjugate of $z$ gives the conjugate of $\boldsymbol{\xi}$. Therefore,

$$\mathbf{L}^T \begin{bmatrix} -0.4+0.2j & -0.4-0.2j \\ 1 & 1 \end{bmatrix} = [0.28+0.06j \quad 0.28-0.06j]$$

Using Problem 4.23, this can be expressed with purely real values: $\mathbf{L}^T \begin{bmatrix} -0.4 & 0.2 \\ 1 & 0 \end{bmatrix} = [0.28 \quad 0.06]$, from which $\mathbf{L}^T = [0.3 \quad 0.4]$. The observer system matrix is $\mathbf{A}_c = \begin{bmatrix} -0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix}$. The final observer is shown in Figure 13.8. ■

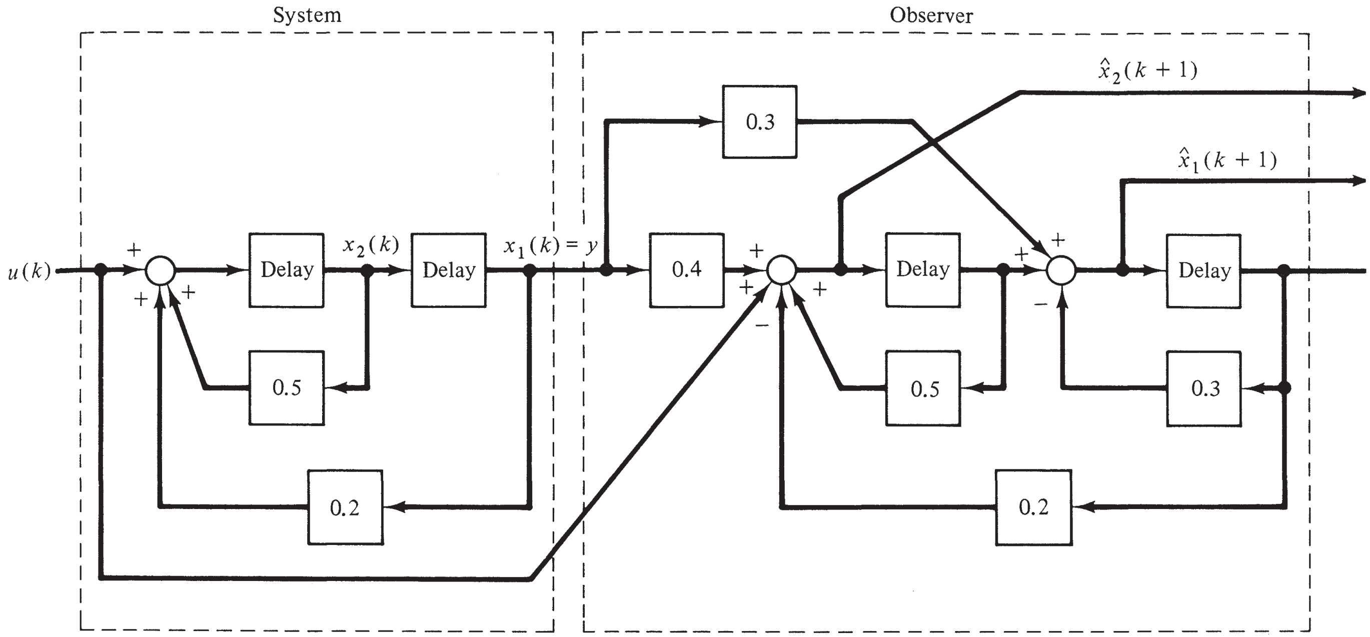



**Figure 13.8**

A second form of the discrete-time full state observer can be postulated as

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}_c\hat{\mathbf{x}}(k) + \mathbf{L}\mathbf{y}(k+1) + \mathbf{z}(k) \tag{13.31}$$

Using the state equations to eliminate the $\mathbf{y}(k+1)$ term gives

$$\begin{aligned}\hat{\mathbf{x}}(k+1) &= \mathbf{A}_c\hat{\mathbf{x}}(k) + \mathbf{L}\{\mathbf{C}\mathbf{x}(k+1) + \mathbf{D}\mathbf{u}(k+1)\} + \mathbf{z}(k)\\ &= \mathbf{A}_c\hat{\mathbf{x}}(k) + \mathbf{LCAx}(k) + \mathbf{LCBu}(k) + \mathbf{LDu}(k+1) + \mathbf{z}(k)\end{aligned}$$

Forming the difference from the true state equation, the error equation is

$$\begin{aligned}\mathbf{e}(k+1) = (\mathbf{A} - \mathbf{LCA})\mathbf{x}(k) - \mathbf{A}_c\hat{\mathbf{x}}(k) + (\mathbf{B} - \mathbf{LCB})\mathbf{u}(k)\\ - \mathbf{LDu}(k+1) - \mathbf{z}(k)\end{aligned} \tag{13.32}$$

As before, the observer input $\mathbf{z}(k)$ is selected to cancel all the $\mathbf{u}$ terms, so $\mathbf{z}(k) = (\mathbf{B} - \mathbf{LCB})\mathbf{u}(k) - \mathbf{LDu}(k+1)$. The observer matrix is selected as

$$\mathbf{A}_c = (\mathbf{A} - \mathbf{LCA}) = (\mathbf{I} - \mathbf{LC})\mathbf{A} \tag{13.33}$$

so that the error equation is once again a simple homogeneous equation. Its response will decay to zero if the eigenvalues of $\mathbf{A}_c$ are all stable. Note, however, that in forcing the eigenvalues of $\mathbf{A}_c$ to prespecified stable locations by choice of the gain $\mathbf{L}$, a different kind of problem is being posed. The question now is whether the original system matrix $\mathbf{A}$ can be *multiplied* by a factor $(\mathbf{I} - \mathbf{LC})$ to force the result, $\mathbf{A}_c$, to have arbitrary eigenvalues. The answer is affirmative if the system $\mathbf{A}$ is nonsingular, as can be seen by defining a new "given" matrix $\mathbf{C}' = \mathbf{CA}$. This allows Eq. (*13.33*) to be written in a form which hides the product problem and forces the equation to appear as the one treated in previous sections, $\mathbf{A}_c = \mathbf{A} - \mathbf{LC}'$. Remember that the ability to arbitrarily prescribe eigenvalues to $\mathbf{A}_c$ required observability. Assuming $\{\mathbf{A}, \mathbf{C}\}$ is observable, is $\{\mathbf{A}, \mathbf{C}'\}$ also observable? Call the observability test matrix of Eq. (*11.6*) $\mathbf{Q}$ when $\mathbf{C}$ is used and $\mathbf{Q}'$

when $\mathbf{C}'$ is used. Clearly $\mathbf{Q}' = \mathbf{A}^T\mathbf{Q}$, so that if $\mathbf{A}$ is nonsingular, $\text{rank}(\mathbf{Q}') = \text{rank}(\mathbf{Q})$. Any discrete system which is derived by sampling a continuous system described by Eq. (*13.1*) will have a nonsingular $\mathbf{A}$ matrix, since $\mathbf{A}$ is then just the transition matrix $\mathbf{\Phi}$ over one sample period and $\mathbf{\Phi}^{-1}$ always exists. Other discrete systems, such as those having pure time delays, need not have $\mathbf{A}$ full rank. When $\mathbf{A}$ is not full rank, complete control over the eigenvalues of $\mathbf{A}_c$ is not possible. By forcing $n - \text{rank}(\mathbf{A})$ eigenvalues to be at the origin, the procedure can still be carried out. See Reference 8 for a further discussion of this point. Figure 13.9 shows two arrangements for this observer. Note

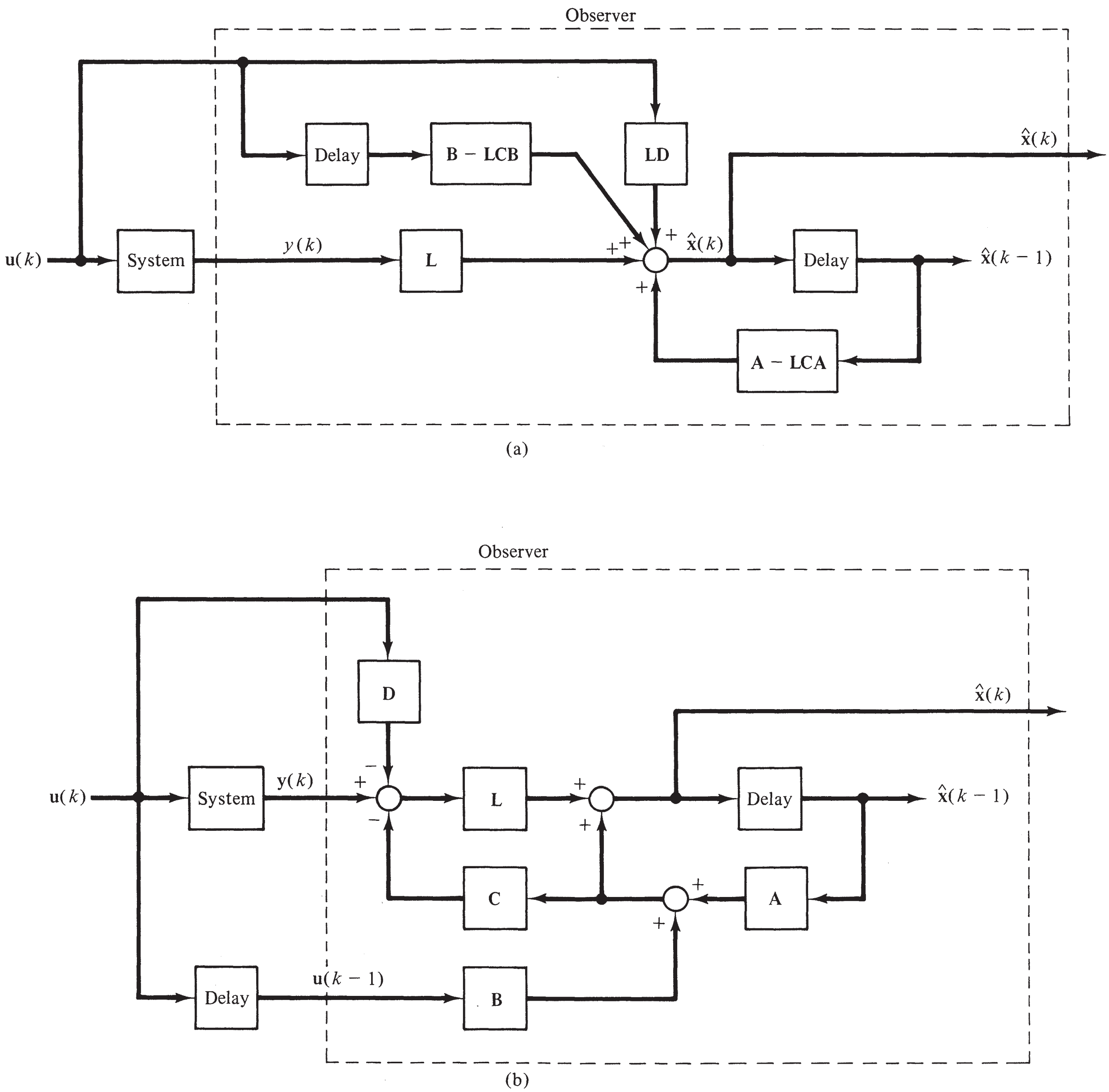



**Figure 13.9**

that whenever **D** is not zero, the input signals **u** at two successive time steps are required.

**EXAMPLE 13.12** Use the same system and eigenvalue specifications as Example 13.11, but use $\mathbf{y}(k+1)$ as the input rather than $\mathbf{y}(k)$.

First calculate $\mathbf{C}' = \mathbf{CA} = [0 \quad 1]$ and then solve

$$\begin{bmatrix} 0.1+0.2j & -0.2 & 0 \\ -1 & -0.4+0.2j & 1 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0}$$

The two conjugate solutions are used to write

$$\mathbf{L}^T \begin{bmatrix} 1 & 1 \\ 0.5+j & 0.5-j \end{bmatrix} = [1.4+0.3j \quad 1.4-0.3j]$$

or $\mathbf{L}^T \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} = [1.4 \quad 0.3]$, giving $\mathbf{L}^T = [1.25 \quad 0.3]$. This version of the observer has

$$\mathbf{A}_c = \begin{bmatrix} 0 & -0.25 \\ 0.2 & 0.2 \end{bmatrix}, \qquad \mathbf{B} - \mathbf{LCB} = \begin{bmatrix} -1.25 \\ 0.7 \end{bmatrix}$$

The implementation is given in Figure 13.10. The time argument $k$ on the input $\mathbf{u}(k)$ may seem inconsistent with Figure 13.9. Actually when $\mathbf{D} \neq [\mathbf{0}]$, both $\mathbf{u}(k)$ and $\mathbf{u}(k+1)$ directly affect $\mathbf{y}(k+1)$, so either notational convention can be used provided the correct representation inside the system block is used. ■

**EXAMPLE 13.13** Suppose the previous model is modified to $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix}$, with **B** and **C** unchanged. Note that **A** is singular and that with $\mathbf{C}' = \mathbf{CA} = [0 \quad 1]$, the system is not observable. Investigate the implications.

Eq. (*13.28*) gives $\begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda - 0.5 & 1 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0}$, from which the requirements are $\lambda \xi_1 = 0$ and

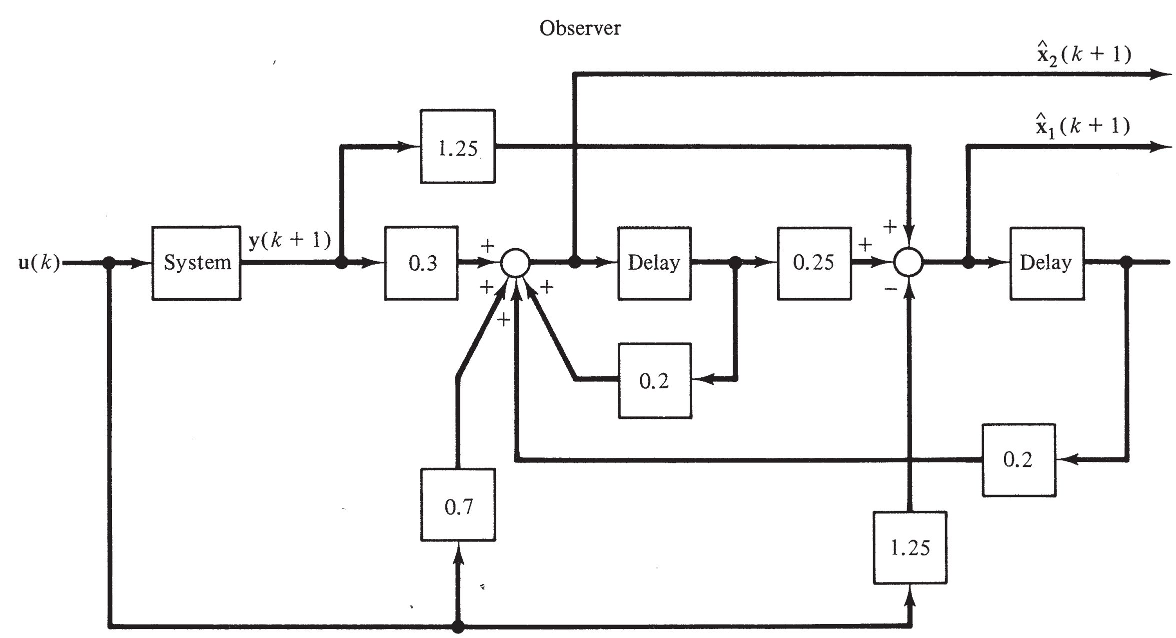



**Figure 13.10** Observer for Example 13.12.

$-\xi_1 + (\lambda - 0.5)\xi_2 + \xi_3 = 0$. If any nonzero eigenvalue is specified, then $\xi_1$ must be zero. Assume this is so and then find that $\xi_2$ can be set arbitrarily to 1 and $\xi_3 = 0.5 - \lambda$. One of the equations for $\mathbf{L}^T$ is

$$\mathbf{L}^T\begin{bmatrix} 0 & \\ 1 & \end{bmatrix} = [0.5 - \lambda \quad\ ] \tag{13.34}$$

Any choice other than 0 for the second eigenvalue will require another zero in the first row of Eq. (*13.34*), resulting in a singular matrix, so $\mathbf{L}^T$ cannot be determined. The only choice is to put an eigenvalue at $z = 0$ (which may have been a good value anyway, but there is no choice in the matter here.) With $\lambda = 0$ it is possible to set $\xi_1 = 1$ and $\xi_2 = 1$ and find $\xi_3 = 1.5$. With these selections, the observer gain matrix is $\mathbf{L}^T = [z_1 + 1 \quad 0.5 - z_1]$, a valid solution for any real value of observer eigenvalue $\lambda = z_1$. The other observer eigenvalue is at $\lambda = 0$. ■

### 13.6.3 Continuous-Time Reduced Order Observers

Assume that the state equations have been selected in such a way that the $m$ outputs constitute the first $m$ states, perhaps modified by a **Du** term. If this is not true, a transformation to a new set of states can make it true. (See Problem 13.12.) Then partitioning the state equations the obvious way gives

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \qquad \mathbf{C} = [\mathbf{I}_m \quad \mathbf{0}]$$

The subset of states $\mathbf{x}_1$ are treated as knowns because they are given by $\mathbf{x}_1(t) = \mathbf{y}(t) - \mathbf{Du}(t)$. The lower half of the partitioned state differential equations are

$$\dot{\mathbf{x}}_2 = \mathbf{A}_{22}\mathbf{x}_2 + \{\mathbf{A}_{21}\mathbf{x}_1 + \mathbf{B}_2\mathbf{u}\} \tag{13.35}$$

The top half of these partitioned equations can be written as

$$\{\dot{\mathbf{x}}_1 - \mathbf{A}_{11}\mathbf{x}_1 - \mathbf{B}_1\mathbf{u}\} = \mathbf{A}_{12}\mathbf{x}_2 \tag{13.36}$$

In both Eqs. (*13.35*) and (*13.36*) all terms inside the braces can be treated as known. Only the $\dot{\mathbf{x}}_1$ may cause some concern, since differentiation seems to be required. This will be dealt with later, in the same way that $\dot{\mathbf{u}}$ terms were avoided in Chapter 3. By way of analogy with the full-state observer, Eq. (*13.35*) represents the dynamics and Eq. (*13.36*) represents the measurements of the reduced observer. Table 13.1 lists the corresponding terms in the two cases.

**TABLE 13.1** SYSTEM ANALOGIES FOR REDUCED ORDER OBSERVER

| Term | Full State Observer | Reduced Order Observer |
|---|---|---|
| State | $\mathbf{x}\ \ (n \times 1)$ | $\mathbf{x}_2\ \ (n-m) \times 1$ |
| Input | $\mathbf{Bu}$ | $\{\mathbf{A}_{21}\mathbf{x}_1 + \mathbf{B}_2\mathbf{u}\}$ |
| System matrix | $\mathbf{A}\ \ (n \times n)$ | $\mathbf{A}_{22}\ \ (n-m) \times (n-m)$ |
| Output matrix | $\mathbf{C}$ | $\mathbf{A}_{12}$ |
| Output signal | $\mathbf{y}$ | $\{\dot{\mathbf{x}}_1 - \mathbf{A}_{11}\mathbf{x}_1 - \mathbf{B}_1\mathbf{u}\}$ |
| Direct signal | $\mathbf{Du}$ | None |

Using these set-up analogies, the reduced order observer equations for the two portions of the partitioned state vector are:

$$\hat{\mathbf{x}}_1 = \mathbf{x}_1 = \mathbf{y} - \mathbf{D}\mathbf{u} \qquad \text{(just use the given data)}$$

$$\dot{\hat{\mathbf{x}}}_2 = \mathbf{A}_r\hat{\mathbf{x}}_2 + \mathbf{L}_r\mathbf{y}_r + \mathbf{z}_r \tag{13.37}$$

where

$$\mathbf{A}_r = \mathbf{A}_{22} - \mathbf{L}_r\mathbf{A}_{12}$$

$$\mathbf{y}_r = \dot{\mathbf{x}}_1 - \mathbf{A}_{11}\mathbf{x}_1 - \mathbf{B}_1\mathbf{u}$$

$$\mathbf{z}_r = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{B}_2\mathbf{u}$$

The reduced order observer gain matrix $\mathbf{L}_r$ can be calculated with the same algorithm as used in the full state observer. It can be selected to give any specified set of observer eigenvalues, provided the system $\{\mathbf{A}_{22}, \mathbf{A}_{12}\}$ is observable. This is automatically true if the original full system is observable. The mechanics of the calculation are exactly as in the full state case. The implementation is, of course, a little different. Two versions are shown in Figure 13.11. Figure 13.11*a* shows the $\dot{\mathbf{x}}_1$ term as it appears in the preceding equations. The need for actually differentiating the data is avoided by moving that signal to the output side of the integrator, as shown in Figure 13.11*b*.

**EXAMPLE 13.14** Consider the system

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$$

Design a first-order observer with an eigenvalue at $\lambda = -5$ to estimate the unmeasured third state component. The partitioned matrix terms for this example are $\mathbf{A}_{22} = -3, \mathbf{A}_{12} = [0 \quad 1]^T$. Therefore, $[s\mathbf{I} - \mathbf{A}_{22}^T \mid \mathbf{A}_{12}^T]\boldsymbol{\xi} = 0$ reduces to a scalar equation $[-2 \quad 0 \quad 1]\boldsymbol{\xi} = 0$. Clearly, $\xi_2$ is arbitrary, and it is set to 0. Also, $\xi_1 = 1$ is selected and then $\xi_3 = 2$ is determined. Thus $\mathbf{L}^T[1] = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, or $\mathbf{L} = [0 \quad 2]$.

The general arrangement of Figure 13.11*b* specializes in this case to the first-order observer of Figure 13.12. Note that there are four inputs and just one nontrivial output, $\hat{\mathbf{x}}_3$. By using Mason's gain formula on Figure 13.12 or the Laplace transform of Eq. (*13.37*), it is found that

$$\hat{\mathbf{x}}_3(s) = [1/(s+5)][-x_1(s) + 2(s-1)x_2(s) - 2u_1(s) + u_2(s)]$$

which shows that the pole at $s = -5$ has been attained.

### *13.6.4 Discrete-Time Reduced Order Observers*

The discrete-time state equations (*13.2*) are considered, but again it is assumed that $\mathbf{C} = [\mathbf{I}_m \quad 0]$. The derivation in this section is almost identical to the continuous case of Sec. 13.6.3. Both forms of full state discrete observers can be developed in the reduced state case, but given here is only the one using outputs that are one sample period old.

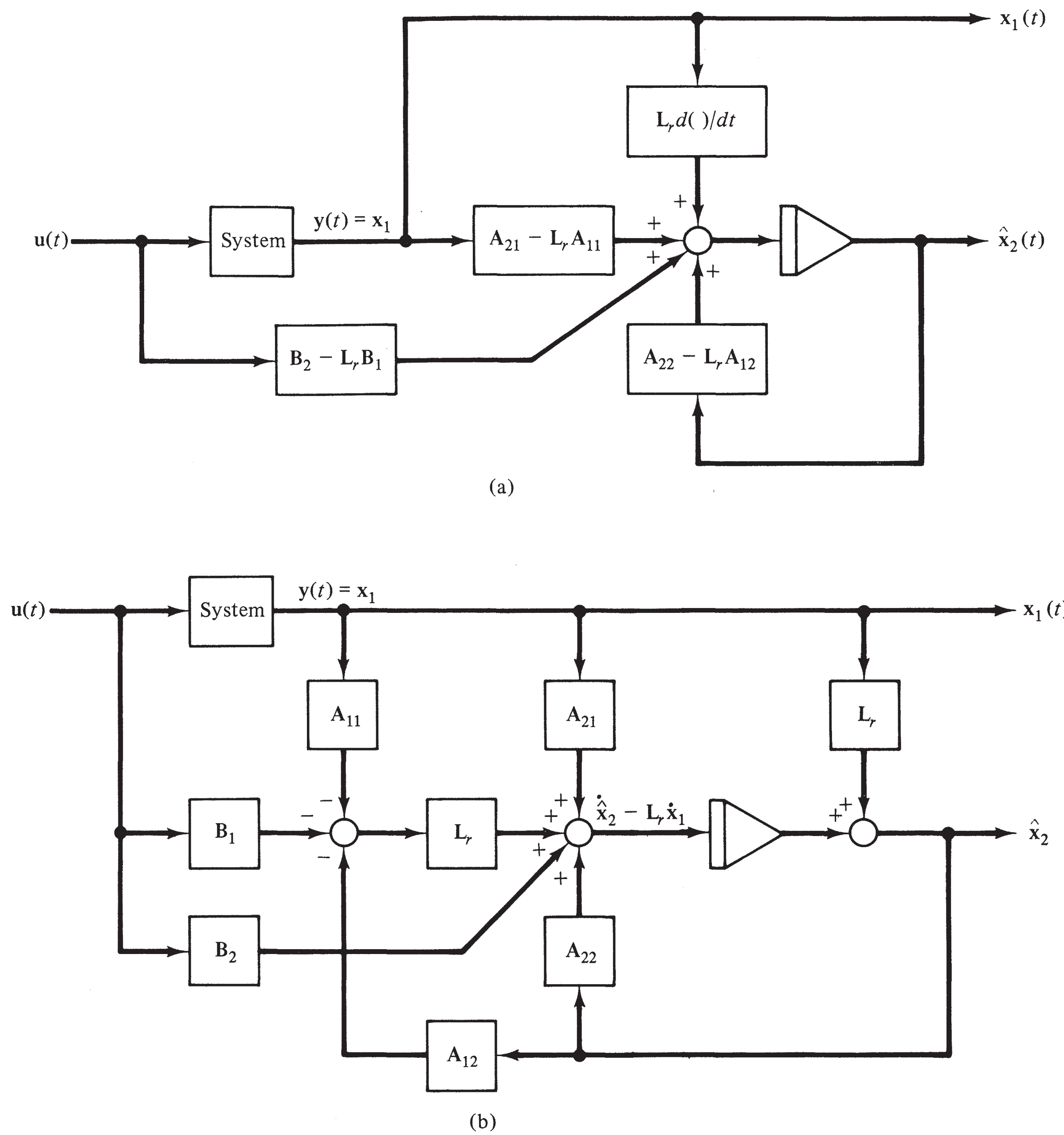



**Figure 13.11**

Partition the state vector into the $m$ measured components and the $n$-$m$ remaining components. Then the state equations become

$$\mathbf{x}_1(k+1) = \mathbf{A}_{11}\mathbf{x}_1(k) + \mathbf{A}_{12}\mathbf{x}_2(k) + \mathbf{B}_1\mathbf{u}(k)$$

$$\mathbf{x}_2(k+1) = \mathbf{A}_{21}\mathbf{x}_1(k) + \mathbf{A}_{22}\mathbf{x}_2(k) + \mathbf{B}_2\mathbf{u}(k)$$

and

$$\mathbf{y}(k) = \mathbf{x}_1(k) + \mathbf{D}\mathbf{u}(k)$$

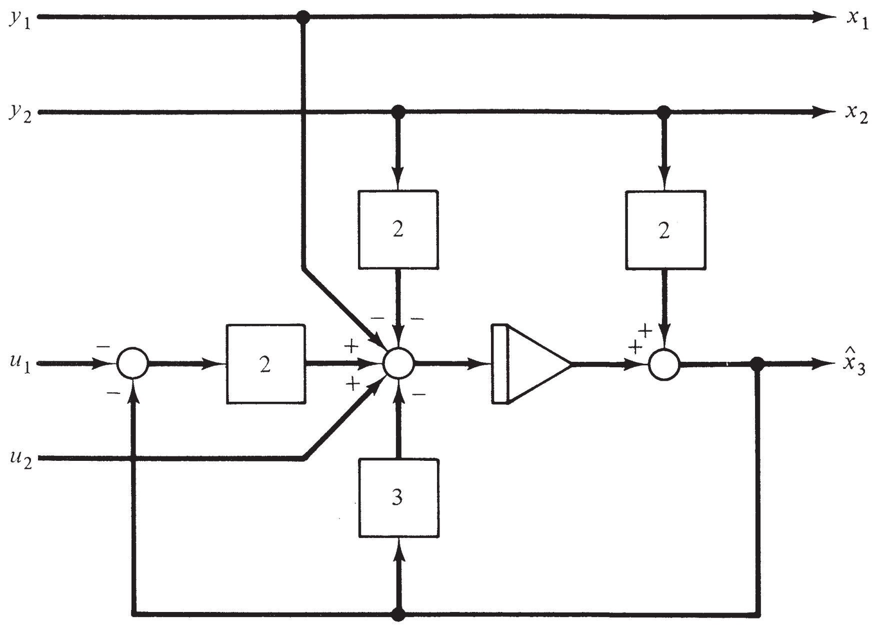



**Figure 13.12**

Since only the noise-free case is being discussed, **y** and **u** are assumed known exactly at all times. The third equation plays only the role of giving $\mathbf{x}_1$ if **D** is nonzero. Therefore $\mathbf{x}_1$ can be treated as known quantity at all times. Define two more "known" quantities

$$\mathbf{y}_r(k) \triangleq \mathbf{x}_1(k+1) - \mathbf{A}_{11}\mathbf{x}_1(k) - \mathbf{B}_1\mathbf{u}(k)$$

$$\mathbf{v}_r(k) \triangleq \mathbf{A}_{21}\mathbf{x}_1(k) + \mathbf{B}_2\mathbf{u}(k) \qquad (13.38)$$

Then the new equivalent dynamic equation is

$$\mathbf{x}_2(k+1) = \mathbf{A}_{22}\mathbf{x}_2(k) + \mathbf{v}_r(k)$$

and the new equivalent measurement equation is

$$\mathbf{y}_r(k) = \mathbf{A}_{12}\mathbf{x}_2(k)$$

A linear observer form is postulated as

$$\hat{\mathbf{x}}_2(k+1) = \mathbf{A}_r\hat{\mathbf{x}}_2(k) + \mathbf{L}_r\mathbf{y}_r(k) + \mathbf{z}_r(k)$$

The error $\mathbf{e}(k) \triangleq \mathbf{x}_2(k) - \hat{\mathbf{x}}_2(k)$ satisfies

$$\mathbf{e}(k+1) = \mathbf{A}_{22}\mathbf{x}_2(k) + \mathbf{v}_r(k) - \mathbf{A}_r\hat{\mathbf{x}}_2 - \mathbf{L}_r\mathbf{y}_r(k) - \mathbf{z}_r(k)$$

If the selections $\mathbf{z}_r(k) = \mathbf{v}_r(k)$ and $\mathbf{A}_r = \mathbf{A}_{22} - \mathbf{L}_r\mathbf{A}_{12}$ are made, then

$$\mathbf{e}(k+1) = (\mathbf{A}_{22} - \mathbf{L}_r\mathbf{A}_{12})\mathbf{e}(k)$$

The convergence rate of $\mathbf{e}(k)$ to zero can be controlled by proper choice of $\mathbf{L}_r$, just as in the case of the full state observer. Figure 13.13 gives the reduced order observer in block diagram form.

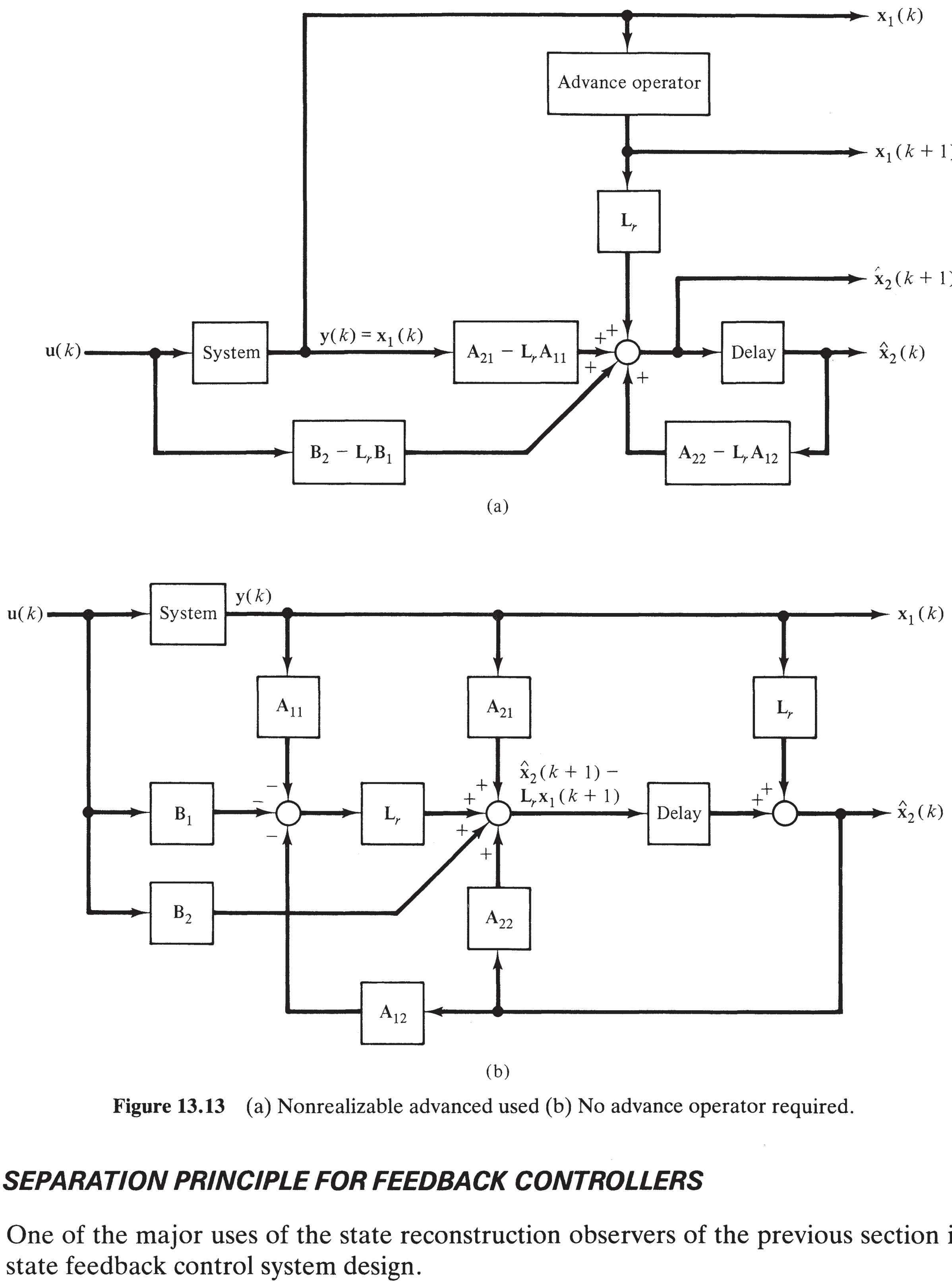



**Figure 13.13** (a) Nonrealizable advanced used (b) No advance operator required.

## 13.7 A SEPARATION PRINCIPLE FOR FEEDBACK CONTROLLERS

One of the major uses of the state reconstruction observers of the previous section is state feedback control system design.

The observers described previously are used to estimate **x.** If a constant-state feedback matrix **K** is then used, with $\hat{\mathbf{x}}$ as input instead of **x,** the system shown in Figure 13.14 is obtained. The composite system is of order $2n$. By proper selection of **K,** $n$ of

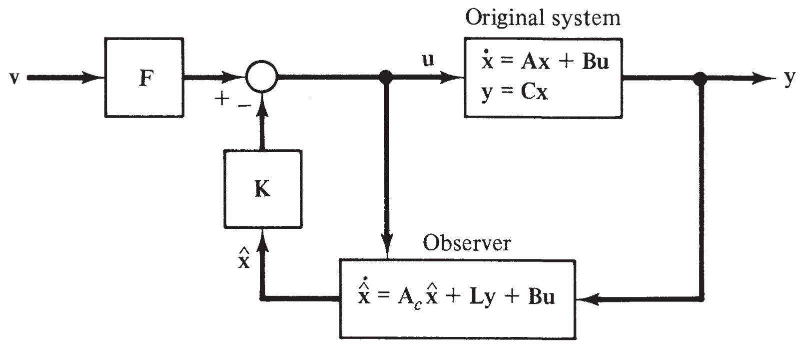



Figure 13.14

the closed-loop eigenvalues can be specified as in Sec. 13.4. By proper selection of **L**, the remaining $n$ eigenvalues of the observer can be specified. This represents a *separation principle*. That is, a feedback system with the desired poles can be designed, proceeding as if all states were measurable. Then a separate design of the observer can be used to provide the desired observer poles. To see this, note that

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \qquad \dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{LCx} + \mathbf{Bu}, \qquad \mathbf{u} = \mathbf{Fv} - \mathbf{K}\hat{\mathbf{x}}$$

can be combined into

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & \mathbf{A}_c - \mathbf{BK} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{BF} \\ \mathbf{BF} \end{bmatrix} \mathbf{v}$$

The $2n$ closed-loop eigenvalues are roots of

$$\Delta_T(\lambda) = \begin{vmatrix} \mathbf{I}_n\lambda - \mathbf{A} & \mathbf{BK} \\ -\mathbf{LC} & \mathbf{I}_n\lambda - \mathbf{A}_c + \mathbf{BK} \end{vmatrix} = 0$$

This can be reduced to a block triangular form by a series of elementary operations. Subtracting rows $i(1 \le i \le n)$ from rows $n + i$, and then adding columns $n + j$ $(1 \le j \le n)$ to columns $j$ gives

$$\Delta_T(\lambda) = \begin{vmatrix} \mathbf{I}_n\lambda - \mathbf{A} & \mathbf{BK} \\ -\mathbf{I}_n\lambda + \mathbf{A} - \mathbf{LC} & \mathbf{I}_n\lambda - \mathbf{A}_c \end{vmatrix} = \begin{vmatrix} \mathbf{I}_n\lambda - \mathbf{A} + \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{I}_n\lambda - \mathbf{A}_c \end{vmatrix}$$

$$= |\mathbf{I}_n\lambda - \mathbf{A} + \mathbf{BK}| \cdot |\mathbf{I}_n\lambda - \mathbf{A}_c| = \Delta'(\lambda) \cdot \Delta_c(\lambda)$$

This verifies that the two sets of $n$ eigenvalues can be specified separately, provided $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is completely controllable and completely observable. Experience indicates that a good design usually results if the continuous-time observer poles are selected to be a little farther to the left in the $s$-plane than the desired closed-loop state feedback poles. The discrete-time observer poles should be somewhat nearer the $Z$-plane origin so that the transients die out faster than the dominant system modes. Sec. 14.7 points out that there are other factors involved in the selection of observer pole locations.

## *13.8 TRANSFER FUNCTION VERSION OF POLE PLACEMENT–OBSERVER DESIGN*

In this section the combined pole placement–observer design problem is reconsidered using the transfer function point of view. For the single-input, single-output case, a complete and very satisfying set of results is obtained. Only a brief introduction to the multiple-input, multiple-output problem is presented. The full treatment is left for the references because the main focus of this book is on state variable methods.

Consider the single-input, single-output $n$th-order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

The Laplace transform input-output relation is

$$y(s) = \{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\}u(s)$$

Throughout this section the input-output transfer function will be written as the ratio of two polynomials $a(s)$ and $b(s)$. That is,

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D = \frac{b(s)}{a(s)}$$

In this case, the degree of the denominator polynomial is $n$, the number of state variables. The degree of the numerator polynomial $b(s)$ will clearly be $n$ or less for any transfer function derived from state equations. If $D = 0$, it will always have a degree less than $n$. Any general transfer function $b(s)/a(s)$, not necessarily derived from state equations, is said to be proper if

$$\text{degree}[b(s)] \le \text{degree}[a(s)]$$

If the strict inequality holds, the transfer function is said to be *strictly proper*. Therefore, assuming that $b(s)/a(s)$ is strictly proper is equivalent to assuming that $D = 0$. For simplicity, this assumption is made throughout most of this section. The treatment of nonzero $D$ is similar to that presented in Sec. 13.5 and 13.6. That is, a modified output $y' = y - Du$ can be used in the implementation.

From the treatment of Sec. 13.6, the form of the full state observer is

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}y + \mathbf{B}u$$

The observer output $\hat{\mathbf{x}}$ is then multiplied by the pole placement–control feedback gain matrix $\mathbf{K}$, a $1 \times n$ row. The final quantity being fed back is a scalar $w = \mathbf{K}\hat{\mathbf{x}}$. Figure 13.15$a$ shows the standard state variable form of the system, the observer, and the control feedback gain. In this section it is convenient to write the transform of the feedback signal $w(s)$ as

$$\begin{aligned} w(s) &= \mathbf{K}(s\mathbf{I} - \mathbf{A}_c)^{-1}[\mathbf{L}y(s) + \mathbf{B}u(s)] \\ &= [q(s)/d(s)]y(s) + [e(s)/d(s)]u(s) \end{aligned}$$

Clearly the denominator of both transfer functions from $y$ to $w$ and from $u$ to $w$ is $d(s) = \det\{s\mathbf{I} - \mathbf{A}_c\}$. Figure 13.15$b$ gives the same pole placement–observer system in transfer function form. Figure 13.15$b, c$, and $d$ give several possible rearrangements of this basic diagram. In each case the final closed-loop transfer function relation between the external reference signal $v(s)$ and the output $y(s)$ is found to be

$$\frac{y(s)}{v(s)} = \frac{b(s)d(s)}{a(s)[d(s) + e(s)] + b(s)q(s)} \tag{13.39}$$

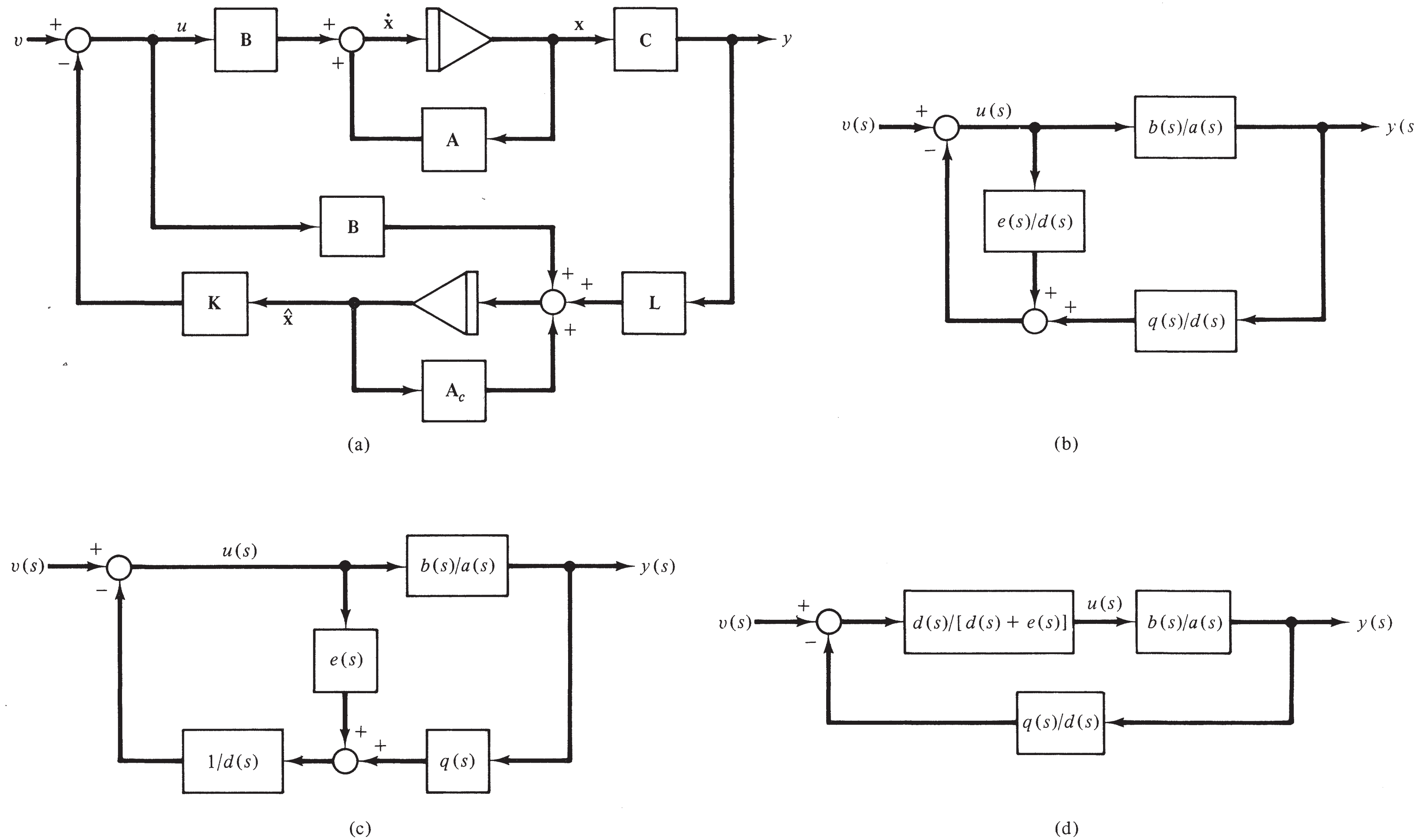



**Figure 13.15** Various rearrangements of the pole placement–observer system.

Obviously, if the controller gain **K**, the observer gain **L**, and the observer system matrix $\mathbf{A}_c$ were known, say from methods presented in Secs. 13.4 and 13.6, then the complete input-output transfer function of Eq. (*13.39*) would be known. The central purpose of this section is to reverse the process. Given a system described by $b(s)/a(s)$, is it possible to determine the polynomials $d(s)$, $e(s)$, and $q(s)$ so that the closed-loop system has arbitrarily specified pole locations? From previous sections the answer is known to be yes if the open-loop system is completely controllable and completely observable. In the present transfer function context, the question can be stated entirely in terms of polynomial algebra. Let $d(s) + e(s)$ be renamed $p(s)$. Let the desired closed-loop characteristic polynomial be $c(s)$. For specified $a(s)$, $b(s)$, and $c(s)$, is it possible to find polynomials $p(s)$ and $q(s)$ which satisfy the Diophantine equation

$$a(s)p(s) + b(s)q(s) = c(s) \tag{13.40}$$

Several things are obvious directly from Eq. (*13.40*). First, if $a(s)$ and $b(s)$ have one or more common factors, then $c(s)$ cannot be specified arbitrarily. Of necessity, $c(s)$ must possess the same common factors as $a$ and $b$. It was shown in Chapter 11 that system pole-zero cancellations indicate a loss either of controllability or observability or both. Common factors do not occur in a completely controllable and completely observable system. The absence of common factors in $a(s)$ and $b(s)$ will be shown to be both necessary and sufficient in the following development, provided that certain meaningful polynomial degree restrictions are enforced. For example, it is obvious from Eq. (*13.40*) that if $a(s)$ is of degree 5 and $b(s)$ is of degree 3, then a $c(s)$ with degree 2 cannot be specified.

### *13.8.1 The Full State Observer*

With the hindsight of the state variable origin of Eq. (*13.40*), it is known that a full state observer $d(s)$ will have the same degree $n$ as $a(s)$. We assume strictly proper transfer functions (for now), so that $b(s)$, $q(s)$, and $e(s)$ are assumed to have degree $n - 1$. Some of their coefficients could be zero, leading to a lower degree. Therefore, $p(s)$ is of degree $n$ also. A meaningfully restricted version of the Diophantine equation then is

| | |
|---|---|
| Given arbitrary polynomials | $a(s)$ of degree $n$, |
| | $b(s)$ of degree $n - 1$ |
| | $c(s)$ of degree $2n$ |
| Find solution polynomials | $p(s)$ of degree $n$ |
| | $q(s)$ of degree $n - 1$ |

That satisfy Eq. (*13.40*)

After determining $p(s)$, a further specification of the observer characteristic equation $d(s)$ allows determination of

$$e(s) = p(s) - d(s)$$

In the state variable treatment of Sec. 13.7, the separation theorem results showed that the $n$ desired control poles $(r(s))$ and the $n$ more rapidly decaying observer poles $(d(s))$

could be specified separately by choice of **K** and **L**. Then the composite system characteristic polynomial is the product $c(s) = d(s)r(s)$. With this (usual) choice, the final closed-loop system transfer function of Eq. (*13.39*) becomes

$$\frac{b(s)d(s)}{r(s)d(s)} = \frac{b(s)}{r(s)}$$

The cancellation leaves the $n$ desired control poles and the original open-loop zeros. The observer modes completely cancel. They are unobservable hidden modes. In the present transfer function treatment, somewhat more freedom of choice exists. It is possible, for example, to select $c(s) = r(s)b(s)$ in order to cancel the open-loop zeros, if stable, and replace them by freely selected poles of the observer, $d(s)$. Other choices for $c(s)$ might cancel only certain factors of $b(s)$ (see also Sec. 14.7 regarding choice of observer poles). The difference between the approach of this section and the earlier sections is that here $c(s)$ and $d(s)$ may be selected arbitrarily, subject only to degree constraints. In the state variable treatment, $d(s)$ and $r(s)$ were selected.

**Solution of the Diophantine equation.** Consider the known polynomials

$$\begin{aligned} a(s) &= s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 \\ b(s) &= b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0 \\ c(s) &= s^{2n} + c_{2n-1}s^{2n-1} + \cdots + c_1 s + c_0 \end{aligned} \tag{13.41}$$

Determine the coefficients of the unknown polynomials of Eq. (*13.40*),

$$\begin{aligned} p(s) &= s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_1 s + p_0 \\ q(s) &= q_{n-1}s^{n-1} + q_{n-2}s^{n-2} + \cdots + q_1 s + q_0 \end{aligned} \tag{13.42}$$

Note that $a(s)$ and $c(s)$ are assumed to be monic (i.e., highest coefficient normalized to unity), and thus $p(s)$ must also be monic. As a result, there are $n$ unknowns in $p$ and $n$ more unknowns in $q$. By substituting the polynomial forms into the Diophantine equation and then equating coefficients of like powers of $s$, $2n$ equations are obtained. These will be referred to as the *matrix form of the Diophantine equation.*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & b_{n-1} & 0 & 0 & \cdots & 0 \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & b_{n-2} & b_{n-1} & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & \cdots & 0 & b_{n-3} & b_{n-2} & b_{n-1} & \cdots & 0 \\ \vdots & & & & & & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \cdots & 1 & b_1 & b_2 & b_3 & \cdots & 0 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} & b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} & 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} & 0 & 0 & b_0 & \cdots & b_{n-3} \\ & \vdots & & & \vdots & & & & & \vdots \\ 0 & 0 & 0 & & a_1 & 0 & 0 & 0 & \cdots & b_1 \\ 0 & 0 & 0 & & a_0 & 0 & 0 & 0 & \cdots & b_0 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ p_{n-2} \\ p_{n-3} \\ p_{n-4} \\ \vdots \\ p_0 \\ q_{n-1} \\ q_{n-2} \\ q_{n-3} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} c_{2n-1} - a_{n-1} \\ c_{2n-2} - a_{n-2} \\ c_{2n-3} - a_{n-3} \\ c_{2n-4} - a_{n-4} \\ \vdots \\ c_n - a_0 \\ c_{n-1} \\ c_{n-2} \\ c_{n-3} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} \tag{13.43}$$

It is known from Chapters 5 and 6 that unique solutions for the $2n$ unknowns exist if and only if the $2n \times 2n$ coefficient matrix is nonsingular. It can be shown that the matrix is nonsingular if and only if the polynomials $a(s)$ and $b(s)$ have no common factors. A direct algebraic proof in the general case is tedious. A demonstration for a second-order system is now given. The general proof, using other methods, may be found in Reference 9.

Consider the second-order transfer function

$$\frac{b(s)}{a(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

In this case there are $2n = 4$ simultaneous equations, given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} c_3 - a_1 \\ c_2 - a_0 \\ c_1 \\ c_0 \end{bmatrix}$$

A unique solution exists if the $4 \times 4$ matrix has a nonzero determinant. Laplace expansion with respect to row 1 reduces the problem to a $3 \times 3$ determinant, which when set to zero gives

$$b_0^2 - a_1 b_1 b_0 + b_1^2 a_0 = 0$$

If $b_1 = 0$, then $b_0$ must also be zero in order to yield a singular matrix. Assume this degenerate case does not apply and solve for the ratio $b_0/b_1 = a_1/2 \pm [(a_1/2)^2 - a_0]^{1/2}$. This ratio for $b_0/b_1$, which forces the $4 \times 4$ determinant to zero, also makes $s + b_0/b_1$ a factor of the quadratic $a(s)$. Aside from the degenerate case where both $b_1$ and $b_0$ are zero, the determinant can vanish only if the numerator term is a factor in the denominator.

**EXAMPLE 13.15** A second-order system has the transfer function

$$\frac{y(s)}{u(s)} = \frac{s + 1}{s^2 + 3s + 2}$$

Can the preceding scheme for designing a pole placement–observer system be used to give an arbitrary fourth-order closed-loop characteristic equation $c(s)$?

The $4 \times 4$ coefficient matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Its determinant is easily seen to be zero, and its rank is 3. This indicates that no unique solution exists for arbitrary $c(s)$ polynomials. However, nonunique solutions will exist for *certain* $c(s)$ polynomials (see Chapter 6). The zero determinant has been caused by the factor $s + 1$, which is common to both $a(s)$ and $b(s)$. ■

**EXAMPLE 13.16** Assume that the system of the previous example is changed so that $b(s) = s + 5$. Design a full state observer-controller which gives closed-loop poles at $s = -3 \pm 4j$, $s = -5$, and $s = -10$.

The desired polynomial $c(s)$ is

$$c(s) = (s^2 + 6s + 25)(s + 5)(s + 10)$$
$$= s^4 + 21s^3 + 165s^2 + 675s + 1250$$

The matrix form of the appropriate Diophantine equation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 3 & 5 & 1 \\ 0 & 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 21-3 \\ 165-2 \\ 675 \\ 1250 \end{bmatrix}$$

Generally a machine solution would be in order, but for low-order problems such as this the solution can easily be obtained manually. It is found that $p(s) = s^2 + 18s + 65$ and $q(s) = 44s + 224$. Knowing $p(s)$, the selection of $d(s)$ will give $e(s)$. Suppose the observer characteristic polynomial $d(s)$ is selected as $(s + 10)(s + 6) = s^2 + 16s + 60$. Note specifically that the factors of $d(s)$ were *not* both selected to be factors of $c(s)$. This will cancel the open-loop zero at $s = -5$ and replace it by a zero at $s = -6$. Finally, $e(s) = p(s) - d(s) = 2s + 5$. By using the $q(s)/d(s)$ and $e(s)/d(s)$ transfer functions in various ways, the final compensated system can be represented in the different forms of Figure 13.15. Note, however, that the hidden modes $(s + 5)$ and $(s + 10)$, while not observable in the output, still exist internally. If parameter errors have caused inexact cancellation, some small contribution from these modes may be observed. In this case, these modes are sufficiently stable so that their contribution due to initial conditions or disturbances would decay rapidly. ■

**EXAMPLE 13.17** Consider the system used earlier in Examples 13.1 and 13.10. For comparison purposes, use the transfer function methods to design a closed-loop system with poles at $s = -3$ and $-4$. Put the two full state observer poles at $s = -8$, as in Example 13.10.

For the given **A**, **B**, and **C**, it is easy to show that the open-loop transfer function is $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = 2/[s(s-3)] = b(s)/a(s)$. The desired closed-loop characteristic equation is

$$c(s) = (s + 3)(s + 4)(s + 8)^2$$
$$= s^4 + 23s^3 + 188s^2 + 640s + 768$$

The matrix form of the Diophantine equation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 26 \\ 188 \\ 640 \\ 768 \end{bmatrix}$$

The solution is especially easy here because both $b_1$ and $a_0$ are zero; it is given by

$$p(s) = s^2 + 26s + 266$$
$$q(s) = 719s + 384$$

Both observer poles are specified to be at $s = -8$, so $d(s) = (s + 8)^2$. This gives $e(s) = 10s + 202$.

The feedback signal is

$$w(s) = \mathbf{K}\hat{\mathbf{x}}(s) = \frac{(719s + 384)y(s) + (10s + 202)u(s)}{s^2 + 16s + 64}$$

and the closed-loop transfer function, after cancellations, is

$$\frac{y(s)}{v(s)} = \frac{2}{s^2 + 7s + 12}$$

For comparison purposes, the observer of Example 13.9 gave

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} -19 & 2 \\ -\frac{121}{2} & 3 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 19 \\ \frac{121}{2} \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The pole placement gain of Example 13.1 was $\mathbf{K} = [6 \quad 10]$. By taking Laplace transforms, it is found that

$$\mathbf{K}\hat{\mathbf{x}} = \mathbf{K}[s\mathbf{I} - \mathbf{A}_c]^{-1}\left\{\begin{bmatrix} 19 \\ \frac{121}{2} \end{bmatrix} y(s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s)\right\}$$

When expanded, this agrees precisely with the preceding expression for $w(s) = q(s)/d(s)y(s) + e(s)/d(s)u(s)$. ■

**EXAMPLE 13.18** Can the desired closed-loop poles of the previous example be achieved using only dynamic output feedback?

Dynamic output feedback means using only the $y(s)$ signal, passed through a strictly proper transfer function $q(s)/d(s)$ but without an $e(s)/d(s)u(s)$ component. This means that $e(s) = 0$ is required; hence $d(s) = p(s)$. If this is the selection, the resulting closed-loop transfer function is found to be

$$\frac{y(s)}{v(s)} = \frac{d(s)b(s)}{c(s)}$$

$$= \frac{2(s^2 + 26s + 266)}{(s^2 + 7s + 12)(s^2 + 16s + 64)}$$

The conclusion is that the desired control poles can be achieved, as well as the other two poles at $s = -8$. The open-loop $b(s)$ is still a numerator factor. However, the feedback path denominator poles (which are no longer associated with an observer) must be accepted as whatever they come out, i.e., $p(s)$. The observer poles are replaced by the dynamic compensator poles, and these are no longer factors in the overall closed-loop denominator. Hence the $d(s) = p(s)$ factor in the numerator no longer cancels. The separation principle property has been lost by not using the extra feedback path from $u(s)$ to $w(s)$. ■

### *13.8.2 The Reduced Order Observer*

From the state variable treatment of the pole placement–observer problem, it is known that it is necessary to estimate only $n - m$ states, where $m$ is rank of the output matrix $\mathbf{C}$. If $\mathbf{C}$ is full rank, this is the same as the number of outputs. The reduced order observer requires a dynamical system of order $n - m$ rather than $n$. In the single output case, this means that $d(s)$ and $p(s)$ need be only of degree $n - 1$, and hence $c(s)$ need be only of degree $2n - 1$. The previous results are only slightly modified by these changes in degree. Thus the modified problem becomes:

Given $a(s) = n$th degree monic polynomial, as before
$b(s) = (n - 1)$st degree polynomial, as before
$c(s) = (2n - 1)$st degree monic polynomial

Find $p(s) = (n-1)$st degree monic polynomial
$q(s) = (n-1)$st degree polynomial

Such that the modified Diophantine equation

$$a(s)p(s) + b(s)q(s) = c(s)$$

is satisfied. The polynomial coefficients are numbered as before; that is, the subscript agrees with the power of $s$ it multiplies so the constant term has the 0 subscript. Substituting the polynomials into the Diophantine equation, expanding, and equating like powers of $s$ gives the following $2n-1$ equations for the $2n-1$ unknowns ($n$ coefficients $q_i$ and $n-1$ coefficients $p_i$).

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & 0 & 0 & \cdots & 0 \\
a_{n-1} & 1 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-1} & 0 & \cdots & 0 \\
a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 & b_{n-3} & b_{n-2} & b_{n-1} & \cdots & 0 \\
\cdot & & & \cdots & 1 & 0 & \vdots & & \vdots & & \vdots \\
\cdot & & & & a_{n-1} & 1 & & & & & 0 \\
\cdot & & & & & a_{n-1} & b_0 & & & & b_{n-1} \\
a_0 & a_1 & a_2 & \cdots & & a_{n-2} & & & & & \vdots \\
\vdots & & & & & \vdots & & & b_1 & b_2 & b_3 \\
0 & 0 & 0 & \cdots & a_1 & a_2 & 0 & \cdots & b_0 & b_1 & b_2 \\
0 & 0 & 0 & \cdots & a_0 & a_1 & 0 & \cdots & 0 & b_0 & b_1 \\
0 & 0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 & 0 & b_0
\end{bmatrix}
\begin{bmatrix}
p_{n-2} \\ p_{n-3} \\ p_{n-4} \\ \vdots \\ p_0 \\ q_{n-1} \\ q_{n-2} \\ \vdots \\ q_2 \\ q_1 \\ q_0
\end{bmatrix}$$

$$= [c_{2n-2} - a_{n-1} \;\; c_{2n-3} - a_{n-2} \;\; c_{2n-4} - a_{n-3} \;\; \cdots \;\; c_n - a_1 \;\; c_{n-1} - a_0 \;\; c_{n-2} \;\; \cdots \;\; c_2 \;\; c_1 \;\; c_0]^T \tag{13.44}$$

As in the full state observer case, it can be shown that the $(2n-1) \times (2n-1)$ coefficient matrix is nonsingular if and only if $a(s)$ and $b(s)$ have no common factors. If this is true, which it will be for all open-loop systems which are completely controllable and completely observable, a unique solution for the coefficients of $p(s)$ and $q(s)$ can be found. As in the full state case, after finding $p(s)$, a specification of the observer denominator $d(s)$ gives $e(s)$ through the polynomial relation $p(s) = d(s) + e(s)$. The block diagram representations of Figure 13.15 remain valid.

**EXAMPLE 13.19** Design a reduced-order observer-pole placement controller for a system with the open-loop transfer function

$$\frac{b(s)}{a(s)} = \frac{1}{s(s+1)}$$

such that the desired closed-loop poles are at $s = -1 \pm j$ and the observer pole is at $s = -2$. This same example is given in References 9 and 10.

The given polynomials have coefficients $a_0 = 0$, $a_1 = 1$, $b_0 = 1$, and $b_1 = 0$. The total system order will be 3, (second-order system plus first-order observer) and the desired $c(s)$ coefficients are $c_0 = 4$, $c_1 = 6$, and $c_2 = 4$. The three simultaneous equations are

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 4-1 \\ 6-0 \\ 4 \end{bmatrix}$$

and the solution gives $p(s) = s + 3, q(s) = 3s + 4$. Since it has been specified that the observer should have a pole at $s = -2$, $d(s) = s + 2$, so $e(s) = 1$. The two transfer functions $q(s)/d(s)$ and $e(s)/d(s)$ can be used to manipulate the composite system into various forms, as before. Figure 13.16 presents the solution in a form which shows a simple lead compensator in the feedback path and another lead compensator in the forward path. After cancellations, the closed-loop transfer function is $y(s)/v(s) = 1/[(s+1+j)(s+1-j)]$, as desired. ■

### 13.8.3 The Discrete-Time Pole Placement–Observer Problem

If a discrete-time system is described by a strictly proper $Z$-transfer function $y(z)/u(z) = b(z)/a(z)$, then all the previous material for continuous-time Laplace-transformed systems carries over exactly. The substitution of the complex variable $z$ for $s$ is the only change required. This is true in both the full state and reduced-order observer cases which have been considered.

**EXAMPLE 13.20** *Discrete-time full state observer:* Consider the third-order system of Problem 13.13 but modified to satisfy the current restriction of a single input and output,

$$\mathbf{B} = [-1 \quad 1 \quad 3]^T$$

The open-loop input-output transfer function is easily found to be

$$\frac{y(z)}{u(z)} = \frac{-z^2 - 0.5z + 0.61}{z^3 - 0.3z^2 - 0.51z + 0.194}$$

Using the transfer function approach, design a system with control poles at $z = 0.5 + 0.4j$, $0.5 - 0.4j$, and $0.3$. Put the full-state observer poles at $z = 0, 0.1$, and $0.2$.

Expanding the desired factors gives the sixth-order characteristic polynomial

$$c(z) = z^6 - 1.6z^5 + 1.12z^4 - 0.362z^3 + 0.0511z^2 - 0.00246z$$

The matrix Diophantine equation to be solved for $p(z)$ and $q(z)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.3 & 1 & 0 & -1 & 0 & 0 \\ -0.51 & -0.3 & 1 & -0.5 & -1 & 0 \\ 0.194 & -0.51 & -0.3 & 0.61 & -0.5 & -1 \\ 0 & 0.194 & -0.51 & 0 & 0.61 & -0.5 \\ 0 & 0 & 0.194 & 0 & 0 & 0.61 \end{bmatrix} \begin{bmatrix} p_2 \\ p_1 \\ p_0 \\ q_2 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} -1.6 + 0.3 \\ 1.12 + 0.51 \\ -.362 - 0.194 \\ 0.0511 \\ -0.00246 \\ 0 \end{bmatrix}$$

The solution gives $q(z) = 1.1559z^2 - 1.6987z + 0.5155$ and $p(z) = z^3 - 1.3z^2 + 2.3959z - 1.621$. Selecting $d(z) = z^3 - 0.3z^2 + 0.02z$ gives $e(z) = -z^2 + 2.3759z - 1.621$, so that

$$\mathbf{K\hat{x}} = \frac{-z^2 + 2.3759z - 1.621}{z^3 - 0.3z^2 + 0.02z} u(z) + \frac{1.1559z^2 - 1.6987z + 0.5155}{z^3 - 0.3z^2 + 0.02z} y(z)$$

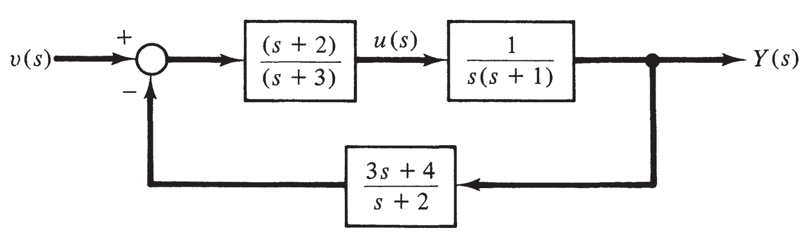



Figure 13.16

As a check on this result, the state variable method of Sec. 13.4 shows that state feedback gains of

$$\mathbf{K} = [0.51923 \quad -3.3423 \quad 0.953846]$$

will give the desired poles at $z = 0.5 \pm 0.4j$ and $z = 0.3$. The same algorithm applied to the dual problem gives observer gains

$$\mathbf{L} = [0 \quad -0.765734 \quad -1.471328]^T$$

to achieve observer poles at $z = 0, 0.1$, and $0.2$. Using these in the observer equation

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}}(k) + \mathbf{L}y(k) + \mathbf{B}u(k)$$

allows verification that $\mathbf{K}(z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{L} = \mathbf{K}\hat{\mathbf{x}}(z)/y(z) = q(z)/d(z)$ and $\mathbf{K}(z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{B} = \mathbf{K}\hat{\mathbf{x}}/u = e(z)/d(z)$ agree with the transfer function results. The closed-loop transfer function for the composite system is given by

$$\frac{y(z)}{v(z)} = \frac{b(z)}{r(z)} = \frac{-z^2 - 0.5z + 0.61}{(z - 0.5 + 0.4j)(z - 0.5 - 0.4j)(z - 0.3)} \tag{13.45}$$

Although this is what was asked for in the design specifications, it gives a less-than-desirable step response. Its time response due to a step input has excessive overshoot and oscillations. ■

**EXAMPLE 13.21** *Discrete-time reduced order observer:* Repeat the previous example, but use a reduced order observer with poles at $z = 0.1$ and $0.2$.

Using the same desired control poles with the two observer poles gives $c(z) = z^5 - 1.6z^4 + 1.12z^3 - 0.362z^2 + 0.0511z - 0.00246$. The matrix version of the Diophantine equation gives five simultaneous equations:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -0.3 & 1 & -0.5 & -1 & 0 \\ -0.51 & -0.3 & 0.61 & -0.5 & -1 \\ 0.194 & -0.51 & 0 & 0.61 & -0.5 \\ 0 & 0.194 & 0 & 0 & 0.61 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 \\ q_2 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} -1.6 + 0.3 \\ 1.12 + 0.51 \\ -0.362 - 0.194 \\ 0.0511 \\ -0.00246 \end{bmatrix}$$

Numerical solution yields $p(z) = z^2 - 3.9573z + 1.0672$ and $q(z) = -2.6573z^2 + 1.9531z - 0.34345$. Using the specified observer poles gives the observer characteristic polynomial $d(z) = z^2 - 0.3z + 0.02$. This, along with $p(z)$, gives $e(z) = -3.6573z + 1.0472$. As usual, the controller-observer system is described by the two compensator transfer functions,

$$w(z) = \mathbf{K}\hat{\mathbf{x}}(z) = [e(z)/d(z)]u(z) + [q(z)/d(z)]y(z) \tag{13.46}$$

The closed-loop transfer function is the same as in Example 13.20, Eq. (*13.45*). ■

**EXAMPLE 13.22** *Reduced order observer, state variable method:* Repeat the controller-observer design, using state variable methods, to verify the results of Example 13.21.

The control gains are unchanged, that is, $\mathbf{K} = [0.51923 \quad -3.3423 \quad 0.953846]$. Let $\mathbf{x}_2 = [x_2 \quad x_3]^T$ denote the states to be estimated by the reduced order observer. Using results from Problem 13.13, but now using only one input $u_1 = u$ (and the corresponding single column $\mathbf{B} = [-1 \quad 1 \quad 3]^T$), the observer equations are

$$\hat{\mathbf{x}}_2(k+1) = \begin{bmatrix} -0.504196 & 0.239161 \\ -1.7902 & 0.804196 \end{bmatrix} \hat{\mathbf{x}}_2(k)$$
$$+ \begin{bmatrix} 0.391608 \\ -1.95804 \end{bmatrix} [y(k+1) + u(k)] + \begin{bmatrix} -0.1 \\ 0.8 \end{bmatrix} y(k) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(k)$$

Bringing $x_1(k) = y(k)$ back in, the entire three-component state reconstruction process is described by

$$\hat{\mathbf{x}}(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.504196 & 0.239161 \\ 0 & -1.77902 & 0.804196 \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} 1 \\ 0.391608 \\ -1.95804 \end{bmatrix} y(k+1)$$

$$+ \begin{bmatrix} 0 \\ -0.1 \\ 0.8 \end{bmatrix} y(k) + \begin{bmatrix} 0 \\ 1.391688 \\ 1.04196 \end{bmatrix} u(k)$$

Taking $Z$-transforms, solving for $\hat{\mathbf{x}}(z)$, and then premultiplying by the gain row matrix $\mathbf{K}$ gives

$$\mathbf{K}\hat{\mathbf{x}}(z) = \mathbf{K}[z\mathbf{I} - \mathbf{A}_c]^{-1} \left\{ \begin{bmatrix} z \\ 0.391608z - 0.1 \\ -1.95804z + 0.8 \end{bmatrix} y(z) + \begin{bmatrix} 0 \\ 1.391608 \\ 1.04196 \end{bmatrix} u(z) \right\}$$

where $\mathbf{A}_c$ has been used to indicate the $3 \times 3$ matrix. Substituting values for $\mathbf{K}$ and simplifying leads to

$$\mathbf{K}\hat{\mathbf{x}}(z) = \frac{(-2.6573z^3 + 1.9531z^2 - 0.34345z)}{(z^2 - 0.3z + 0.02)z} y(z)$$

$$+ \frac{(-3.65756z + 1.0473)z}{(z^2 - 0.3z + 0.02)z} u(z)$$

After canceling a factor of $z$, this is the same result as obtained in Example 13.21, Eq. (*13.46*). ■

## *13.9 DESIGN OF DECOUPLED OR NONINTERACTING SYSTEMS* [11,12]

A system with an equal number $m$ of inputs and outputs is considered in this section. If the $m \times m$ transfer matrix $\mathbf{H}(s)$ is diagonal and nonsingular, the system is said to be *decoupled* because each input affects one and only one output. When the number of inputs and outputs are not equal, $\mathbf{H}(s)$ is not square, and thus cannot be diagonal. Various other kinds of decoupling and partial decoupling have been defined where $\mathbf{H}(s)$ is triangular or block diagonal and so on [13, 14].

The problem of reducing a system with $m$ inputs and $m$ outputs to decoupled form, using a state feedback control law $\mathbf{u} = -\mathbf{K}_d\mathbf{x} + \mathbf{F}_d\mathbf{v}$, is considered. Assuming that $\mathbf{D} = [\mathbf{0}]$, the transfer matrix for the state feedback system of Eqs. (*13.4*) and (*13.5*) is

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_d]^{-1}\mathbf{B}\mathbf{F}_d \tag{13.47}$$

The decoupling problem is that of selecting matrices $\mathbf{F}_d(m \times m)$ and $\mathbf{K}_d(m \times n)$ so that $\mathbf{H}(s)$ is diagonal and nonsingular. Consider the inverse transform of equation (*13.47*) and the Cayley-Hamilton theorem applied to $e^{[\mathbf{A} - \mathbf{B}\mathbf{K}_d]t}$. This leads to an alternative statement of decoupling. The matrices

$$\mathbf{C}[\mathbf{A} - \mathbf{B}\mathbf{K}_d]^j\mathbf{B}\mathbf{F}_d, \qquad j = 0, 1, \ldots, n-1 \tag{13.48}$$

must all be diagonal if the system is decoupled. Let the $i$th row of $\mathbf{C}$ be $\mathbf{c}_i$ and define a set of $m$ integers by

$$d_i = \min_j \{j | \mathbf{c}_i \mathbf{A}^j \mathbf{B} \neq \mathbf{0}, j = 0, 1, \ldots, n-1\} \tag{13.49}$$

or

$$d_i = n - 1 \qquad \text{if } \mathbf{c}_i \mathbf{A}^j \mathbf{B} = \mathbf{0} \text{ for all } j$$

The original system can be decoupled using state feedback [11] if and only if the following $m \times m$ matrix is nonsingular:

$$\mathbf{N} = \begin{bmatrix} \mathbf{c}_1 \mathbf{A}^{d_1} \mathbf{B} \\ \mathbf{c}_2 \mathbf{A}^{d_2} \mathbf{B} \\ \vdots \\ \mathbf{c}_m \mathbf{A}^{d_m} \mathbf{B} \end{bmatrix}$$

In particular, one set of decoupling matrices is

$$\mathbf{F}_d = \mathbf{N}^{-1} \quad \text{and} \quad \mathbf{K}_d = \mathbf{N}^{-1} \begin{bmatrix} \mathbf{c}_1 \mathbf{A}^{d_1+1} \\ \vdots \\ \mathbf{c}_m \mathbf{A}^{d_m+1} \end{bmatrix} \tag{13.50}$$

**EXAMPLE 13.23** Determine whether the open-loop system of Example 13.9 can be decoupled using state feedback. Assume now that $\mathbf{D} = [\mathbf{0}]$.

Since $\mathbf{c}_1 \mathbf{A}^0 \mathbf{B} = [1 \quad 0] \neq \mathbf{0}$, set $d_1 = 0$. Also, $\mathbf{c}_2 \mathbf{A}^0 \mathbf{B} = \mathbf{0}$, but $\mathbf{c}_2 \mathbf{A}\mathbf{B} = [0 \quad 1] \neq \mathbf{0}$, so that $d_2 = 1$. Therefore, $\mathbf{N} = \begin{bmatrix} \mathbf{c}_1 \mathbf{B} \\ c_2 \mathbf{A}\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular and decoupling is possible. The decoupling matrices are $\mathbf{F}_d = \mathbf{N}^{-1} = \mathbf{I}$ and $\mathbf{K}_d = \mathbf{N}^{-1} \begin{bmatrix} \mathbf{c}_1 \mathbf{A} \\ \mathbf{c}_2 \mathbf{A}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$. Using these, the decoupled transfer matrix is obtained from Eq. (*13.47*):

$$\mathbf{H}(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s^2 \end{bmatrix}$$

■

Results of the previous example are typical in that all the poles of the decoupled system are at the origin. This is always the result if Eq. (*13.50*) is used. In fact, the general result is $\mathbf{H}(s) = \text{diag}[s^{-d_1-1} \quad s^{-d_2-1} \quad \cdots \quad s^{-d_m-1}]$. This system is said to be *integrator decoupled*. The peformance of such a decoupled system would usually not be acceptable. Two questions naturally arise. First, can decoupling be accomplished by using only the available outputs? The answer to this is yes, provided dynamic feedback compensators (observers for example) of sufficiently high dimension are allowed [15]. Second, can other state feedback matrices or output feedback matrices be used to prespecify closed-loop pole locations while preserving the decoupled nature of the system? The answer to this question is yes in some cases, no in others. At least $m + \sum_{i=1}^{m} d_i$ poles can be prespecified, but not necessarily all of the poles [11].

**EXAMPLE 13.24** Consider the decoupled system found in Example 13.23. Find a constant feedback matrix which moves the closed-loop poles from zero to $\{-5, -5, -10\}$. Is the resultant system still decoupled?

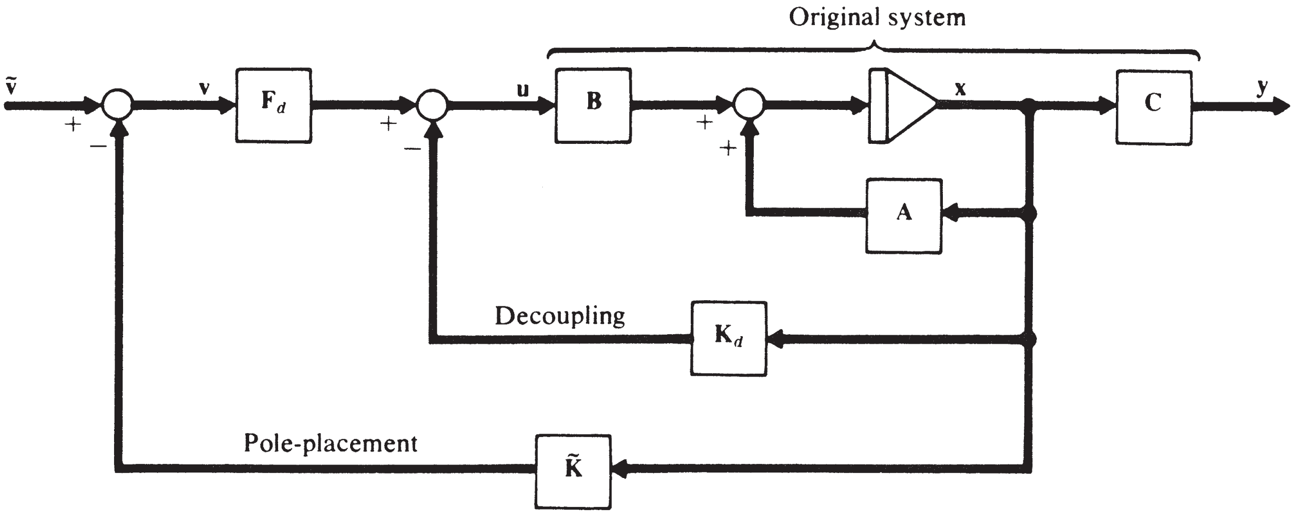



**Figure 13.17**

The decoupled system of Example 13.23 is treated as the open-loop system. That is,

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}_d = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \mathbf{B}\mathbf{F}_d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Since $\mathbf{F}_d$ is nonsingular, this system is still completely controllable (Sec. 13.3). Thus the specified eigenvalues can be achieved using a constant state feedback matrix $\tilde{\mathbf{K}}$. The method of Sec. 13.4 gives $\tilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 5 \\ 20 & 15 & 0 \end{bmatrix}$.

The combined system, using $\mathbf{K}_d$ and $\mathbf{F}_d$ to achieve decoupling and $\tilde{\mathbf{K}}$ to achieve pole placement, is shown in Figure 13.17. In this case the resulting system is still decoupled and has closed-loop poles, as specified, at $-5, -5, -10$. The closed-loop transfer function is

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_d + \mathbf{B}\mathbf{F}_d\tilde{\mathbf{K}}]^{-1}\mathbf{B}\mathbf{F}_d = \begin{bmatrix} 1/(s+5) & 0 \\ 0 & 1/[(s+5)(s+10)] \end{bmatrix}$$ ■

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## ILLUSTRATIVE PROBLEMS

### *State Feedback*

**13.1** A system is described by $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find a constant state feedback matrix $\mathbf{K}$ which yields closed-loop poles $\Gamma = \{-2, -3, -4\}$.

Define $\mathbf{X}(\lambda) = [\mathbf{I}\lambda - \mathbf{A} \mid \mathbf{B}]$ and $\boldsymbol{\xi}^T = [\boldsymbol{\psi}^T \quad (\mathbf{K}\boldsymbol{\psi})^T]$, as in Eqs. (*13.12*) and (*13.13*). Then

$$\mathbf{X}(\lambda) = \begin{bmatrix} \lambda+2 & -1 & 0 & 0 & 0 \\ 0 & \lambda+2 & 0 & 0 & 1 \\ 0 & 0 & \lambda-4 & 1 & 0 \end{bmatrix}$$

One solution to $\mathbf{X}\boldsymbol{\xi} = \mathbf{0}$ is needed for each of the three desired eigenvalues, such that the resulting three $\boldsymbol{\psi}$ columns form a nonsingular matrix. With $\lambda = -2$, $\mathbf{X}\boldsymbol{\xi} = \mathbf{0}$ expands into the component equations $-\xi_2 = 0$, $\xi_5 = 0$, and $-6\xi_3 + \xi_4 = 0$. Clearly, $\xi_1$ is arbitrary, and it is selected as 0. Furthermore, $\xi_4 = -1$ is selected, giving $\boldsymbol{\xi} = [0 \quad 0 \quad -\frac{1}{6} \quad -1 \quad 0]^T$. A multitude of other choices could have been made. With $\lambda = -3$, the corresponding component equations are $-\xi_1 - \xi_2 = 0$, $-\xi_2 + \xi_5 = 0$, and $-7\xi_3 + \xi_4 = 0$. One valid solution is $\boldsymbol{\xi} = [1 \quad -1 \quad 0 \quad 0 \quad -1]^T$. With $\lambda = -4$, $\mathbf{X}\boldsymbol{\xi} = \mathbf{0}$ expands into $-2\xi_1 - \xi_2 = 0$, $-2\xi_2 + \xi_5 = 0$ and $-8\xi_3 + \xi_4 = 0$. One solution is $\boldsymbol{\xi} = [\frac{1}{4} \quad -\frac{1}{2} \quad 0 \quad 0 \quad -1]^T$. These three columns are a subset of the possible columns which form the matrix $\mathbf{U}(\lambda)$, and the top $3\times 3$ partition $\mathbf{G}=\begin{bmatrix} 0 & 1 & \frac{1}{4} \\ 0 & -1 & -\frac{1}{2} \\ -\frac{1}{6} & 0 & 0 \end{bmatrix}$ is nonsingular. The desired gain is given by the bottom $2\times 3$ partition $\mathcal{J}$, postmultiplied by $\mathbf{G}^{-1}$, or $\mathbf{K}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\mathbf{G}^{-1}=\begin{bmatrix} 0 & 0 & 6 \\ 2 & 3 & 0 \end{bmatrix}$, as specified in Eq. (*13.17*).

**13.2** Repeat Problem 13.1 if $\Gamma=\{-2,-2,-20\}$.

Since $\lambda=-2$ is requested as a double eigenvalue, two independent $\boldsymbol{\xi}$ solutions must be found. One is available from Problem 13.1. Another is obtained from the same component equations by selecting $\xi_1=-1$ and $\xi_3=0$, giving $\boldsymbol{\xi}=[-1\ \ 0\ \ 0\ \ 0\ \ 0]^T$. Substituting $\lambda=-20$ into $\mathbf{X}(\lambda)$ leads to the third column. One valid choice is found by setting $\xi_3=0$ and $\xi_5=-1$. Solving for the remaining components gives $\boldsymbol{\xi}=[1/(18)^2\ \ -1/(18)\ \ 0\ \ 0\ \ -1]^T$. Then Eq. (*13.17*) gives the desired gain as $\mathbf{K}=\begin{bmatrix} 0 & 0 & 6 \\ 0 & 18 & 0 \end{bmatrix}$.

**13.3** Repeat Problem 13.1 if $\Gamma=\{-3,-2+j,-2-j\}$.

$\boldsymbol{\xi}=[1\ \ -1\ \ 0\ \ 0\ \ -1]^T$ was selected for $\lambda=-3$ in Problem 13.1. With $\lambda=-2+j$, the component equations are $j\xi_1-\xi_2=0$, $j\xi_2+\xi_5=0$, and $(-6+j)\xi_3+\xi_4=0$. One easy choice is $\xi_3=0, \xi_1=1$. Then $\boldsymbol{\xi}=[1\ \ j\ \ 0\ \ 0\ \ 1]^T$. With $\lambda=-2-j$, the solution $\boldsymbol{\xi}$ is the complex conjugate, so all three solutions have been found. Note, however, that row 3 of the $5\times 3$ matrix is all zeros, so $\mathbf{G}$ is not invertible. An alternate solution for $\lambda=-3$ is easily found to be $[0\ \ 0\ \ -\frac{1}{7}\ \ -1\ \ 0]$. Then

$$\mathbf{K}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & j & -j \\ -\frac{1}{7} & 0 & 0 \end{bmatrix}^{-1}$$

In general it is easier to avoid the complex matrix inverse, and by the results of Problem 4.23

$$\mathbf{K}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{7} & 0 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & 7 \\ 1 & 0 & 0 \end{bmatrix}$$

**13.4** Repeat Problem 13.1 if $\Gamma=\{-3,-3,-3\}$.

Since the rank of $\mathbf{X}(-3)$ is three, there are only two independent solutions to $\mathbf{X}(-3)\boldsymbol{\xi}=0$. One was given in Problem 13.1 and another in Problem 13.3. A triple eigenvalue is requested, so the third column vector must be a generalized eigenvector, as in Eq. (*13.19*). It will satisfy $\mathbf{X}(-3)\boldsymbol{\xi}_g=-\boldsymbol{\psi}$, where $\boldsymbol{\psi}=[1\ \ -1\ \ 0]^T$ or $[0\ \ 0\ \ -\frac{1}{7}]$—i.e., the top partitions of the two $\boldsymbol{\xi}$ solutions. The second choice must be ruled out because it does not give an independent solution. Using the first choice gives $\boldsymbol{\xi}_g=[2\ \ -1\ \ 0\ \ 0\ \ 0]^T$. The gain is then given by Eq. (*13.17*) as

$$\mathbf{K}=-\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & -1 \\ 0 & -\frac{1}{7} & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & 7 \\ 1 & 2 & 0 \end{bmatrix}$$

**13.5** A system is described by Eq. (*13.1*) with $\mathbf{A}=\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find a constant state feedback matrix $\mathbf{K}$ which gives closed-loop eigenvalues $\lambda_1=\lambda_2=\lambda_3=-1$.

Form $\mathbf{X}(-1)=[\lambda\mathbf{I}-\mathbf{A}\ \vdots\ \mathbf{B}]=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Its rank is 3, so there is only one independent solution to $\mathbf{X}\boldsymbol{\xi} = \mathbf{0}$, $\boldsymbol{\xi} = [1 \quad 1 \quad 1 \quad -1]^T$. A chain of two generalized eigenvectors can be found. $\mathbf{X}\boldsymbol{\xi}_{g1} = -[1 \quad 1 \quad 1]^T$ gives the first generalized vector as $\boldsymbol{\xi}_{g1} = -[3 \quad 2 \quad 1 \quad 0]^T$. Using this in the second generalized equation $\mathbf{X}\boldsymbol{\xi}_{g2} = [3 \quad 2 \quad 1]^T$ gives the solution $\boldsymbol{\xi}_{g2} = [6 \quad 3 \quad 1 \quad 0]^T$. The top $3 \times 3$ partition forms the matrix $\mathbf{G}$, and row four forms the matrix $\mathcal{J}$. Then Eq. (*13.17*) gives the gain matrix as

$$\mathbf{K} = [-1 \quad 0 \quad 0]\begin{bmatrix} 1 & -3 & 6 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = [-1 \quad 3 \quad -3]$$

### *Output Feedback*

**13.6** Specify a constant output feedback matrix so that a system with $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{C} = [0 \quad 1]$, and $\mathbf{D} = [0 \quad 0]$ will have a closed-loop eigenvalue at $-10$.

Form $[\lambda\mathbf{I} - \mathbf{A} \mid \mathbf{B}] = \left[\begin{array}{cc|cc} -9 & 0 & 0 & 1 \\ 0 & -6 & 1 & 0 \end{array}\right]$ and find $\boldsymbol{\xi} = [0 \quad 1 \quad 6 \quad 0]^T$ as a nontrivial solution.

The meaning of the vector $\boldsymbol{\xi}$ in the case of output feedback is $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\psi} \\ \mathbf{K}'\mathbf{C}\boldsymbol{\psi} \end{bmatrix}$, as shown in Eq. (*13.22*) (with $\mathbf{D} = 0$). Thus $\mathbf{K}'\mathbf{C}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$. Using $\mathbf{C} = [0 \quad 1]$ gives $\mathbf{K}' = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

**13.7** Another output measurement is added to the system of Problem 13.6, so that now $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is it possible to use constant output feedback so that both $\lambda_1$ and $\lambda_2$ equal $-10$ for the closed-loop system? If yes, find the gain matrix $\mathbf{K}'$.

Since rank $\mathbf{C} = 2 = n$, both eigenvalues can be prescribed, and two independent solutions of $\mathbf{X}\boldsymbol{\xi} = \mathbf{0}$ are needed. A second solution for $\boldsymbol{\xi}$ in Problem 13.6 is $\boldsymbol{\xi} = [1 \quad 0 \quad 0 \quad 9]^T$. This solution could not have been used there to solve for $\mathbf{K}'$ (Why?), but with the modified $\mathbf{C}$ we can write $\mathbf{K}'\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 9 & 0 \end{bmatrix}$, from which $\mathbf{K}' = \begin{bmatrix} 0 & 6 \\ 9 & -9 \end{bmatrix}$.

**13.8** A system [6] is described by Eq. (*13.1*) with $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since this system is completely controllable and rank $\mathbf{C} = m = 2$, two closed-loop eigenvalues can be made arbitrarily close to any specified values by using output feedback. Find $\mathbf{K}'$ so that $\lambda_1 = 1$, $\lambda_2 = \epsilon$, where $\epsilon$ is arbitrarily close to zero.

Nontrivial solutions to $[\lambda\mathbf{I} - \mathbf{A} \mid \mathbf{B}]\boldsymbol{\xi} = \mathbf{0}$ must be found. For $\lambda = 1$, $\boldsymbol{\xi} = [-1 \quad -1 \quad -1 \quad 0]^T$. With $\lambda = \epsilon$, the component equations are $\epsilon\xi_1 - \xi_2 = 0$, $\epsilon\xi_2 - \xi_3 + \xi_4 = 0$, and $-\xi_1 + \epsilon\xi_3 = 0$. Arbitrarily selecting $\xi_1 = 1$ gives $\xi_2 = \epsilon$, $\xi_3 = 1/\epsilon$, and $\xi_4 = 1/\epsilon - \epsilon^2$. The gain can be determined from

$$\mathbf{K}'\mathbf{C}\begin{bmatrix} -1 & 1 \\ -1 & \epsilon \\ -1 & \dfrac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{\epsilon} - \epsilon^2 \end{bmatrix} \quad \text{or} \quad \mathbf{K}'\begin{bmatrix} -1 & 1 \\ -2 & 1+\epsilon \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{\epsilon} - \epsilon^2 \end{bmatrix}$$

Matrix inversion gives $\mathbf{K}' = [2/(1/\epsilon - \epsilon^2) \quad (\epsilon^2 - 1/\epsilon)]/(1 - \epsilon)$, which for very small $\epsilon$ approaches $\mathbf{K}' = [2/\epsilon \quad -1/\epsilon]$. This result indicates that $\lambda_2 = 0$ can be achieved only by using infinite feedback gains. This is analogous to the well-known result for single-input, single-output systems. An infinite gain is required to cause a closed-loop pole to coincide with an open-loop zero.

### *Observers*

13.9 A system described by Eq. (*13.1*) has $\mathbf{A} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbf{D} = [\mathbf{0}]$. Design an observer whose output settles to the actual state vector within 1 s.

It should be verified that the system is completly observable. The determination of the observer gain is carried out by using the duality with the pole placement problem. Using $\mathbf{A}^T$ instead of $\mathbf{A}$ and $\mathbf{C}^T$ instead of $\mathbf{B}$ will give $\mathbf{L}^T$ in place of $\mathbf{K}$. Proceeding, $\mathbf{X}(\lambda) = \begin{bmatrix} \lambda+2 & 0 & 0 & 1 & 0 \\ 2 & \lambda & 3 & 0 & 1 \\ 0 & -1 & \lambda+4 & 1 & 0 \end{bmatrix}$. With $\lambda = -5$, its rank is 3, so it is possible to find two independent solutions to $\mathbf{X}(-5)\boldsymbol{\xi} = \mathbf{0}$. They are found numerically as $\boldsymbol{\xi}_1 = [0.33333 \;\; 0.45833 \;\; 0.541667 \;\; 1 \;\; 0]^T$ and $[0 \;\; -0.125 \;\; 0.125 \;\; 0 \;\; -1]^T$. With $\lambda = -6$, a single solution is needed, and one is $\boldsymbol{\xi} = [0 \;\; -0.13333 \;\; 0.066667 \;\; 0 \;\; -1]^T$, leading to

$$\mathbf{L}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0.33333 & 0 & 0 \\ 0.45833 & -0.125 & -0.133333 \\ 0.541667 & 0.125 & 0.066667 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ -8 & 7 & -1 \end{bmatrix}$$

and then $\mathbf{A}_c = \begin{bmatrix} -5 & 6 & -3 \\ 0 & -7 & 1 \\ 0 & -2 & -4 \end{bmatrix}$. The observer is defined by $\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{y} + \mathbf{B}\mathbf{u}$, where $\mathbf{u}$ and $\mathbf{y}$ are the input and output of the system being observed.

13.10 Consider the open-loop system described by the matrices **A, B, C,** and **D** of Example 13.9. Design an observer with eigenvalues of $-8$, $-8$, and $-10$.

Since the output is $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$, the observer equations are modified to take into account the nonzero **D** matrix. Let $\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{y} + \mathbf{z} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} + \mathbf{L}\mathbf{D}\mathbf{u} + \mathbf{z}$. It is seen that setting $\mathbf{z} = (\mathbf{B} - \mathbf{L}\mathbf{D})\mathbf{u}$ again leads to the error Eq. (*13.30*). Therefore, the observer gain matrix **L** is designed using the dual pole-placement procedure—that is, $\mathbf{A}^T$ and $\mathbf{C}^T$ are used in place of **A** and **B.** Solving Eq. (*13.12*) for each desired $\lambda$ leads to $\mathbf{L} = \begin{bmatrix} 0 & 14 \\ 0 & 48 \\ 12 & 0 \end{bmatrix}$. Then using $\mathbf{A}_c = \mathbf{A} - \mathbf{L}\mathbf{C}$ and $\mathbf{z} = (\mathbf{B} - \mathbf{L}\mathbf{D})\mathbf{u}$ gives the observer shown in Figure 13.18.

13.11 Consider the open-loop unstable system of Example 13.9. Design a closed-loop system with dynamic feedback, which has eigenvalues $-5$, $-6$, and $-6$. The eigenvalues added by the feedback compensation should have real parts more negative than $-7$.

It is first assumed that all state variables are available for feedback. A state feedback matrix **K** is selected to satisfy the specification $\lambda_i \in \{-5, -6, -6\}$. This is found from Eqs. (*13.16*) and (*13.17*) using

$$\mathbf{G} = [\boldsymbol{\psi}(-5) \quad \boldsymbol{\psi}_1(-6) \quad \boldsymbol{\psi}_2(-6)] = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 0 & -4 \\ 0 & -1 & 0 \end{bmatrix}$$

and

$$\mathcal{F} = [\mathbf{f}(-5) \quad \mathbf{f}_1(-6) \quad \mathbf{f}_2(-6)] = \begin{bmatrix} 0 & -1 & 0 \\ -9 & 0 & -16 \end{bmatrix}$$

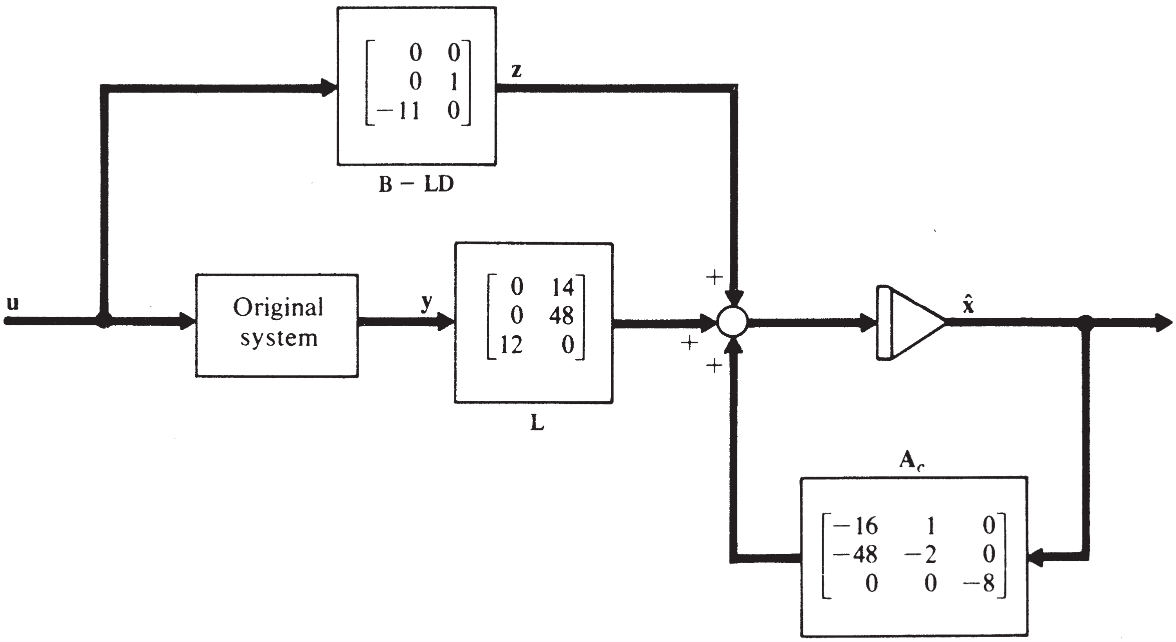


Figure 13.18

Then $\mathbf{K} = \mathscr{G}\mathbf{G}^{-1} = \begin{bmatrix} 0 & 0 & 10 \\ 12 & 7 & 0 \end{bmatrix}$. This result cannot be used directly since all state variables are not available. However, an observer was designed for this system in Problem 13.10. Since that observer meets the specification, it can be used to provide an estimate of **x**. Then **K** is applied to that estimate, as shown in Figure 13.19.

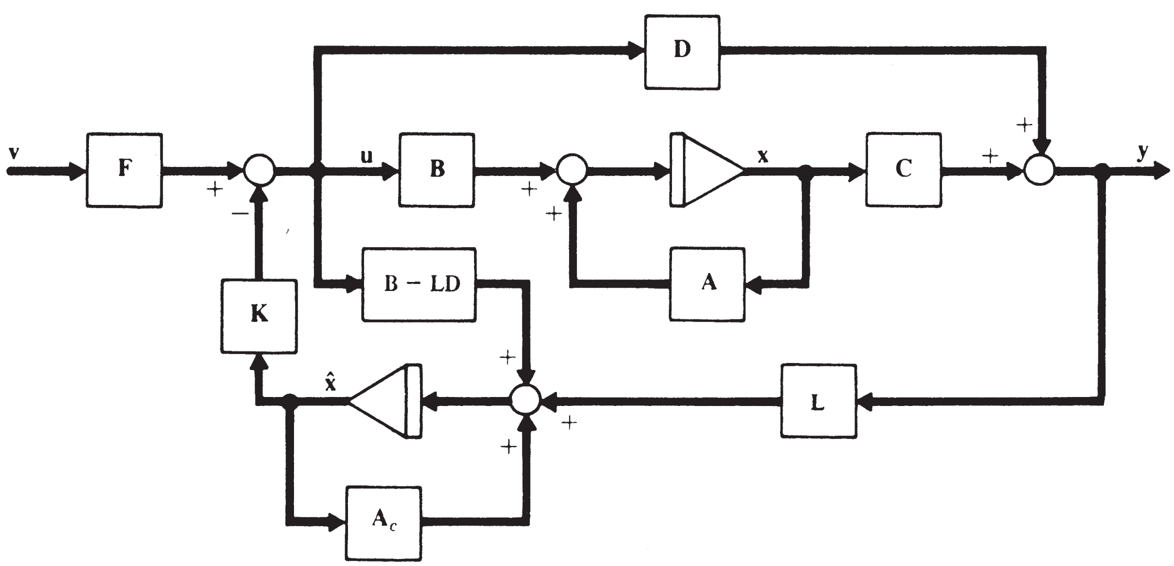


Figure 13.19

**13.12** When can a system of the general form of Eq. (*13.1*) or (*13.2*) be transformed so that $\mathbf{C} = [\mathbf{I} \mid \mathbf{0}]$?

From Sec. 4.10 it is clear that the first requirement is that **C** must have full rank $m$. Assume that this is so and that the first $m$ columns of **C** are linearly independent. Then by a series of row operations (equivalent to premultiplication by a nonsingular transformation matrix $\mathbf{T}_1$) **C** can be brought into row-reduced-echelon form. That is

$$\mathbf{T}_1\mathbf{y} = \mathbf{T}_1\mathbf{C}\mathbf{x} + \mathbf{T}_1\mathbf{D}\mathbf{u}$$

or

$$\mathbf{y}' = [\mathbf{I} \mid \mathbf{C}']\mathbf{x} + \mathbf{D}'\mathbf{u}$$

To further reduce **C** (i.e., to eliminate **C**′), column operations are required. These are accomplished by postmultiplying by a nonsingular transformation matrix $\mathbf{T}_2$. The only way this can be accomplished is to redefine the state vector according to

$$\mathbf{x} = \mathbf{T}_2\mathbf{x}'$$

Then $\mathbf{y}' = [\mathbf{I} \mid \mathbf{0}]\mathbf{x}' + \mathbf{D}'\mathbf{u}$. The dynamics equation must similarly be transformed into $\mathbf{x}'$ variables

$$\mathbf{x}'(k+1) = \mathbf{T}_2^{-1}\mathbf{A}\mathbf{T}_2\mathbf{x}'(k) + \mathbf{T}_2^{-1}\mathbf{B}\mathbf{u}(k)$$
$$= \mathbf{A}'\mathbf{x}'(k) + \mathbf{B}'\mathbf{u}(k)$$

**13.13** A third-order discrete-time system with two inputs and one output is described by

$$\mathbf{A} = \begin{bmatrix} 0 & -0.5 & -0.1 \\ -0.1 & -0.7 & 0.2 \\ 0.8 & -0.8 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{C} = [1 \quad 0 \quad 0]$$

Design a reduced-order observer to estimate $x_2$ and $x_3$. Place its poles at $z = 0.1$ and 0.2.

Using the algorithm of Sec. 13.4 with the equivalent **A** and **B** matrices selected as

$$\mathbf{A}_{22}^T = \begin{bmatrix} -0.7 & -0.8 \\ 0.2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{12}^T = \begin{bmatrix} -0.5 \\ -0.1 \end{bmatrix}$$

gives a gain matrix **K**, which is

$$\mathbf{L}_r^T = [0.391608 \quad -1.95804]$$

Using this to form $\mathbf{A}_r = \mathbf{A}_{22} - \mathbf{L}_r\mathbf{A}_{12}$ gives the observer the result shown in Figure 13.20.

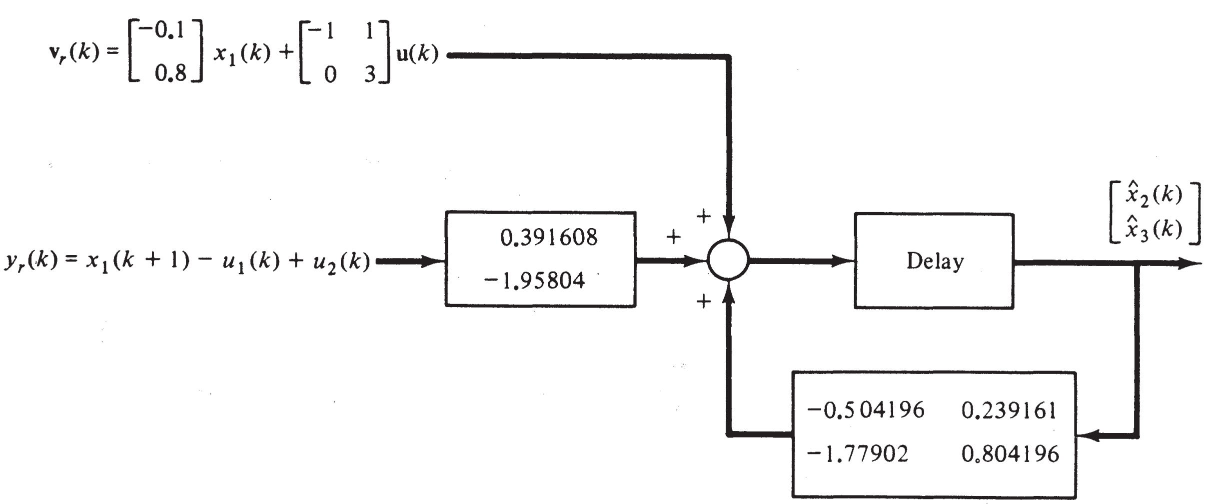


**Figure 13.20**

## *Decoupling*

**13.14** Use state feedback to decouple the system [11] described by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}]$$

Using Eq. (*13.49*), $\mathbf{c}_1\mathbf{B} = [0 \quad 1] \neq \mathbf{0}$, so $d_1 = 0$. $\mathbf{c}_2\mathbf{B} = [1 \quad 0] \neq \mathbf{0}$, so $d_2 = \mathbf{0}$. Then $\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is nonsingular and the system can be decoupled. $\mathbf{F}_d = \mathbf{N}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{K}_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{c}_1\mathbf{A} \\ \mathbf{c}_2\mathbf{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$. Using these results in Eq. (*13.47*) gives the integrator decoupled result, $\mathbf{H}(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$.

Note that the original system is not completely controllable, and thus the decoupled system is not controllable either. Any attempt to specify all three eigenvalues using the method of Sec. 13.4 will fail.

**13.15** Can the system of Example 12.8 be decoupled using state feedback?

The irreducible realization found in Example 12.8 is used. The product $\mathbf{c}_1\mathbf{B} = [0 \quad 1]$ is not zero, so $d_1 = 0$. Likewise, $\mathbf{c}_2\mathbf{B} = [1 \quad 0]$, so $d_2 = 0$. Then $\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is nonsingular, so the system can be decoupled.

The integrator decoupled system is obtained by using $\mathbf{F}_d = \mathbf{N}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{K}_d = \mathbf{N}^{-1}\begin{bmatrix} \mathbf{c}_1\mathbf{A} \\ \mathbf{c}_2\mathbf{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ -\frac{2}{3} & \frac{5}{9} & -\frac{5}{27} & -5 \end{bmatrix}$. Equation (*13.47*) gives the decoupled closed-loop transfer function $\mathbf{H}(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$.

**13.16** Consider the open-loop, decoupled system described by $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ of Example 13.24 and with $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Find an output feedback matrix which places closed-loop poles at $\lambda_1 = -5$, $\lambda_2 = -10$. Is the resultant system still decoupled? Is it stable?

We first find the gain $\mathbf{K}_*$ which will multiply $\mathbf{y}' = \mathbf{y} - \mathbf{D}\mathbf{u}$. The desired values of $\lambda$ are $-5$ and $-10$. One solution of $[\lambda\mathbf{I} - \mathbf{A} \mid \mathbf{B}]\boldsymbol{\xi} = \mathbf{0}$ is found for each $\lambda$. Then the $\boldsymbol{\xi}$ vectors are partitioned to give $\mathbf{K}_*\mathbf{C}[\boldsymbol{\psi}(-5) \quad \boldsymbol{\psi}(-10)] = [\boldsymbol{\psi}(-5) \quad \boldsymbol{\psi}(-10)]$. The numerical values are $\mathbf{K}'\mathbf{C}\begin{bmatrix} 0 & 0.01 \\ 0 & 0.08 \\ -0.2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Using $\mathbf{C}$ and taking the inverse of a $2 \times 2$ matrix gives $\mathbf{K}_* = \begin{bmatrix} 5 & 0 \\ 0 & -100 \end{bmatrix}$. The gain which multiplies the output $\mathbf{y}$ is $\mathbf{K}' = [\mathbf{I} - \mathbf{K}_*\mathbf{D}]^{-1}\mathbf{K}_* = \begin{bmatrix} -\frac{5}{4} & 0 \\ 0 & -100 \end{bmatrix}$. When this feedback is used, the characteristic equation (*13.21*) is

$$\Delta_0'(\lambda) = |\lambda\mathbf{I}_3 - \tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{K}'[\mathbf{I}_2 + \tilde{\mathbf{D}}\mathbf{K}']^{-1}\tilde{\mathbf{C}} = \left|\begin{bmatrix} \lambda+2 & -1 & 0 \\ -96 & \lambda-2 & 0 \\ 0 & 0 & \lambda+5 \end{bmatrix}\right|$$

$$= (\lambda+5)(\lambda^2 - 100) = 0$$

Therefore, the eigenvalues are at $-5$, $-10$, and $+10$ and the system is unstable. Using Eq. (*13.6*) and (*13.7*), and the fact that $\mathbf{F}' = \mathbf{I}_2$ leads to

$$\mathbf{H}(s) = [\mathbf{I}_2 + \mathbf{D}\mathbf{K}']^{-1}\{\tilde{\mathbf{C}}\{s\mathbf{I}_3 - \tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{K}'[\mathbf{I}_2 + \mathbf{D}\mathbf{K}']^{-1}\tilde{\mathbf{C}}\}^{-1}\tilde{\mathbf{B}}[\mathbf{I}_2 + \mathbf{K}'\mathbf{D}]^{-1} + \mathbf{D}\}$$

$$= \begin{bmatrix} -4(s+1)/(s+5) & 0 \\ 0 & 1/(s^2 - 100) \end{bmatrix}$$

The system is still uncoupled. If $\mathbf{D} = [\mathbf{0}]$, as in Example 13.23, then $\mathbf{K}' = \begin{bmatrix} 5 & 0 \\ 0 & -100 \end{bmatrix}$ and $\mathbf{H}(s) = \begin{bmatrix} 1/(s+5) & 0 \\ 0 & 1/(s^2 - 100) \end{bmatrix}$.

**13.17** Assume that a particular system can be decoupled by use of state feedback. Assume that the system is also completely observable. Show that the system can be decoupled by using an observer to estimate the state and then by using the estimated state with the decoupling matrices.

Let the system be described by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \mathbf{y} = \mathbf{C}\mathbf{x}$, and the decoupling state feedback is $\mathbf{u} = \mathbf{F}_d\mathbf{v} - \mathbf{K}_d\mathbf{x}$. Let the observer be described by $\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{y} + \mathbf{B}\mathbf{u}$. It is to be shown that $\mathbf{x}$ can be replaced by $\hat{\mathbf{x}}$ in the control law without altering the decoupling. Writing $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$ and using $\mathbf{u} = \mathbf{F}_d\mathbf{v} - \mathbf{K}_d\hat{\mathbf{x}} = \mathbf{F}_d\mathbf{v} - \mathbf{K}_d\mathbf{x} + \mathbf{K}_d\mathbf{e}$ gives

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}\mathbf{K}_d & \mathbf{B}\mathbf{K}_d \\ \hline \mathbf{0} & \mathbf{A}_c \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{F}_d \\ \mathbf{0} \end{bmatrix} \mathbf{v}$$

Using Laplace transforms gives

$$\begin{bmatrix} \mathbf{x}(s) \\ \mathbf{e}(s) \end{bmatrix} = \left[\begin{array}{c|c} s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_d & -\mathbf{B}\mathbf{K}_d \\ \hline \mathbf{0} & s\mathbf{I} - \mathbf{A}_c \end{array}\right]^{-1} \begin{bmatrix} \mathbf{B}\mathbf{F}_d \\ \mathbf{0} \end{bmatrix} \mathbf{v}(s)$$

Only the upper-left block of the inverse is needed in order to solve for $\mathbf{x}(s)$. Using results of Sec. 4.9, this reduces to $\mathbf{x}(s) = [s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_d]^{-1}\mathbf{B}\mathbf{F}_d\mathbf{v}(s)$ and the output is $\mathbf{y} = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_d]^{-1}\mathbf{B}\mathbf{F}_d\mathbf{v}(s)$. Comparing this with Eq. (*13.47*) shows that the system is still decoupled.

**13.18** Figure 13.21 shows a system described by $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$, an observer described by $\{\mathbf{A}_c, \mathbf{L}\}$, decoupling matrices $\{\mathbf{K}_d, \mathbf{F}_d\}$, and a pole placement feedback matrix $\tilde{\mathbf{K}}$. Let the system be the one in Example 13.9, but with $\mathbf{D} = [\mathbf{0}]$. The decoupling matrices are calculated in Example 13.23, and the pole placement matrix is the one found in Example 13.24. Show that using the output and an observer yields the same decoupled system transfer matrix as was obtained in Example 13.24.

The dynamic equations are $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} + \mathbf{B}\mathbf{u}$. The control law is $\mathbf{u} = \mathbf{F}_d\tilde{\mathbf{v}} - (\mathbf{F}_d\tilde{\mathbf{K}} + \mathbf{K}_d)\hat{\mathbf{x}}$. Using $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ to eliminate $\hat{\mathbf{x}}$ leads to

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}(\mathbf{F}_d\tilde{\mathbf{K}} + \mathbf{K}_d) & \mathbf{B}(\mathbf{F}_d\tilde{\mathbf{K}} + \mathbf{K}_d) \\ \hline \mathbf{0} & \mathbf{A}_c \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{F}_d \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{v}}$$

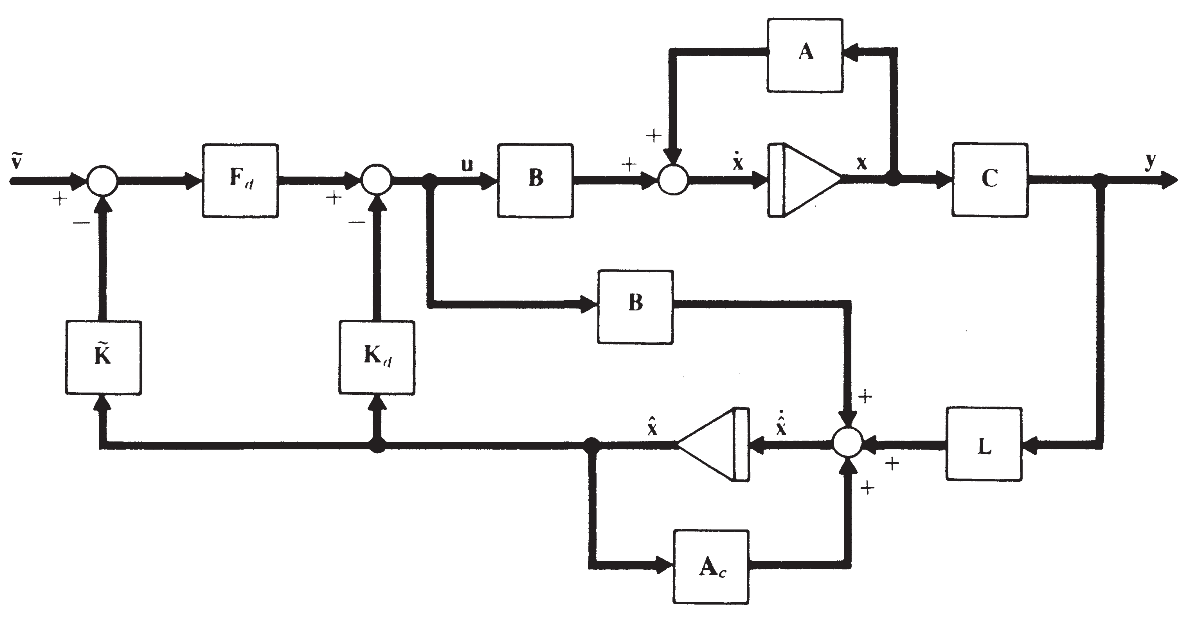


**Figure 13.21**

Taking the Laplace transform and solving for $\mathbf{x}(s)$ gives

$$\mathbf{y}(s) = \mathbf{C}\mathbf{x}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A} + \mathbf{B}(\mathbf{F}_d\tilde{\mathbf{K}} + \mathbf{K}_d)]^{-1}\mathbf{B}\mathbf{F}_d\tilde{\mathbf{v}}$$

This is the same result as in Example 13.24 and gives

$$\mathbf{H}(s) = \begin{bmatrix} 1/(s+5) & 0 \\ 0 & 1/[(s+5)(s+10)] \end{bmatrix}$$

**13.19** Design a controller/observer for the discrete-time system whose open-loop transfer function is

$$\frac{y(z)}{u(z)} = \frac{z + 0.866}{(z-1)(z^2 - 0.8z + 0.32)}$$

Put the closed-loop system poles at $z = 0$ and $0.2 \pm 0.6j$ and put the observer poles at $z = 0$ and $\pm 0.2j$.

The numerator and denominator polynomials are

$$b(z) = z + 0.866$$

$$a(z) = z^3 - 1.8z^2 + 1.12z - 0.32$$

Let the desired closed-loop poles define the polynomials $r(z)$ and $d(z)$, respectively. Then the desired closed-loop characteristic equation is

$$\begin{aligned} c(z) &= (z^3 - 0.4z^2 + 0.4z)(z^3 + 0.04z) = r(z)d(z) \\ &= z^6 - 0.4z^5 + 0.44z^4 - 0.016z^3 + 0.016z^2 \end{aligned}$$

The coefficients of these polynomials are used in Eq. (*13.43*) in order to determine the unknown polynomials $p(z)$ and $q(z)$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1.18 & 1 & 0 & 0 & 0 & 0 \\ 1.12 & -1.8 & 1 & 1 & 0 & 0 \\ -0.32 & 1.12 & -1.8 & 0.866 & 1 & 0 \\ 0 & -0.32 & 1.12 & 0 & 0.866 & 1 \\ 0 & 0 & -0.32 & 0 & 0 & 0.866 \end{bmatrix} \begin{bmatrix} p_2 \\ p_1 \\ p_0 \\ q_2 \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 1.4 \\ -0.68 \\ 0.304 \\ 0.016 \\ 0 \\ 0 \end{bmatrix}$$

Numerical solution gives the coefficients for the polynomials $p(z) = z^3 + 1.4z^2 + 1.84z + 0.92346$ and $q(z) = 1.1245z^2 - 0.90843z + 0.34123$. Using the polynomial $d(z)$ of the desired observer poles gives $e(z) = p(z) - d(z) = 1.4z^2 + 1.8z + 0.92346$. All needed terms are available, and the system can be expressed in any of the optional configurations of Figure 13.15*b, c,* or *d*. (For this discrete system, the polynomials are functions of $z$ instead of $s$). After simplification and cancellation, the closed-loop transfer function is $y(z)/v(z) = b(z)/r(z) = (z + 0.866)/(z^3 - 0.4z^2 + 0.4z)$

**13.20** Show that state feedback does not alter the transfer function zeros.

It is assumed that there are an equal number of inputs and outputs, so the zeros of the square transfer function matrix can be defined as the values of $s$ (or $z$) which give a zero value to the transfer function determinant. The open-loop transfer function is $\mathbf{H}_0 = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ and the closed-loop transfer function under state feedback is $\mathbf{H}_c = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$. This can be rearranged as follows:

$$\begin{aligned} \mathbf{H}_c &= \mathbf{C}\{(s\mathbf{I} - \mathbf{A})[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}]\}^{-1}\mathbf{B} \\ &= \mathbf{C}[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}]^{-1}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}[\mathbf{I} + \mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]^{-1} \end{aligned}$$

The last step made use of a rearrangement identity proven in Problem 4.4. Taking the determinant gives

$$|\mathbf{H}_c(s)| = |\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}|\,|[\mathbf{I}+\mathbf{K}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}]^{-1}|$$
$$= |\mathbf{H}_0(s)|/|[\mathbf{I}+\mathbf{K}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}]|$$

In this form it is obvious that the open- and closed-loop transfer function determinants vanish for the same values of $s$, so the zeros are the same. Note the similarity with the single-input, single-output results of Chapter 2. There it was found that the closed-loop zeros are the union of the forward path (open-loop) transfer function zeros and the poles of the feedback path transfer function. When only constants occur in the feedback path, the open- and closed-loop zeros are the same.

**13.21** Discuss the design techniques of this chapter as contrasted to classical techniques.

In the classical methods, reviewed in Chapter 2, there are usually just one input and one output. Feedback is always output feedback, but perhaps it involves dynamic compensation (lead networks, lead-lag networks, etc.). The design is often developed (at the introductory level) as a trial and error process. The goal is to obtain satisfactory locations for the dominant poles, to ensure acceptable overshoot, settling time, natural frequency, etc. In addition, sensitivity to parameter variations and disturbances is to be reduced and steady-state accuracy must be acceptable. See Reference 16 for a treatment of classical problems using the state variable approach. See also Reference 17.

In this chapter a systematic means of locating *all* poles is presented, provided the system is completely controllable and observable. Multiple inputs and outputs can be treated, but a computer will be required in analyzing most realistic applications. When the feedback is restricted to output signals, observers (or reduced dimensional observers or other dynamic compensators) will be required. This is analogous to using dynamic network compensators in classical design. Sensitivity reduction is directly related to the concept of return difference and the return difference matrix [18].

The design procedures in this chapter assume that the system is controllable and observable. The decomposition techniques of Chapters 11 and 12 allow any linear system to be decomposed into a controllable and an uncontrollable subsystem. The poles of the controllable part can be relocated as desired. If all unstable open-loop poles are associated with the controllable part, the system can be stabilized by use of state feedback. The poles associated with the uncontrollable part are not affected by state feedback. This is what motivated the definition of the property of stabilizability in Chapter 11. Likewise, unobservable states cannot be reconstructed by observers. The consideration of similar decompositions into observable and unobservable subsystems is what prompted the definition of the detectibility property of Chapter 11. These issues did not arise in Chapter 2 because the classical transfer function description of a system, after canceling any common factors, describes the controllable and observable portion of the system, as discussed in Chapter 12.

## PROBLEMS

**13.22** Assuming that all state variables can be measured, find a state feedback matrix $\mathbf{K}$ for the system of Problem 13.9. The closed-loop poles are to be placed at $\lambda_i = -3, -3, -4$.

**13.23** The open-loop system of Problem 13.9 is compensated by adding feedback consisting of the observer designed in Problem 13.9 followed by the constant matrix $\mathbf{K}$ of Problem 13.22. Verify that the sixth-order closed-loop system has eigenvalues at $\{-3, -3, -4, -5, -5, -6\}$.

**13.24** What should the constant output feedback matrix $\mathbf{K}'$ be if the system in Example 13.7 is to have a closed-loop eigenvalue at $-20$?

**13.25** If Problem 13.6 is reconsidered, with $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $\mathbf{K}'$ so that $\lambda_1 = \lambda_2 = -10$.

**13.26** Repeat Problem 13.8 with the desired eigenvalues $\lambda_1 = \epsilon$, $\lambda_2 = -\epsilon$.

**13.27** Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Can this system be decoupled?

**13.28** Can the system described by $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ be decoupled?

**13.29** Find the matrices $\mathbf{F}_d$ and $\mathbf{K}_d$ such that state feedback reduces the system of Problem 13.9 to integrator decoupled form. Find the decoupled transfer matrix also.

**13.30** Find a state feedback gain matrix which will give the following discrete-time system closed-loop poles at $z = 0$ and 0.5.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

**13.31** Repeat Problem 13.30 but force the poles to be at $z = 0.3 \pm j0.5$.

**13.32** A discrete-time system has

$$\mathbf{A} = \begin{bmatrix} 0.65 & -0.15 \\ -0.15 & 0.65 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Find two different feedback gain matrices which will both give closed-loop poles at $z = 0$ and 0.2.

**13.33** A system is described by

$$\dot{y}_1 + 3y_1 - y_2 = u_1$$

$$\ddot{y}_2 + 2(\dot{y}_1 + \dot{y}_2 - \dot{y}_3) + 4(y_2 - y_1) = u_2 + 5u_1$$

$$\ddot{y}_3 + 6\dot{y}_3 - 2\dot{y}_1 + y_3 = u_2$$

Put the system into state variable form, with the first three states being $y_1$, $y_2$, and $y_3$ respectively. Design a state feedback controller which will give closed-loop poles at $-3$, $-4$, $-4$, $-5$, and $-5$. Simulate the system for various test inputs, assuming all states are available for use. Then design a full state observer, with poles at $-6$, $-7$, $-8$, $-9$, and $-10$. Again on a simulation, compare the transient response of the cascaded observer, state feedback system with full state feedback results.

**13.34** Design a reduced-order observer to estimate only the nonmeasured states of Problem 13.33. Use this observer, along with the previous state feedback controller, and compare transient behavior with that obtained in the previous problem.

**13.35** Using the continuous system model of Problem 11.9, show that the closed-loop poles will be at $\lambda_i = -2, -5, -8$, and $-10$ if any of the following state feedback matrices are used. In each case the columns selected for use in Eq. (*13.6*) are shown. Compare these results with the discrete equivalent of Example 13.6.

The Feedback Gain Matrix Using Columns 1, 3, 6, 8

$$\mathbf{K} = \begin{bmatrix} -1.5000001\text{E}-01 & 7.4249178\text{E}-01 & -3.1340556\text{E}+00 & 5.0000000\text{E}-01 \\ 0.0000000\text{E}+90 & -5.3035126\text{E}+00 & 1.8814680\text{E}+01 & 0.0000000\text{E}+00 \end{bmatrix}$$

The Feedback Gain Matrix Using Columns 2, 4, 5, 7

$$\mathbf{K} = \begin{bmatrix} 4.0000007\text{E}-01 & 2.5047928\text{E}-01 & -3.4747612\text{E}-01 & 4.0000005\text{E}+00 \\ 0.0000000\text{E}+00 & -1.7891376\text{E}+00 & -1.0894562\text{E}+00 & 0.0000000\text{E}+00 \end{bmatrix}$$

The Feedback Gain Matrix Using Columns 1, 4, 6, 7

$$\mathbf{K} = \begin{bmatrix} 9.9999994\mathrm{E}-02 & 5.1884985\mathrm{E}-01 & -1.5014696\mathrm{E}+00 & 1.0000000\mathrm{E}+00 \\ 0.0000000\mathrm{E}+00 & -3.7060702\mathrm{E}+00 & 7.1533542\mathrm{E}+00 & 0.0000000\mathrm{E}+00 \end{bmatrix}$$

**13.36** An open-loop system has the transfer function

$$\frac{y(z)}{u(z)} = \frac{z}{(z-1)(z-0.5)}$$

Design a controller and full state observer such that the closed-loop transfer function has poles at $z = 0.6 \pm 0.3j$. Put the observer poles at $z = 0.1$ and $-0.1$.

**13.37** Repeat the previous problem, but use a reduced-order observer. Put the observer pole at $z = -0.1$.