دانشگاه صنعتی شاهرود

دانشکده معدن، نفت و ژئوفیزیک

درس روش های عددی در ژئومکانیک

مدرس

مرتضی جوادی اصطهباناتی

پاییز و زمستان ۱۳۹۶

رئوس مطالب

تعریف مدل

- انتزاع یا الگویی از یک واقعیت (پدیده فیزیکی، رفتار): **Entity**
- هدف: ساده سازی و تکرارپذیری واقعیت‌های جهان هستی جهت امکان مطالعه
- چرا مدل سازی:
  - عدم دسترسی و تکرار پذیری
  - از بین رفتن اصل و موجودیت پدیده در هنگام مطالعه
  - هزینه و زمان
  - امکان مطالعه همزمان پدیده های مختلف و تحلیل حساسیت
  - خطر و ریسک بالا

انواع مدل

- هر نوع و شکل از سیستم یا پدیده اصلی- هر الگویی از اصل
- فیزیکی
- نمایشی و گرافیکی
- ریاضیاتی
- عددی و محاسباتی

تفاوت مدل سازی و شبیه سازی

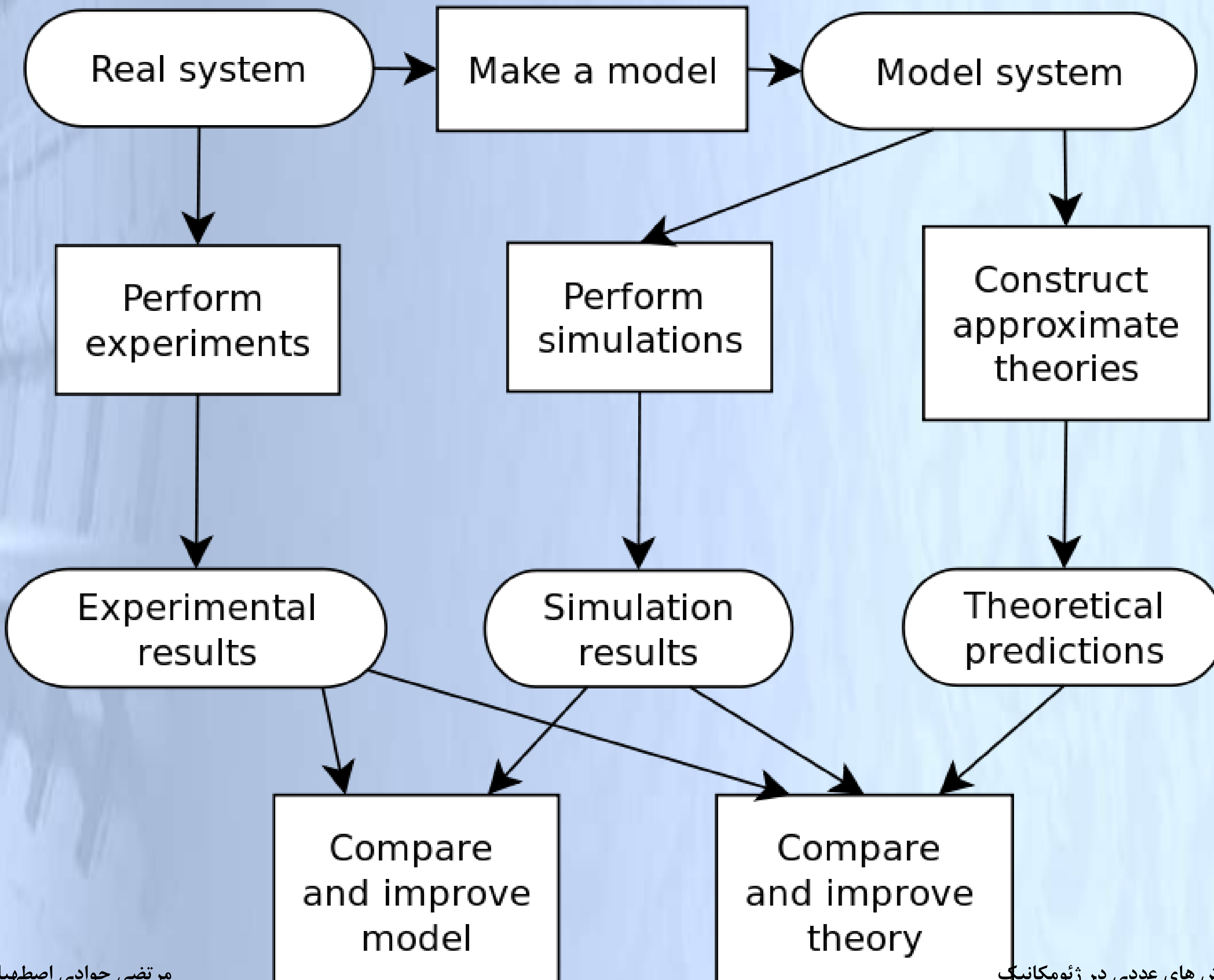
Simulation is a subset of Model

A **simulation** is the process of using a **model** to study the behavior and performance of an actual or theoretical system. ... While a **model** aims to be true to the system it represents, a **simulation** can use a **model** to explore states that would not be possible in the original system.

In a simulation, one or more variable of the mathematical model is changed and resulted changes in other variables are observed.

Simulation sometimes **considers** with visualization of a mathematical model

تفاوت مدل سازی و شبیه سازی

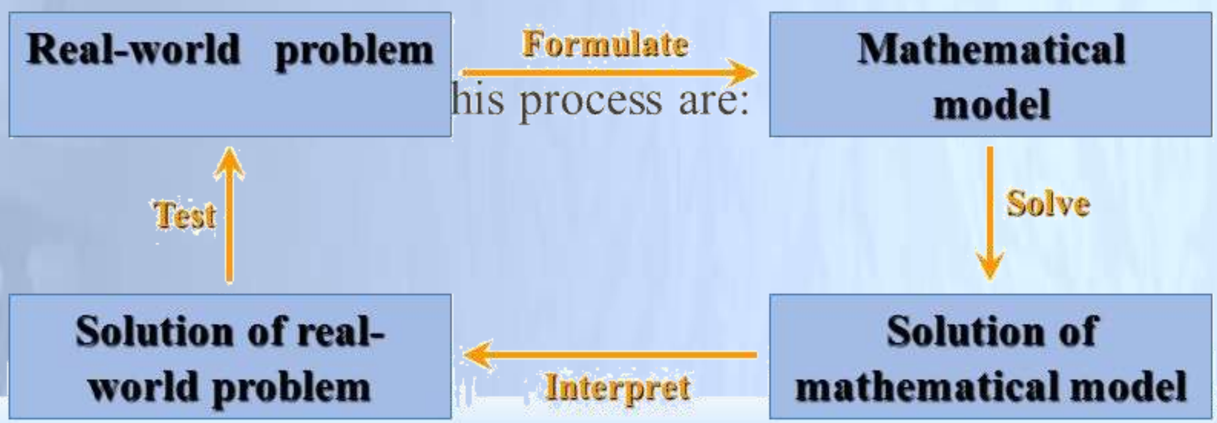


مدل سازی عددی

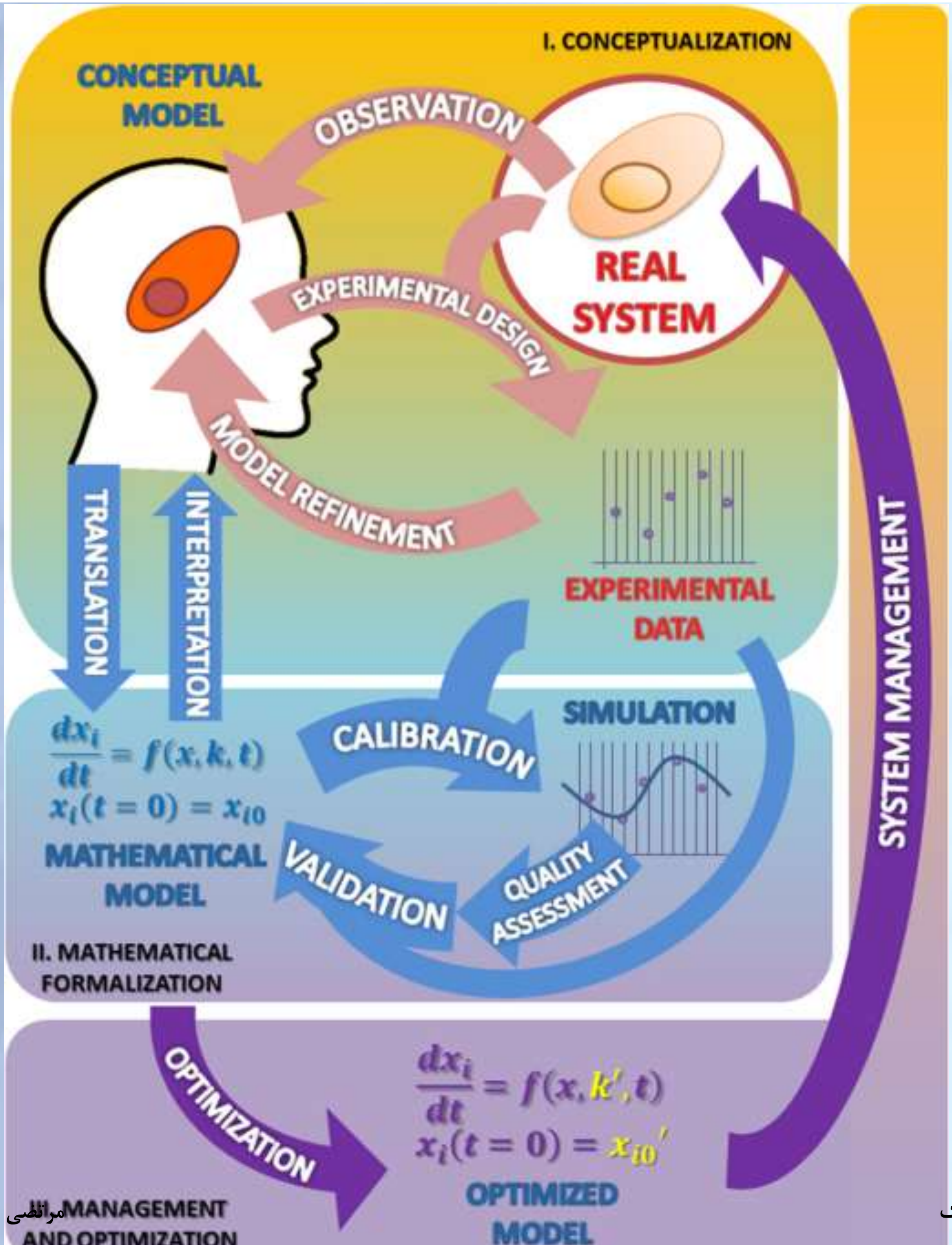
Mathematical Modeling

- As we have seen, mathematics can be used to solve real-world problems.
- Regardless of the field from which the real-world problem is drawn, the problem is analyzed by using a process called mathematical modeling.

his process are:



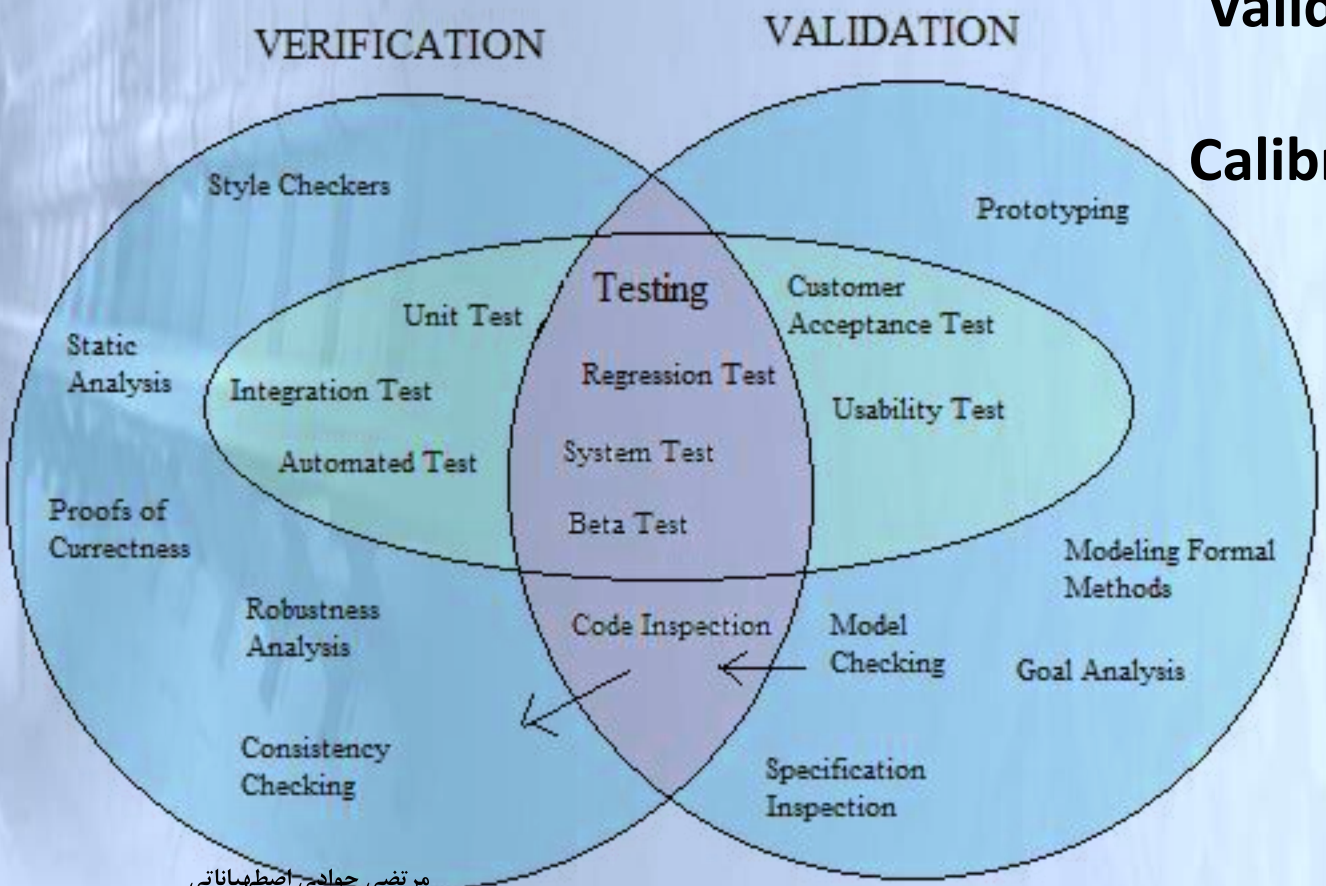
مدل سازی عددی



صحت سنجی و اعتبار سنجی مدل سازی عددی

- صحت سنجی Verification
- اعتبارسنجی Validation
- کالیبراسیون Calibration

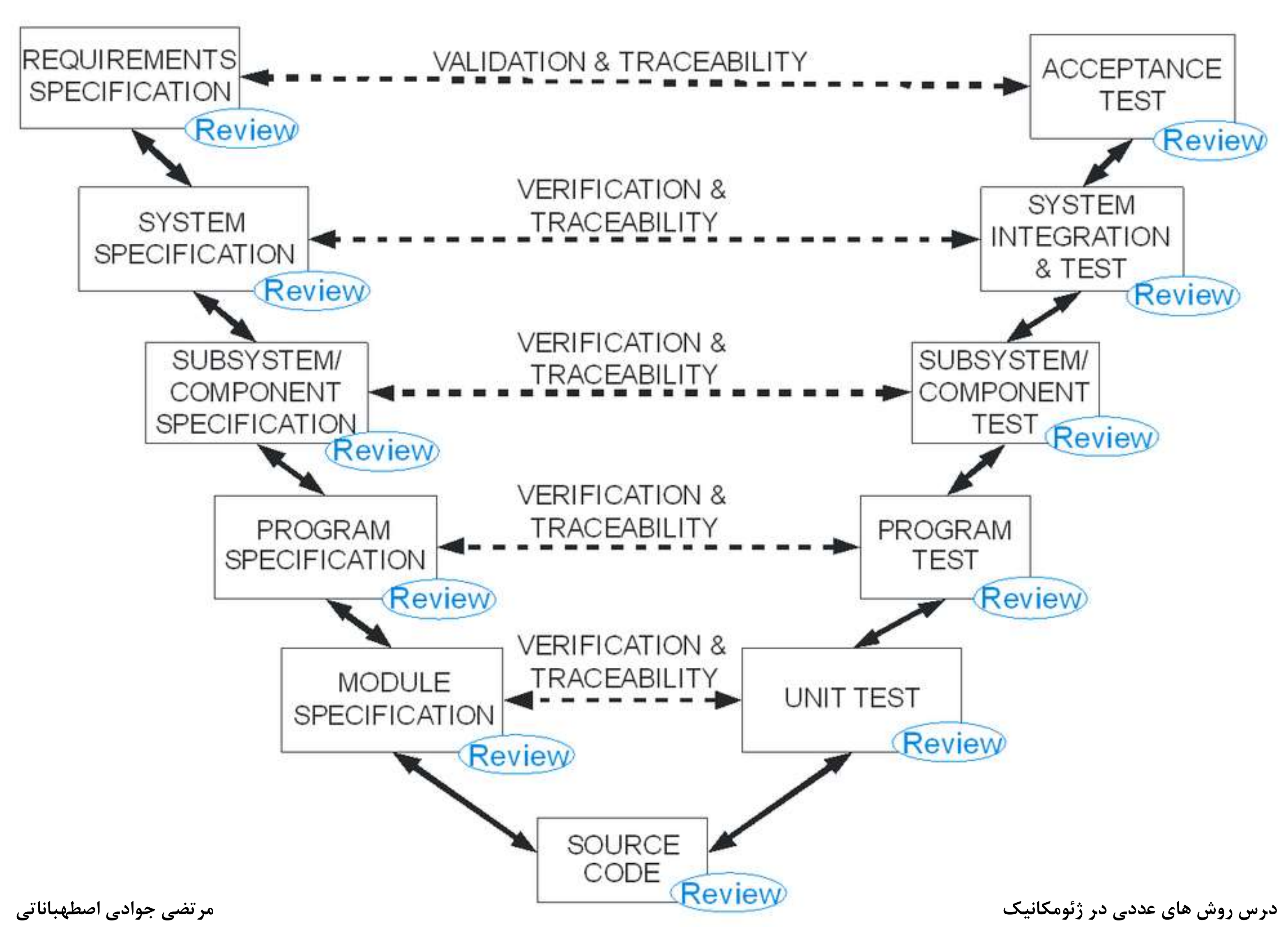
Verification: Are we building the system right?
Validation: Are we building the right system?



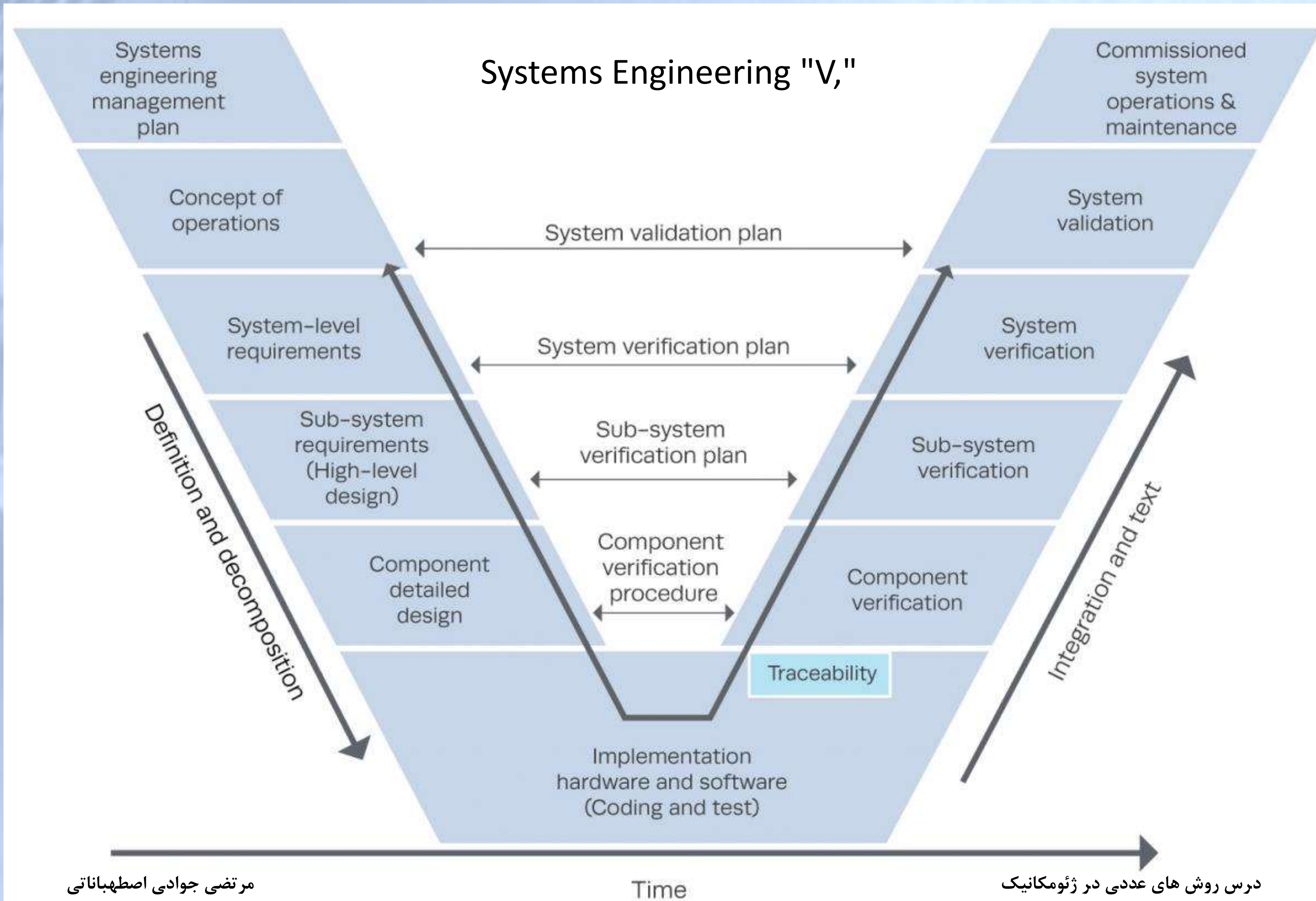
صحت سنجی و اعتبار سنجی مدل سازی عددی

Verification	Validation
Are you building it right?	Are you building the right thing?
Ensure that the software system meets all the functionality.	Ensure that functionalities meet the intended behavior.
Verification takes place first and includes the checking for documentation, code etc.	Validation occurs after verification and mainly involves the checking of the overall product.
Done by developers.	Done by Testers.
Have static activities as it includes the reviews, walkthroughs, and inspections to verify that software is correct or not.	Have dynamic activities as it includes executing the software against the requirements.
It is an objective process and no subjective decision should be needed to verify the Software.	It is a subjective process and involves subjective decisions on how well the Software works.

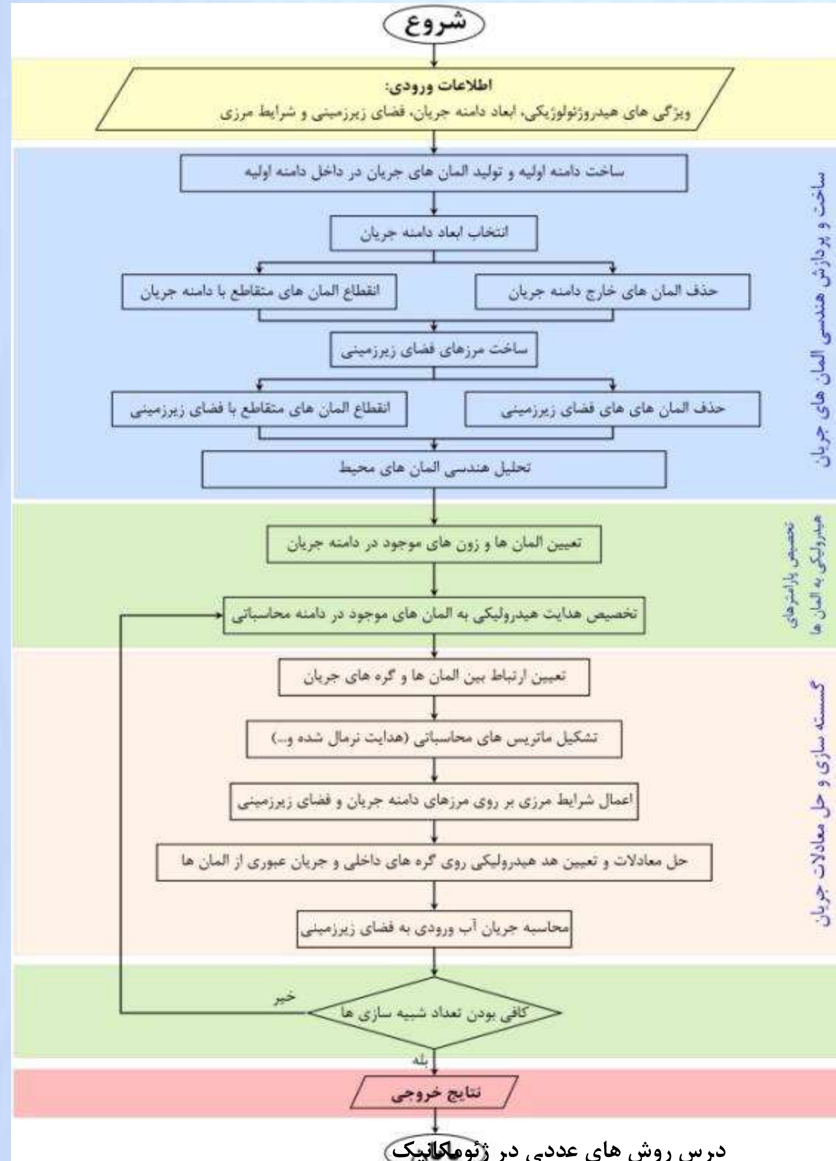
صحت سنجی و اعتبار سنجی مدل سازی عددی

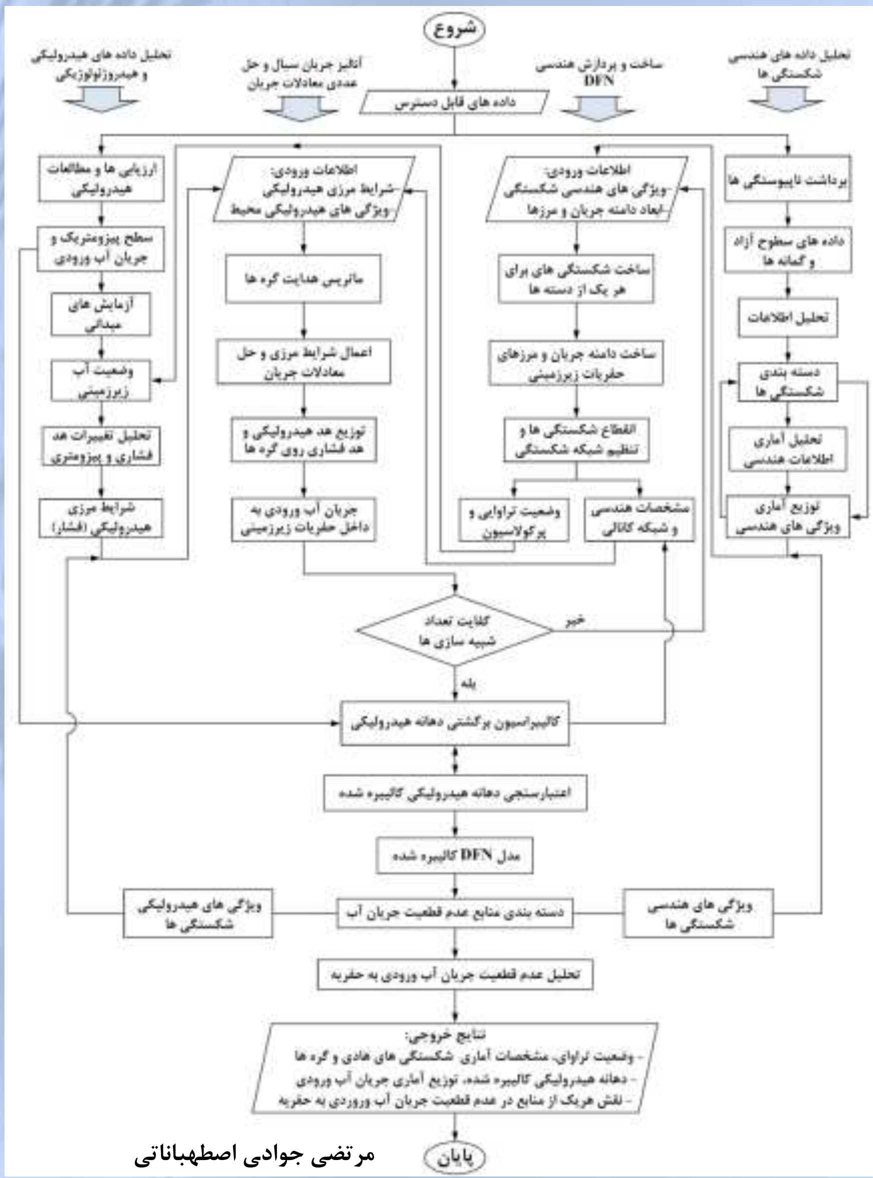


صحت سنجی و اعتبار سنجی مدل سازی عددی

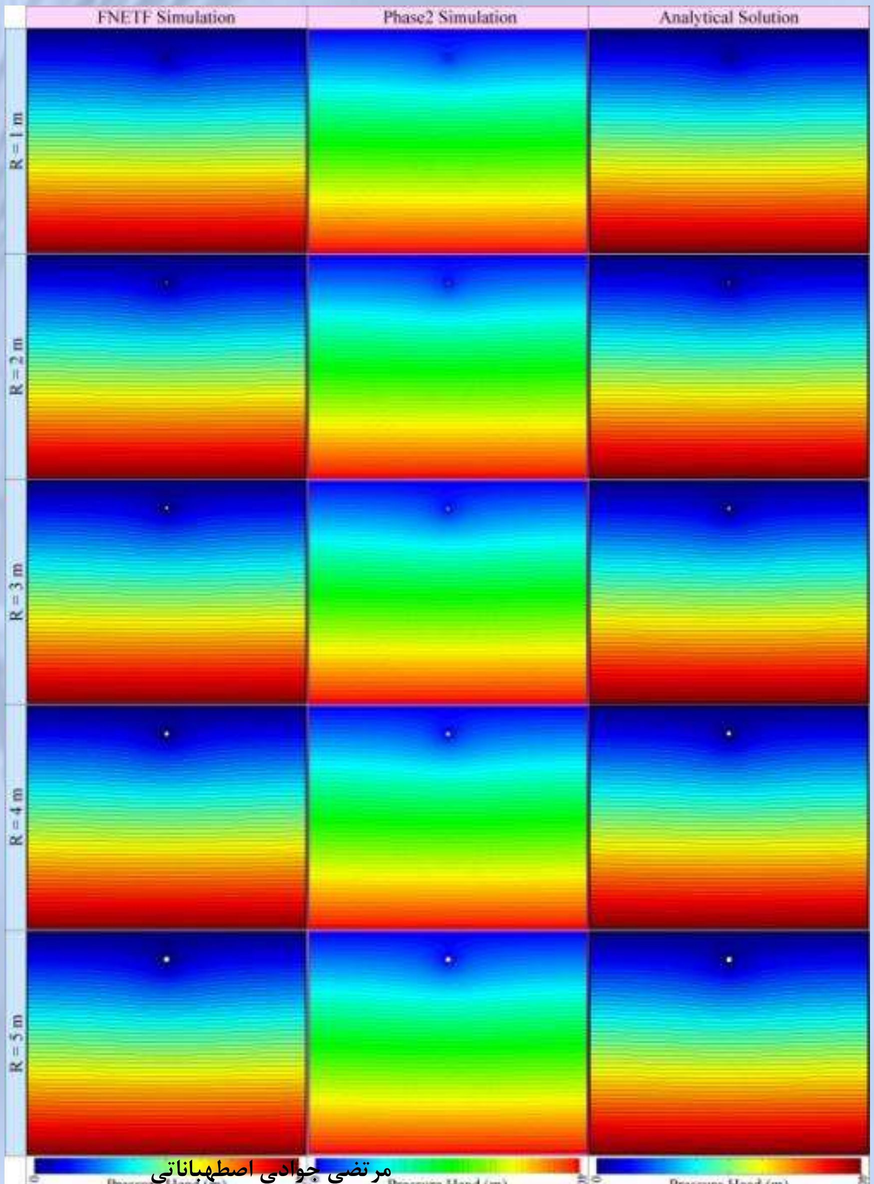


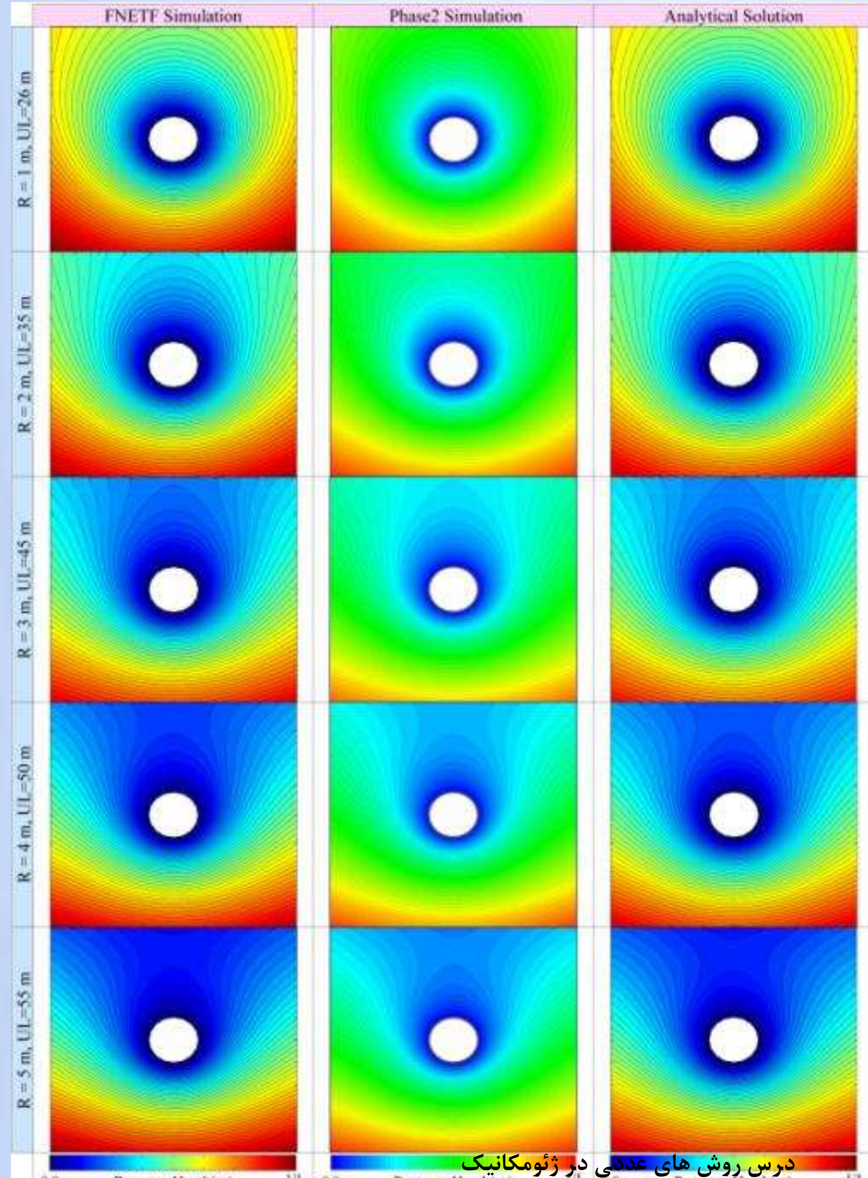
صحت سنجی و اعتبار سنجی مدل سازی عددی



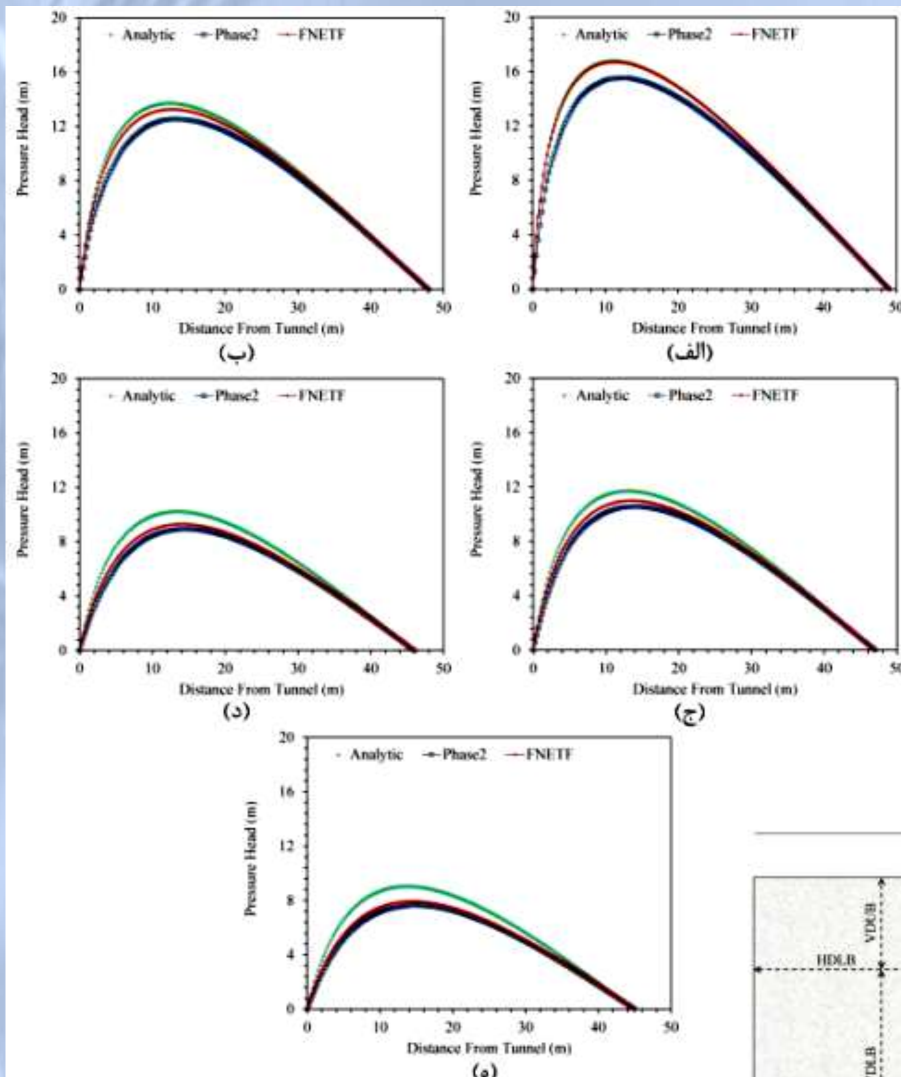


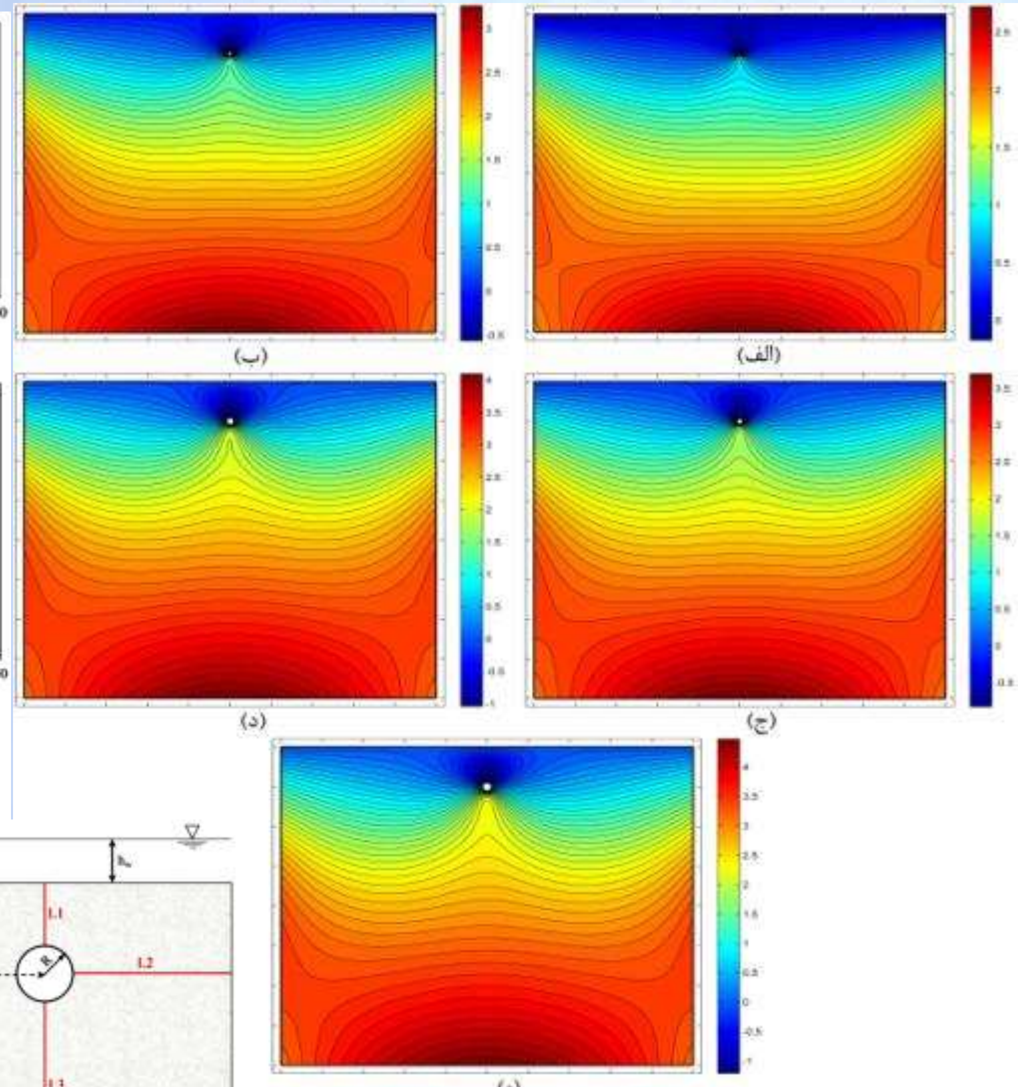
مثال: صحت سنجی مدل سازی عددی

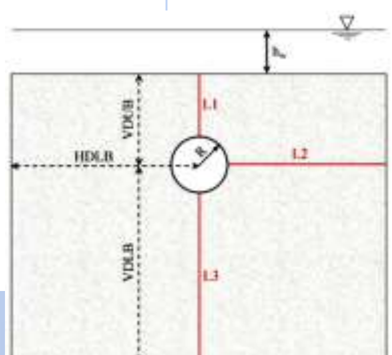




مثال: صحت سنجی مدل سازی عددی







تمرین

- از متون موجود در زمینه ژئومکانیک : یک کیس انتخاب شده که شامل حل عددی یا توسعه روش عددی، صحت سنجی و اعتبارسنجی باشد. مراحل این فرآیند بصورت یک گزارش ارائه شود.

دانشگاه صنعتی شاهرود
دانشکده معدن، نفت و ژئوفیزیک

درس روش های عددی در ژئومکانیک

مدرس
مرتضی جوادی اصطهباناتی

پاییز و زمستان ۱۳۹۶

رئوس مطالب

➢ طراحی در مهندسی سنگ و ژئومکانیک

- ❖ تفاوت مهندسی سنگ با سایر علوم
- ❖ ابزارها و روش های مختلف طراحی
- ❖ رفتارهای مختلف زمین و فضای زیرزمینی

➢ روش های عددی

- ❖ دسته بندی بر مفاهیم مختلف
- ❖ انتخاب نوع روش های عددی
- ❖ مفهوم REV، عدم قطعیت، غیر یکنواختی و ناهمگنی

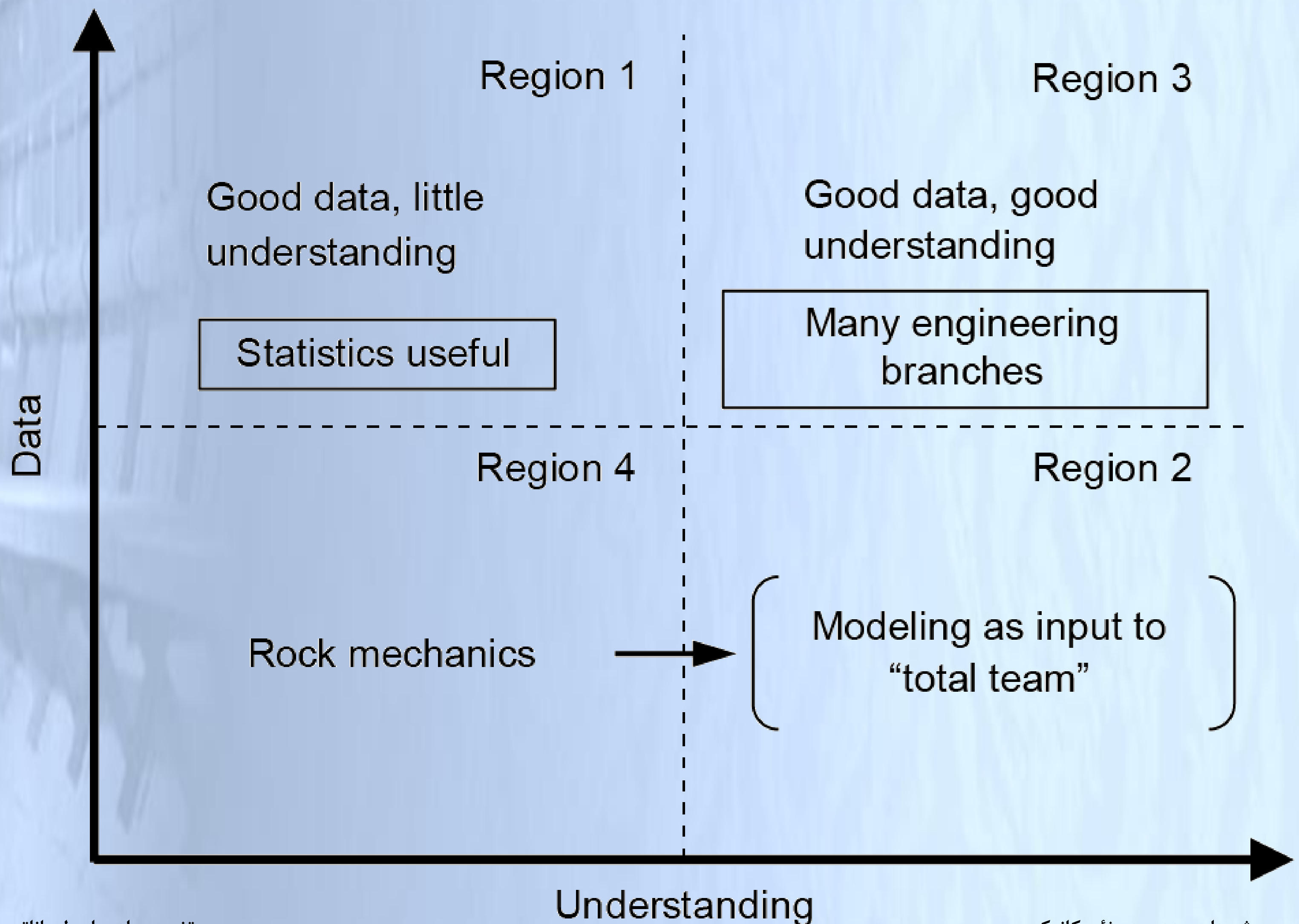
➢ فرآیند عمومی در مدل سازی عدد

طراحی در مهندسی سنگ و ژئومکانیک

- هدف از توسعه علم مهندسی سنگ و ژئومکانیک
  - طراحی و بهره برداری از انواع سازه ها
  - استفاده کاربردی از رفتار زمین
- رفتار زمین (سنگ، خاک، محیط های مخلوط)
  - گستردگی رفتارهای زمین (تنوع رفتاری)
  - پیچیدگی و ارتباطات متقابل بین پدیده ها
- روش های متعدد طراحی و ابزارهای مختلف

طراحی در مهندسی سنگ و ژئومکانیک

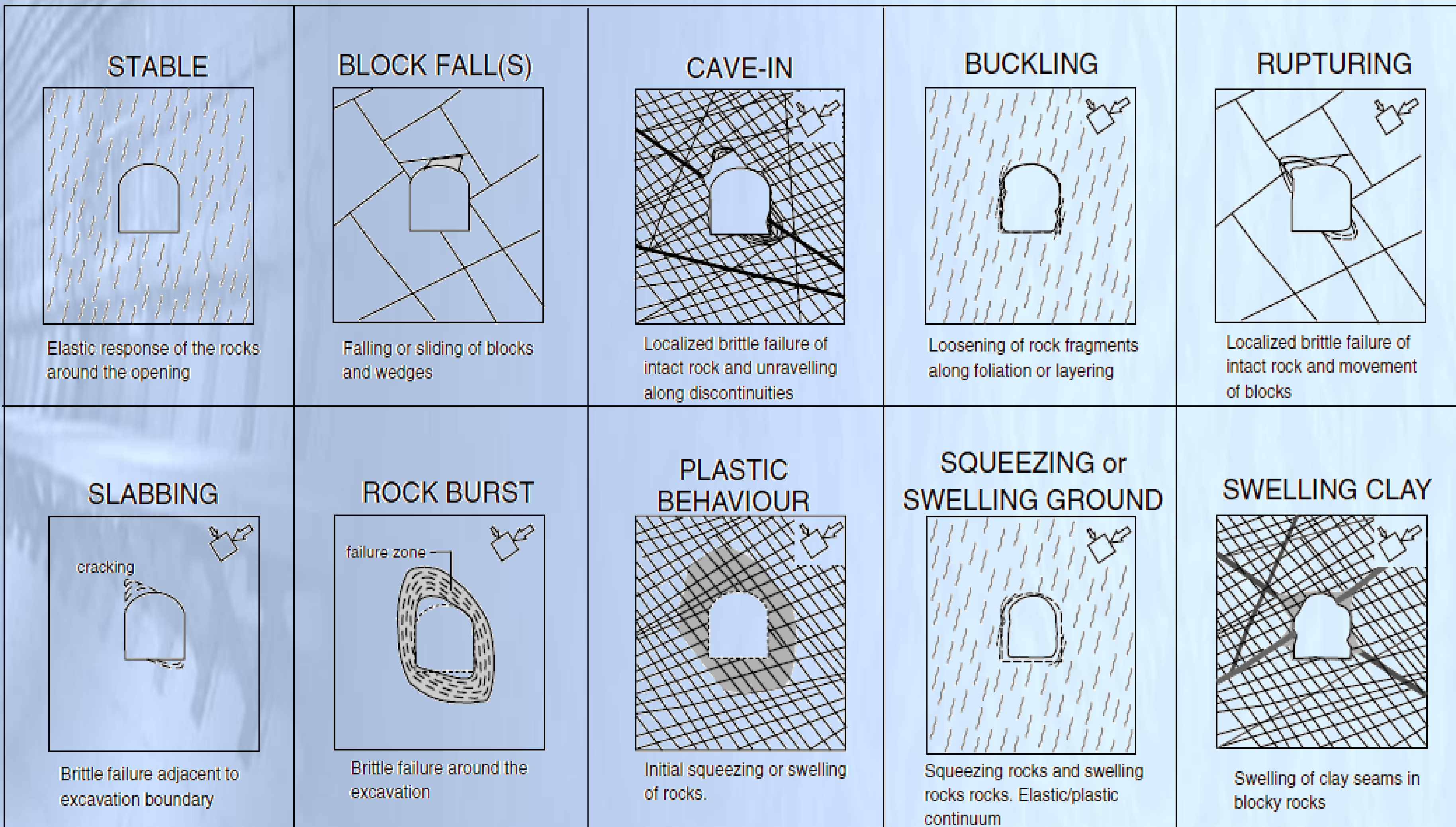
- تفاوت مهندسی سنگ با سایر علوم



طراحی در مهندسی سنگ و ژئومکانیک

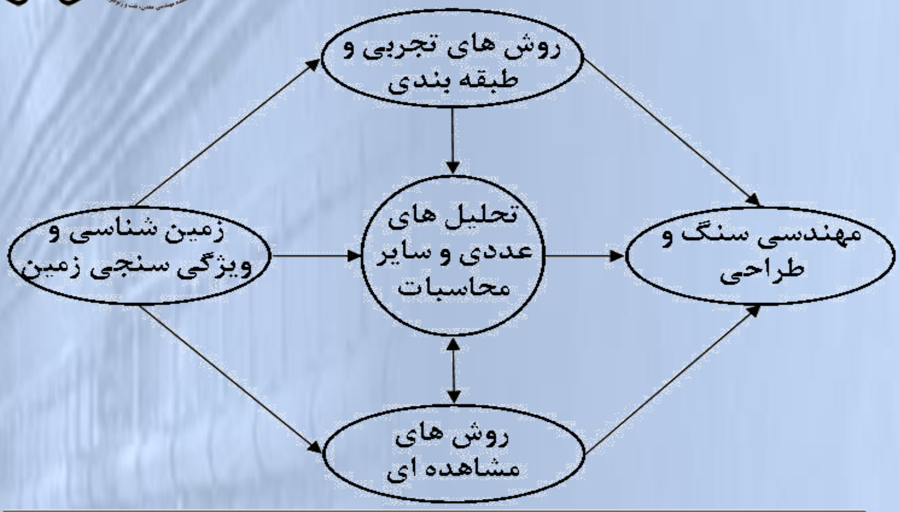
ROCK MASS BEHAVIOR

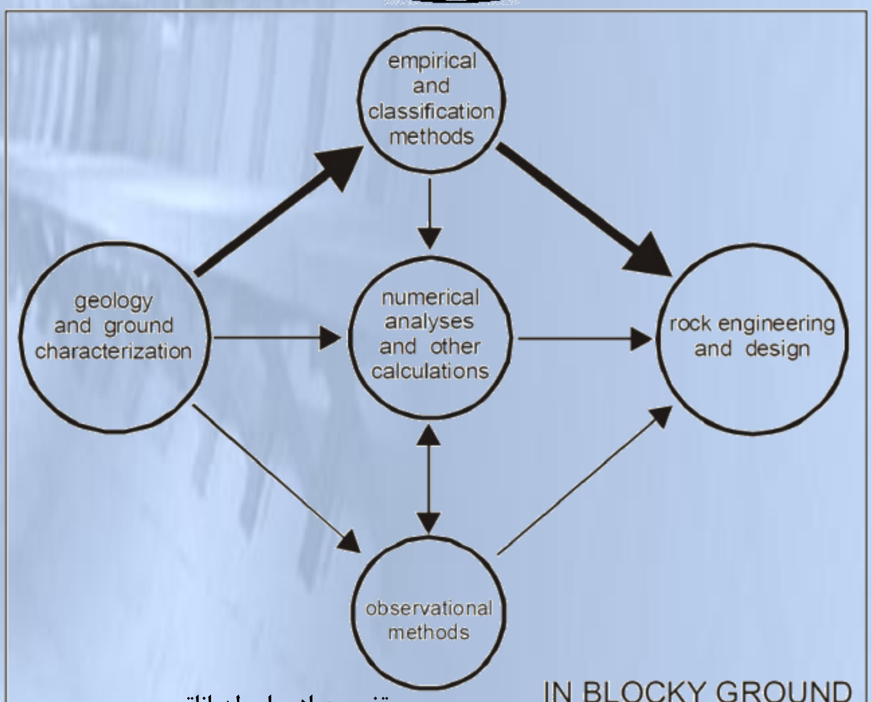
➢ رفتارهای مختلف زمین و فضای زیرزمینی

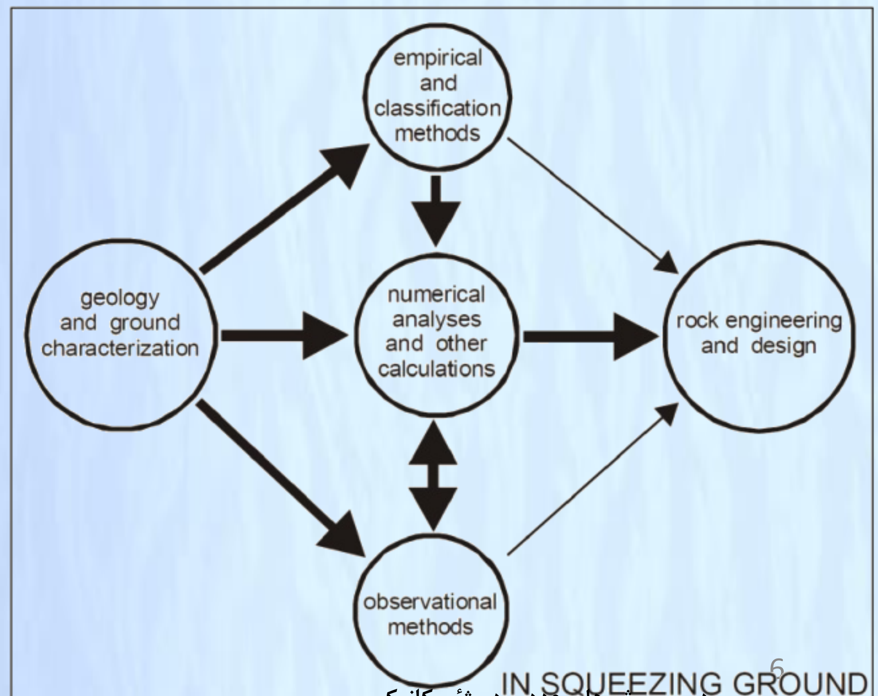


طراحی در مهندسی سنگ و ژئومکانیک

- ابزارها و روش های مختلف طراحی







طراحی در مهندسی سنگ و ژئومکانیک

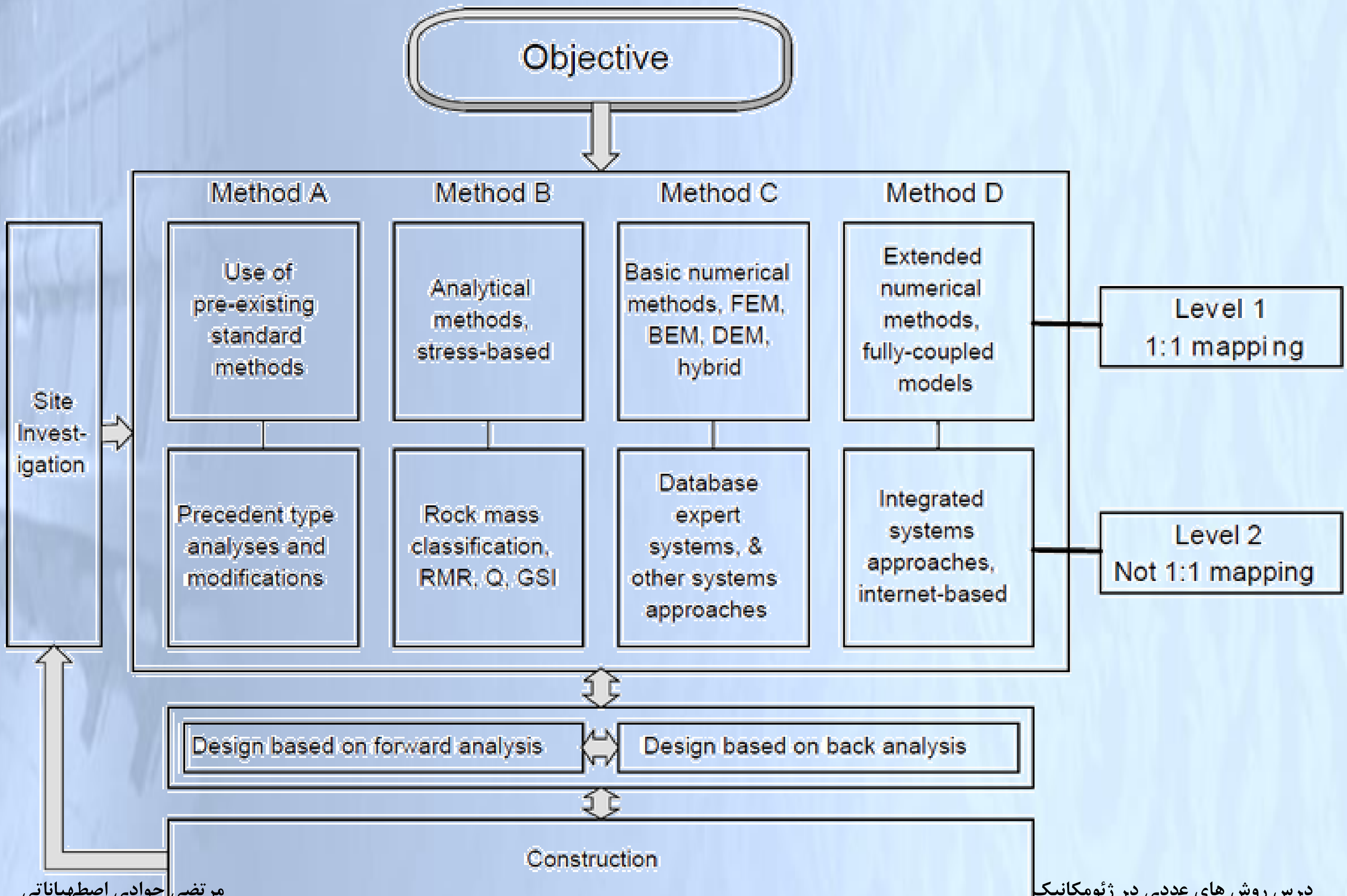
ابزارها و روش های مختلف طراحی

رفتار زمین	سیستم‌های طبقه‌بندی			مهندسی سنگ و ابزارهای طراحی					فاکتور موثر
	RMR	Q	RMi	NATM	عددی	تحلیلی	مشاهده‌ای	قضاوت مهندسی	
پایدار	۲	۲	۲–۱	۱	۱	۲	۱	۱	نیروی گرانش
سقوط بلوک	۲–۱	۲–۱	۲–۱	۲–۱	۲	۲	۲	۱	نیروی گرانش
به شدت ناپایدار	۳	۳–۲	۲	۳	۳	۲	۳	۲	نیروی گرانش
زمین‌های روان	۴	۴	۴	۴	۴	۴	۳	۲	نیروی گرانش
خردشونده	۴	۳	۳	۳	۲	۲	۲	۲	توزیع مجدد تنش
پوسته شوندگی (تدریجی)	۴	۳	۳	۲	۲	۳	۲	۲	توزیع مجدد تنش
پوسته شوندگی (ناگهانی)	۴	۲	۲	۳–۲	۲	۲	۲	۲	توزیع مجدد تنش
انفجار زمین	۴	۴–۳	۲	۳	۲	۲	۲–۱	۲	توزیع مجدد تنش
پلاستیک	۴	۴–۳	۳	۳–۲	۲	۲	۳	۲	توزیع مجدد تنش
لهیدگی	۴	۳	۳	۲–۱	۲	۲	۲	۳	توزیع مجدد تنش
آب شستگی	۴	۴	۴	۳	۴	۴	۲	۲	آب
آماسی	۴	۳	۳	۳	۳	۳	۲	۲	آب
غرقابی (گل و سنگ)	۴	۴	۴	۴–۳	۴	۴	۳	۳	آب
حجوم آب	۴	۴	۴	۴	۳	۲	۲	۳	آب

امتیاز مربوط به ابزارها، ۱: مناسب، ۲: متوسط، ۳: ضعیف، ۴: غیرقابل کاربرد

طراحی در مهندسی سنگ و ژئومکانیک

➢ ابزارها و روش های مختلف طراحی



روش های عددی

- دسته بندی بر اساس ماهیت رفتاری
  - روش های پیوسته
  - روش های ناپیوسته
  - روش های هیبرید
- دسته بندی بر اساس روش حل
  - FEM, FDM, BEM,EXFEM,
  - DEM, PBM,DDA,
- دسته بندی بر اساس هموژنیتی
  - Deterministic
  - Probabilistic
  - Gray Number

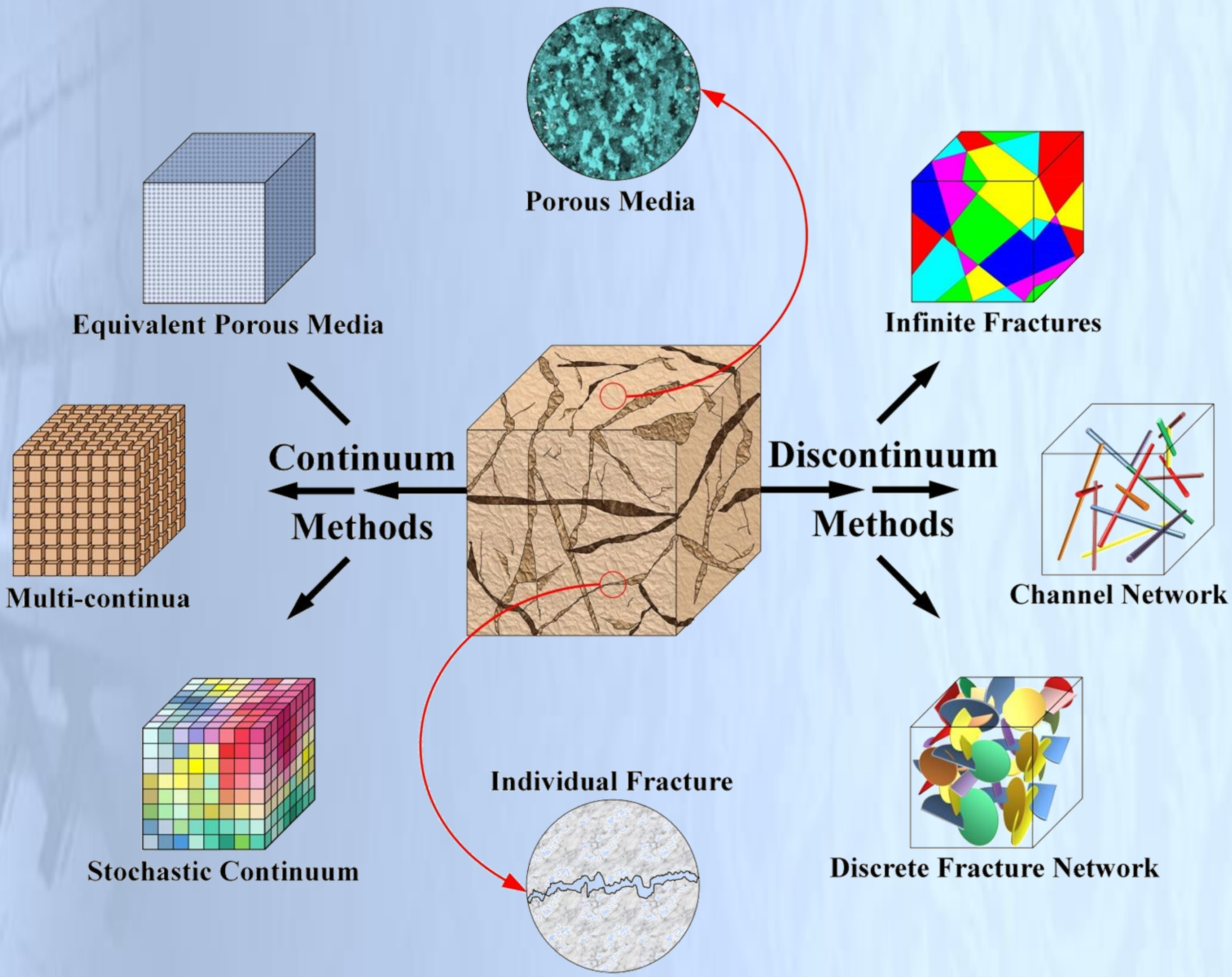
نوع ساختار

مقیاس بررسی ها

اهمیت مطالعه

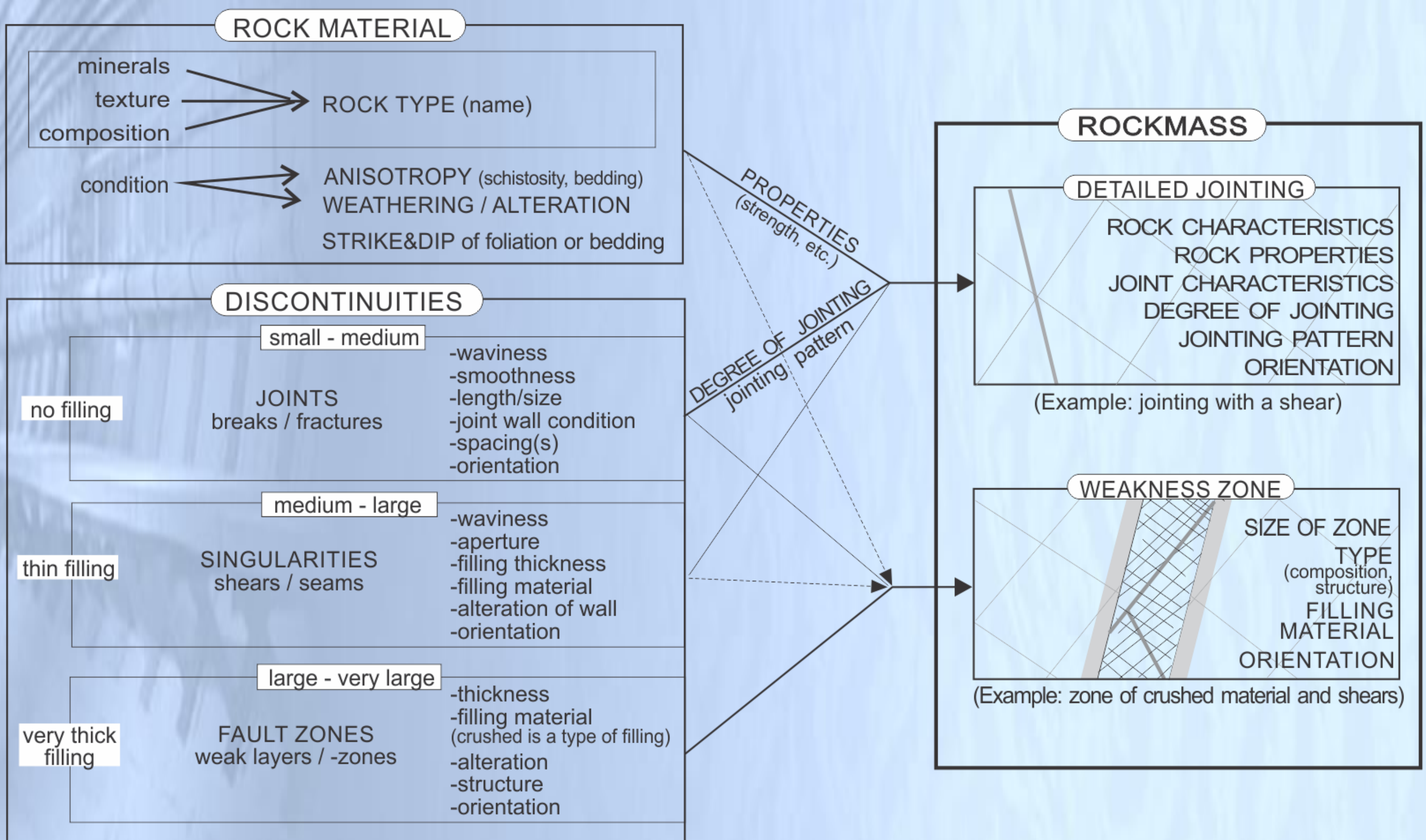
روش های عددی

- انتخاب نوع روش های عددی



روش های عددی

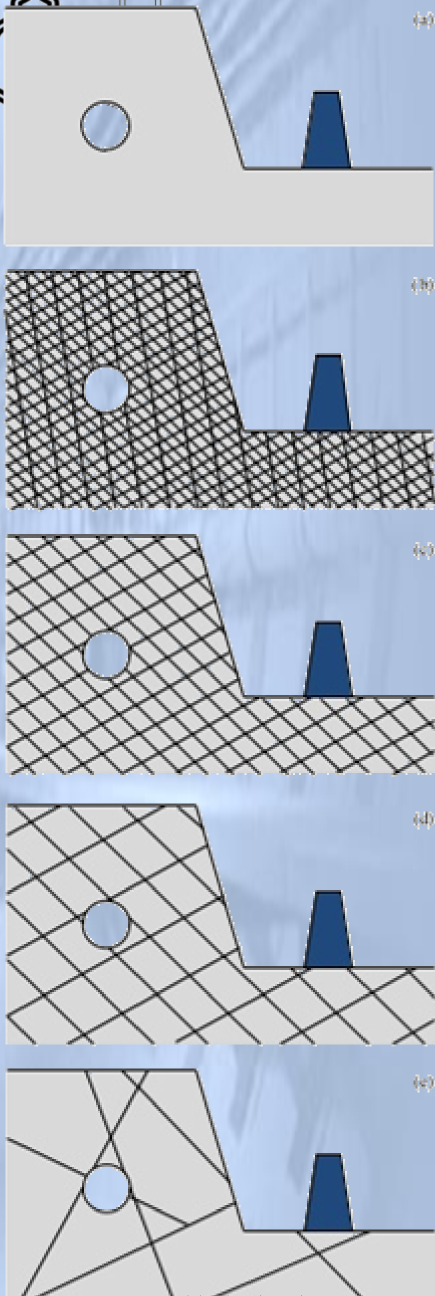
➢ انتخاب نوع روش های عددی و نقش ناپیوستگی ها



روش های عددی

➢ انتخاب نوع روش های عددی

ROCK MASS CONDITION



(a) No major joints:

Rock mass deformability is isotropic and is mainly governed by intact rock mechanical properties

Rock mass behaviour can be described by classical continuum models

(b) Small spacing joint sets:

Rock mass deformability is mainly isotropic. Compaction of rock fragments is the governing deformation mode

Rock mass behaviour can be described by equivalent parameters models

(c) Medium spacing joint sets:

Rock mass deformability is anisotropic. Normal closure/opening and sliding along joints are the governing deformation modes

Rock mass behaviour can be described by equivalent constitutive models

(d) Large spacing joint sets:

Rock mass deformability is anisotropic.

Intact rock deformation and rock block rotation/dislocation are the governing deformation modes.

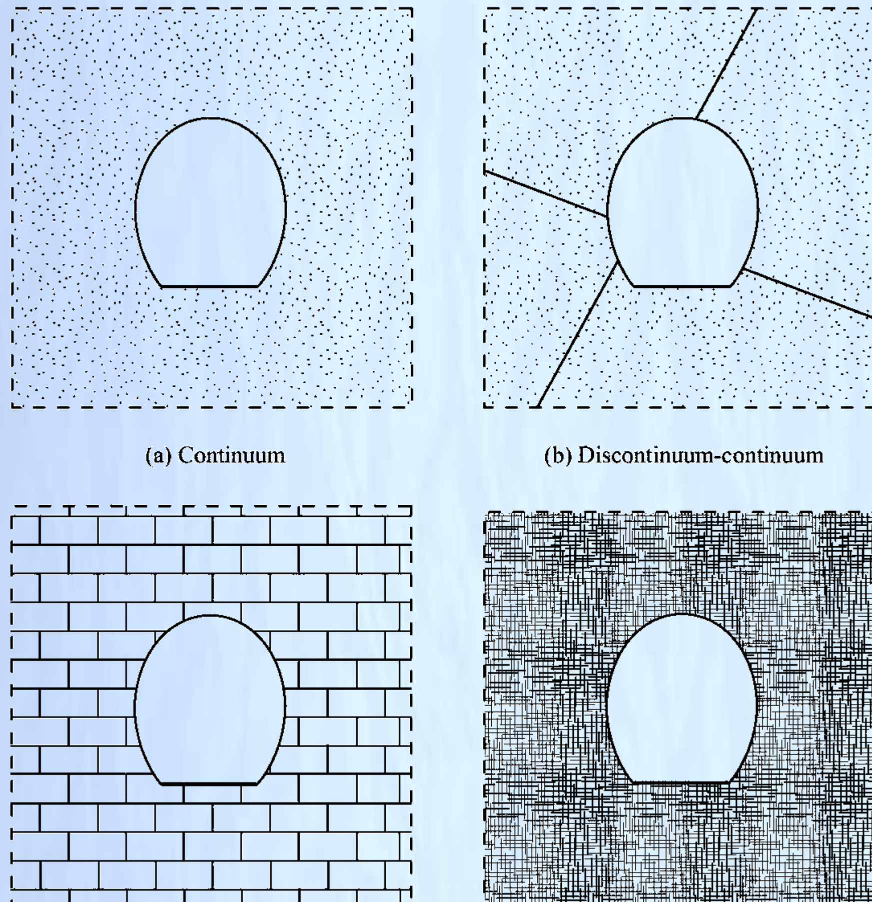
Rock mass behaviour can be described by discrete element models

(e) Random persistent discontinuities:

Rock mass behaviour depends on boundary conditions and geometry of discontinuities relative to the location of structures

Rock mass behaviour can be described by either joint/interface models or the discrete element method

Figure 2.9: Discontinuity patterns and applicability of analysis techniques



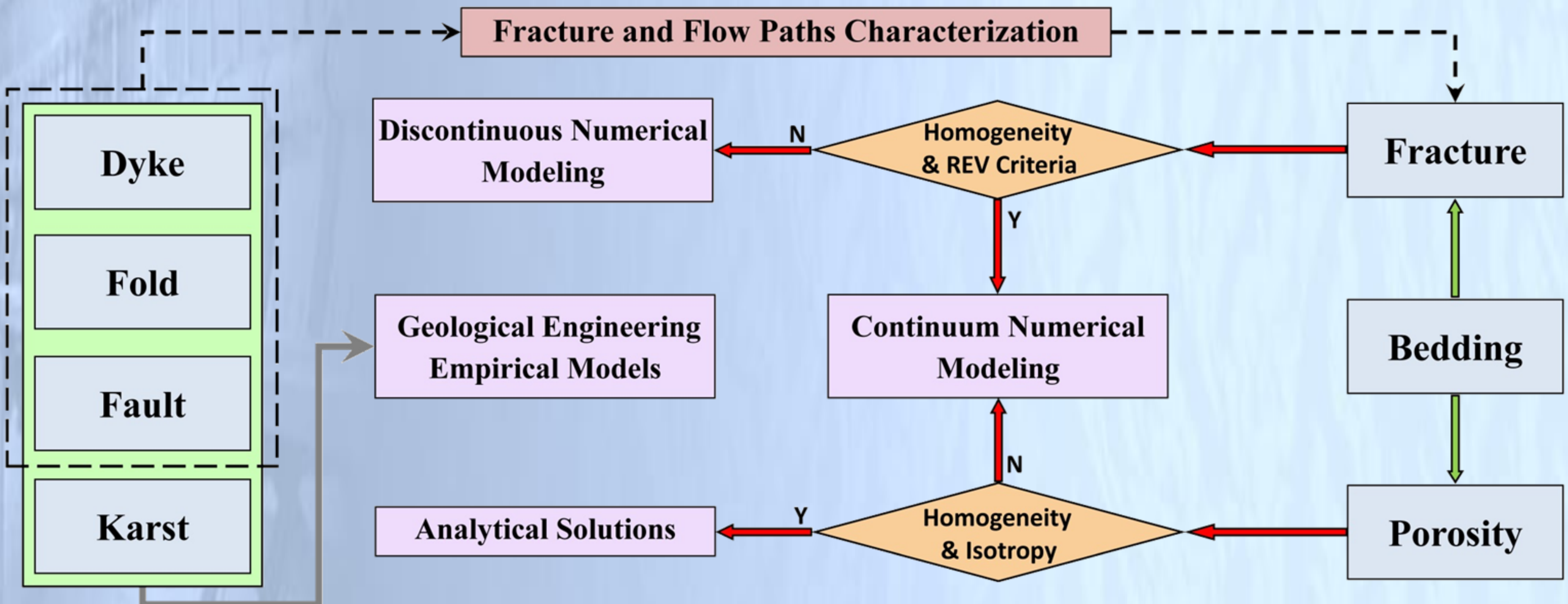
(a) Continuum 	(b) Discontinuum-continuum

(c) Discontinuum 	(d) Pseudo-continuum

روش های عددی

- انتخاب نوع روش های عددی

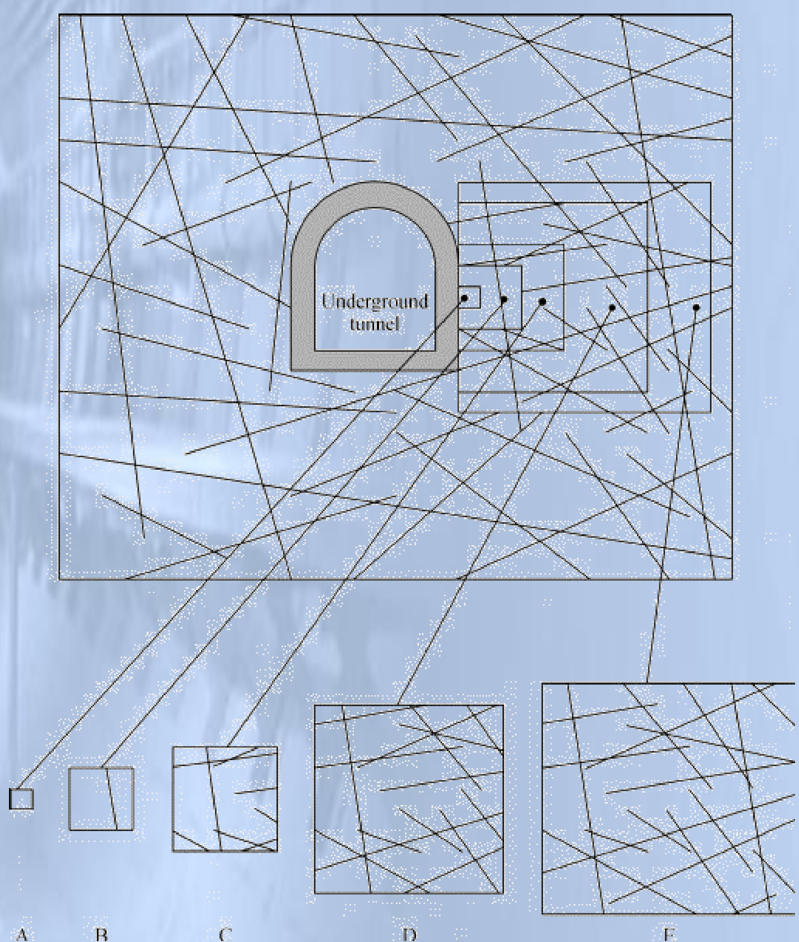
TYPE OF STRUCTURES

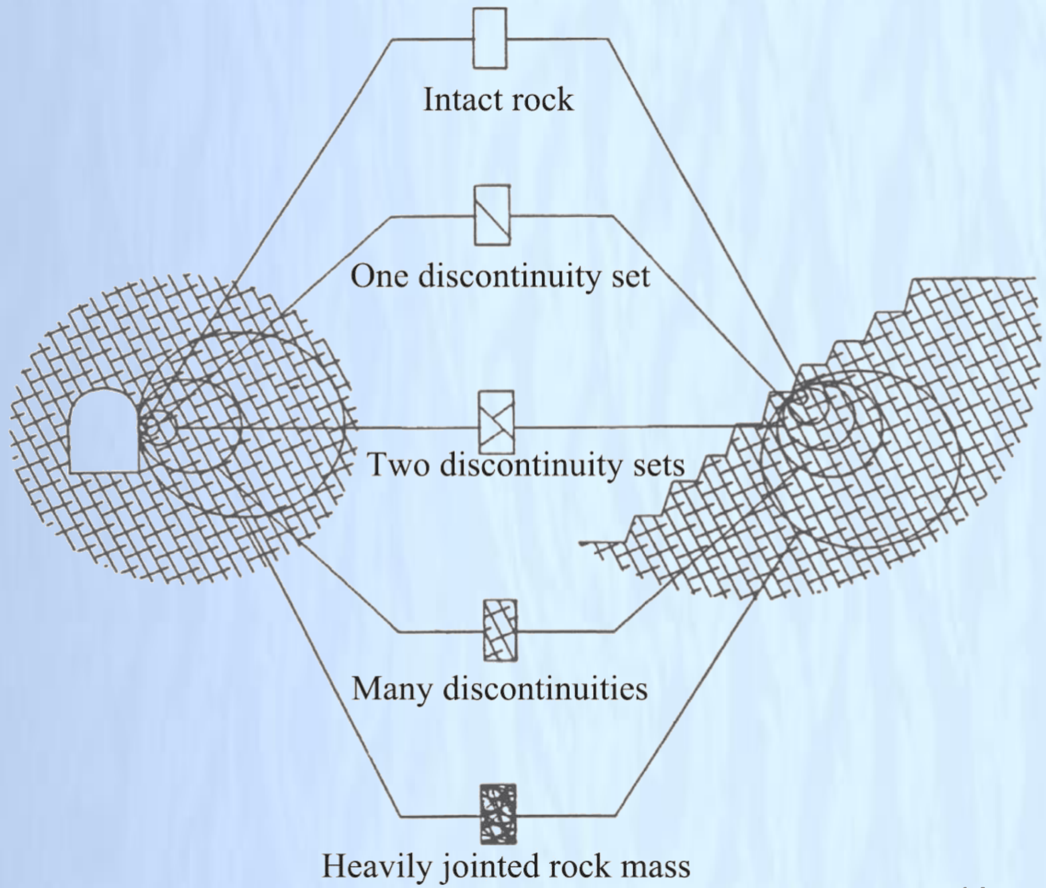


روش های عددی

➢ انتخاب نوع روش های عددی

SCALE OF STUDY

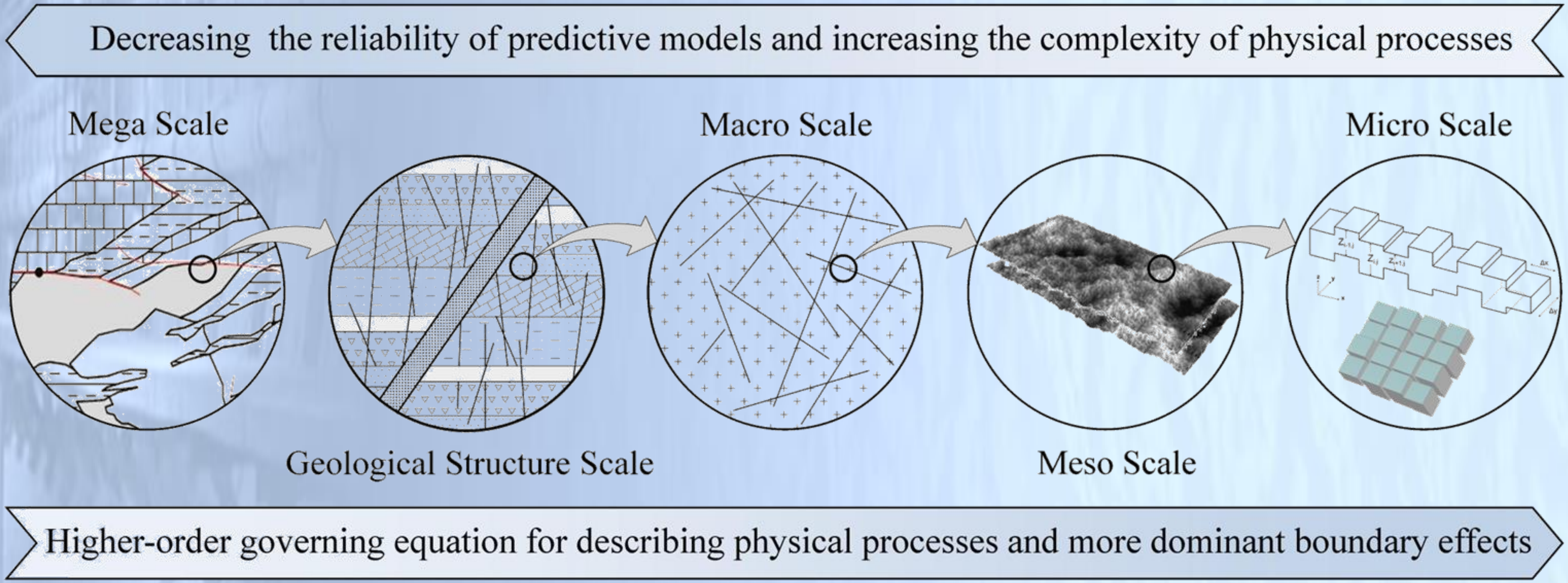




روش های عددی

➢ انتخاب نوع روش های عددی

SCALE OF STUDY

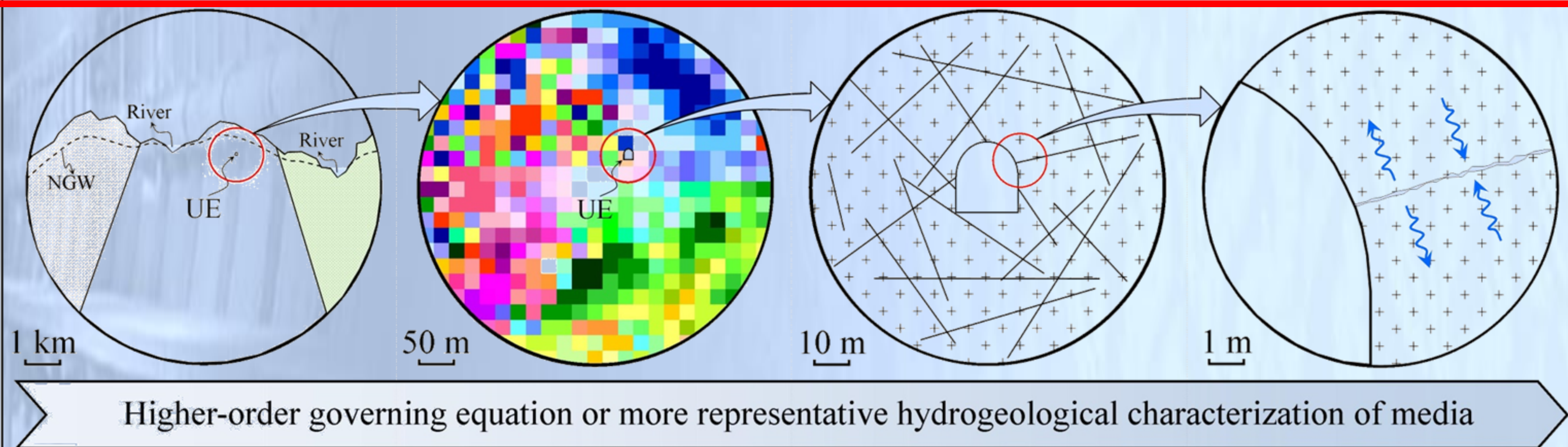


روش های عددی

➢ انتخاب نوع روش های عددی

SCALE AND IMPORTANCY OF STUDY

Very Far Field	Far Field	Near Field	Very Near Field

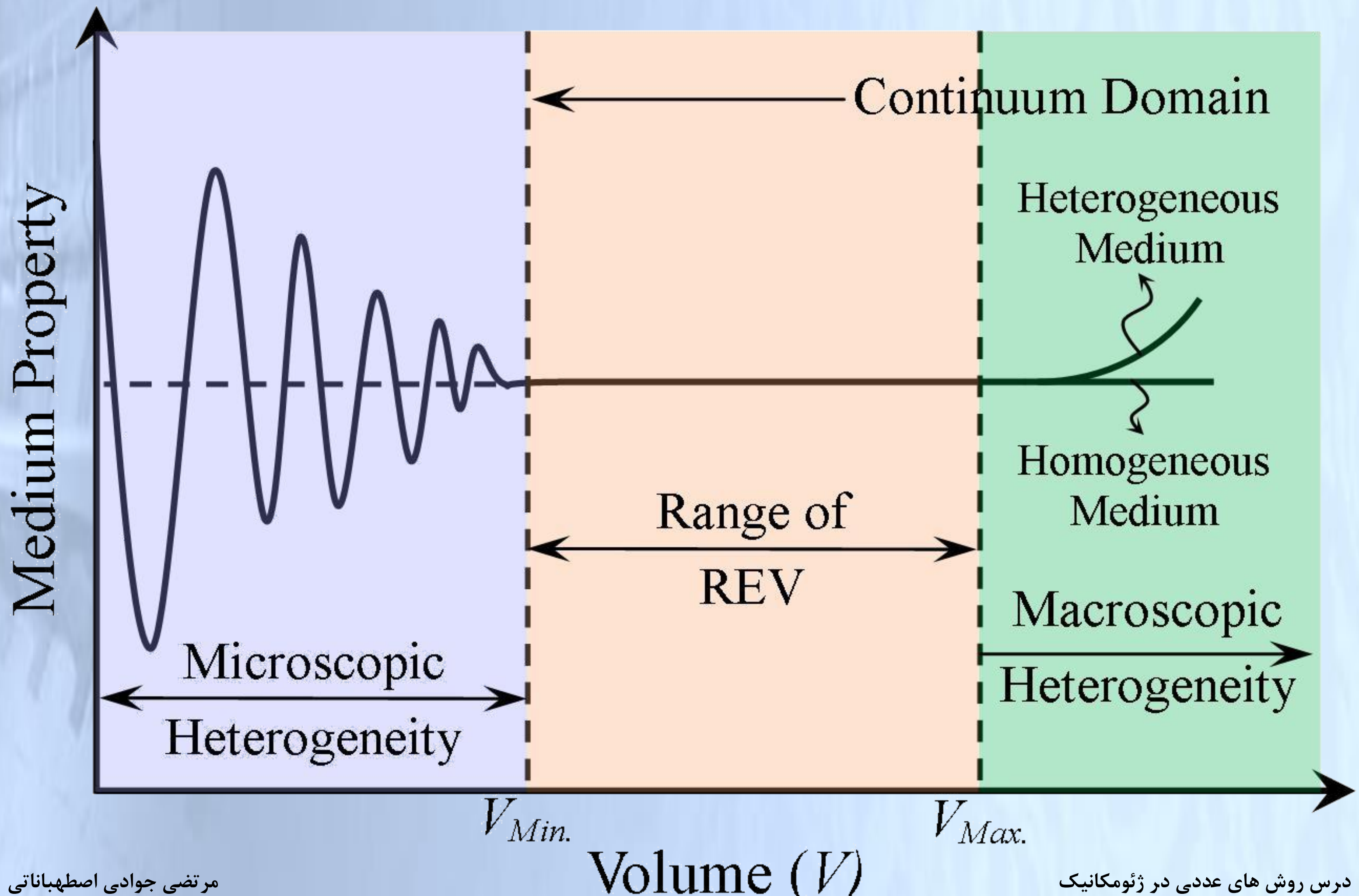


Method of Analysis:			
Continuum Analysis with Equivalent Porous media	Stochastic Continuum Flow Analysis	Discrete Fracture Flow (Discontinuum Models)	Experimental or High-order Numerical Models
Some Applications:			
Effect of tunneling on regional groundwater	Uncertainty of groundwater head and flow around tunnel	Hydromechanical interactions induced by tunneling	Precise fluid flow and contaminant transport

روش های عددی

➢ انتخاب نوع روش های عددی و مفهوم REV

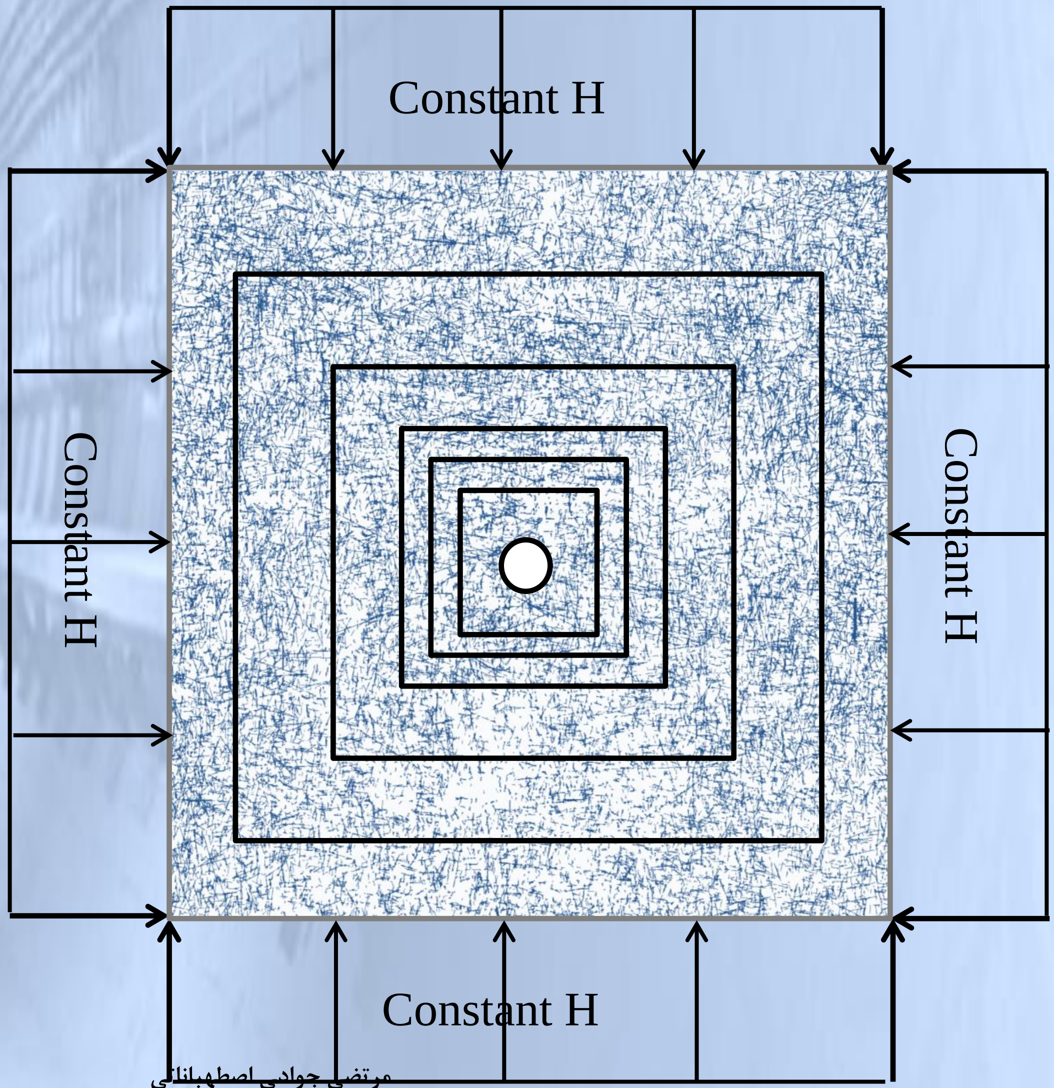
SCALE DEPENDENCY OF BEHAVIOR

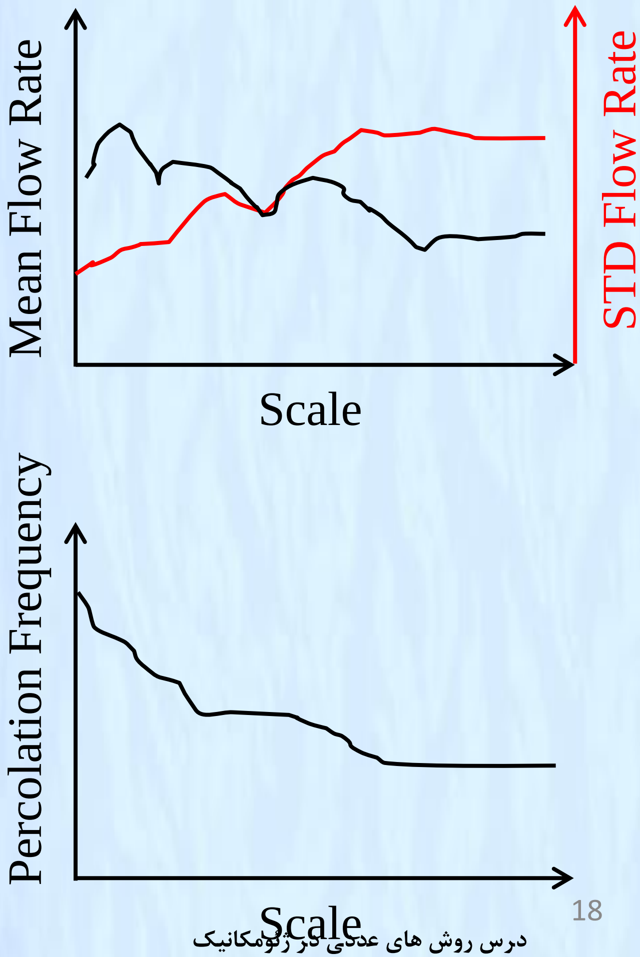


روش های عددی

➢ انتخاب نوع روش های عددی و مفهوم REV

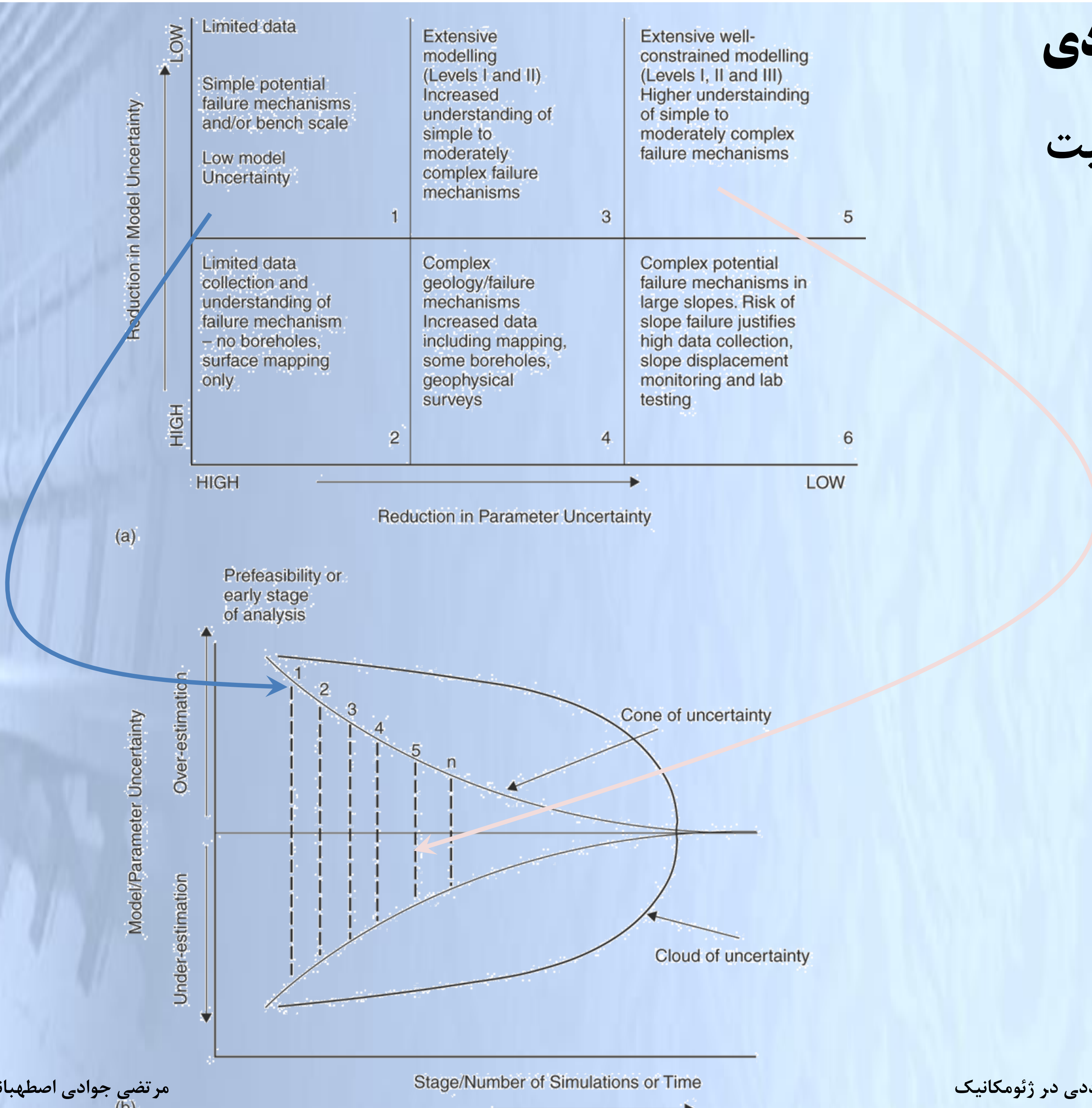
SCALE DEPENDENCY OF BEHAVIOR





روش های عددی

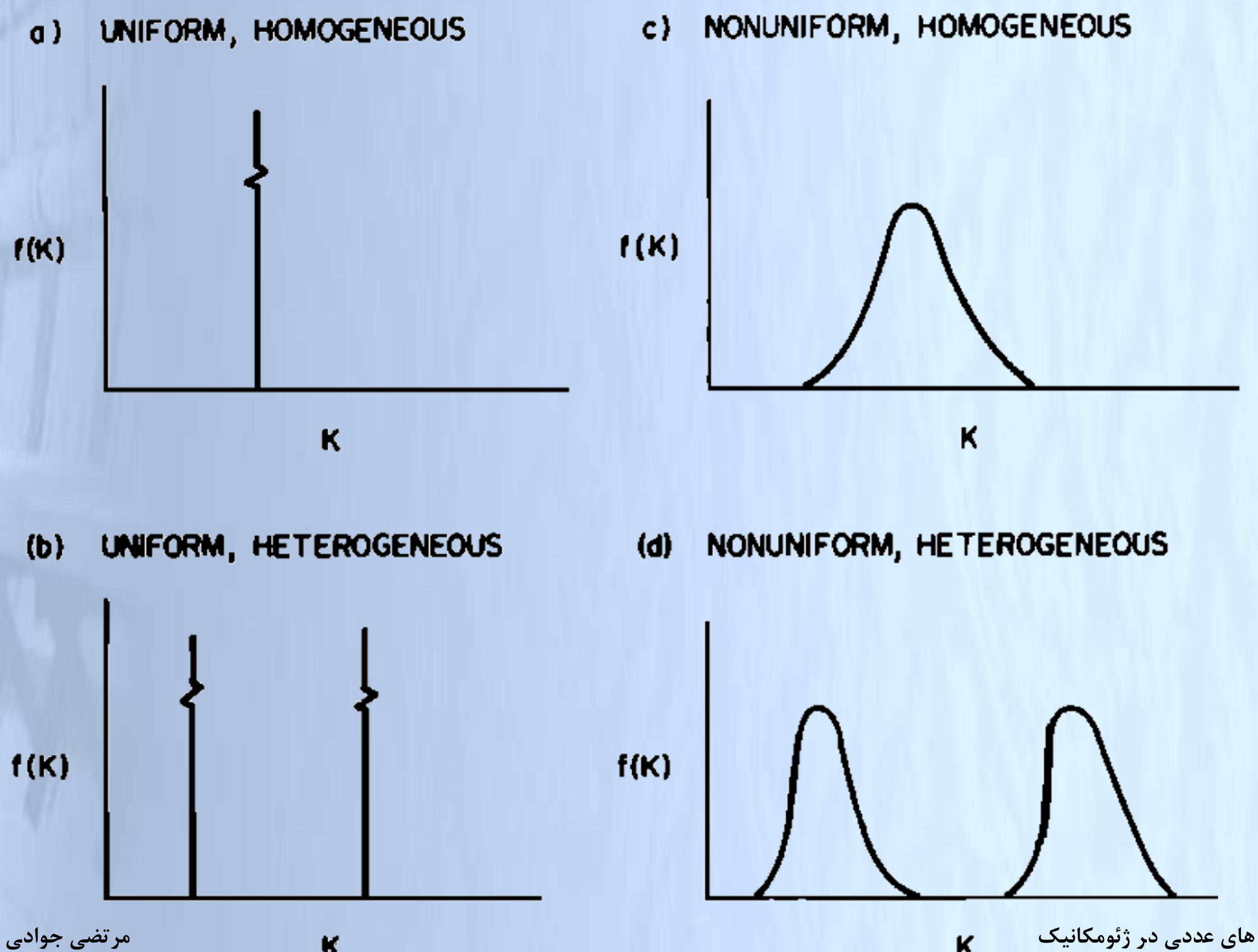
- عدم قطعیت



روش های عددی

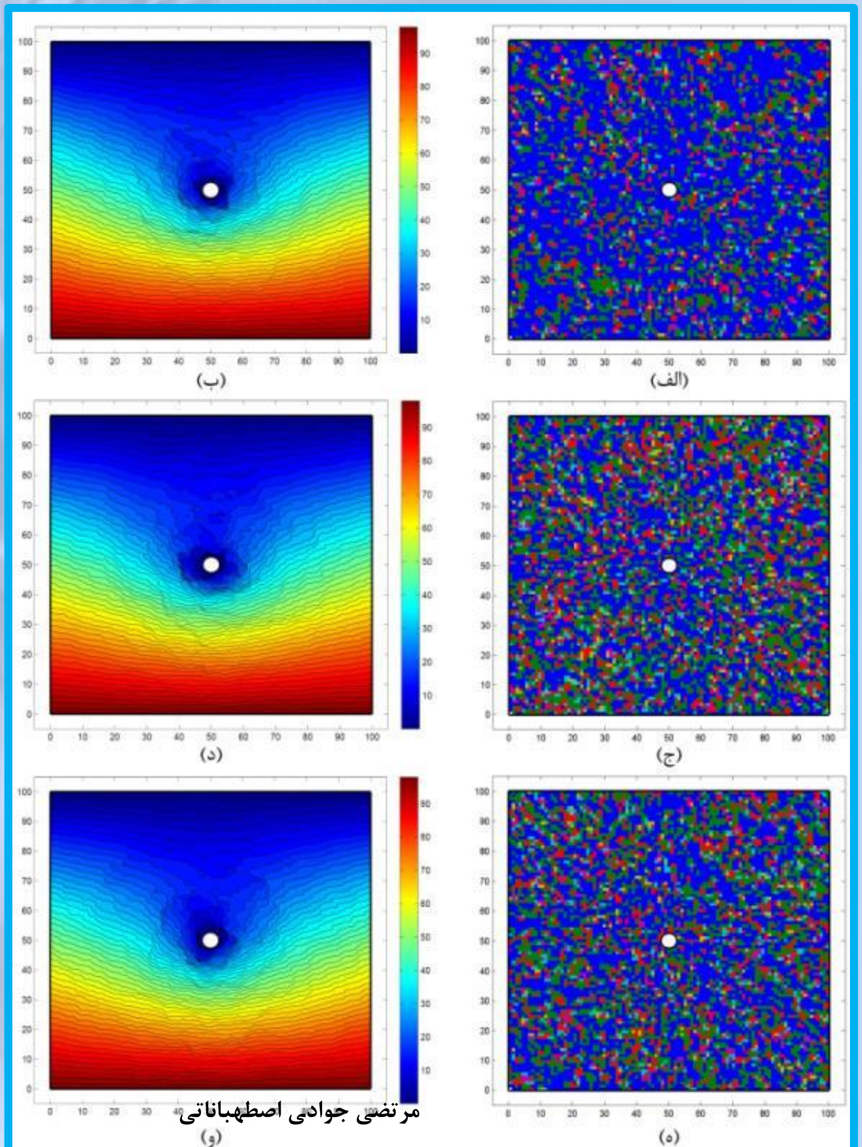
- انتخاب نوع روش های عددی و مفهوم یکنواختی، همگنی، ناهمگنی

VARIATION OF PARAMETERS AND CHARECTERISTICS



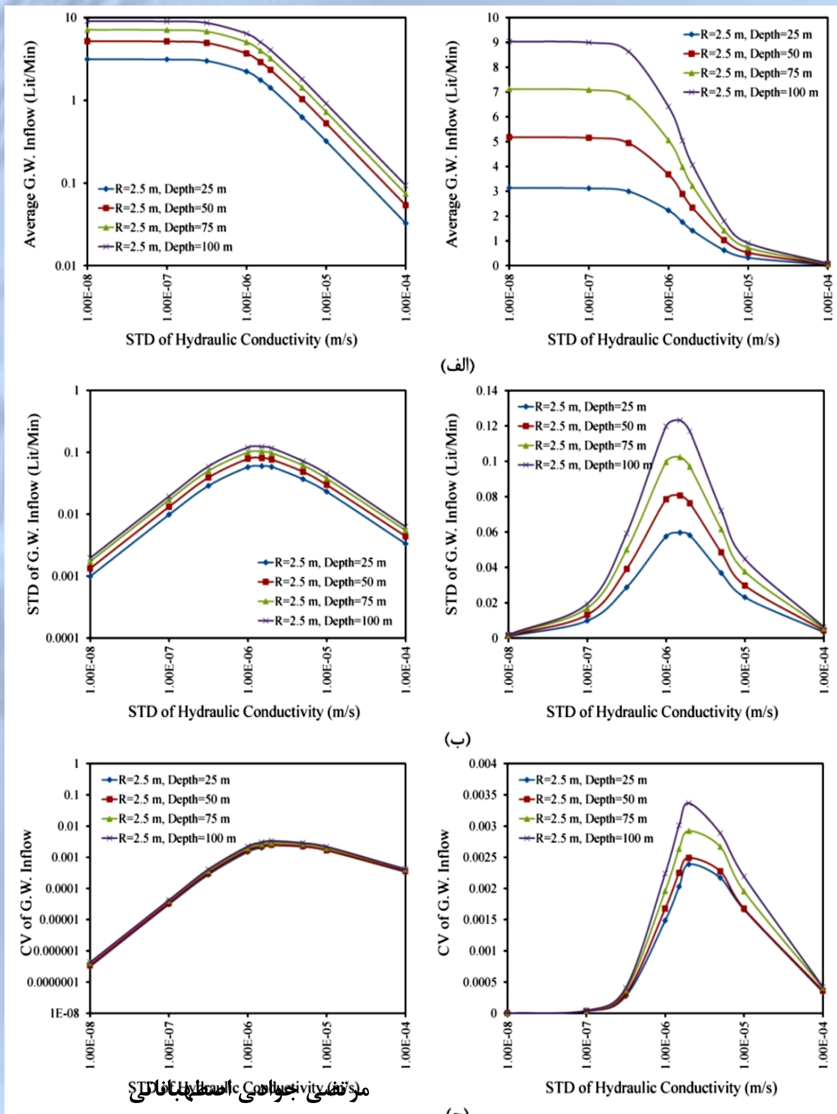
روش های عددی

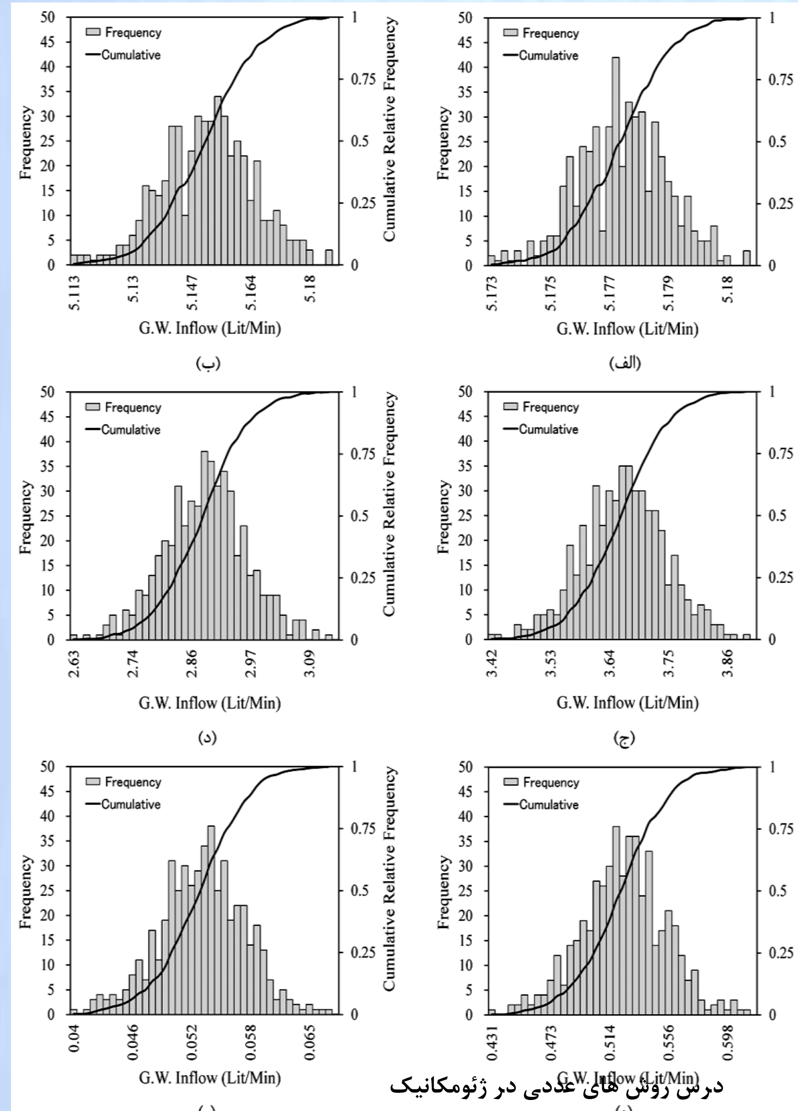
- انتخاب نوع روش های عددی و مفهوم یکنواختی، همگنی، ناهمگنی



روش های عددی

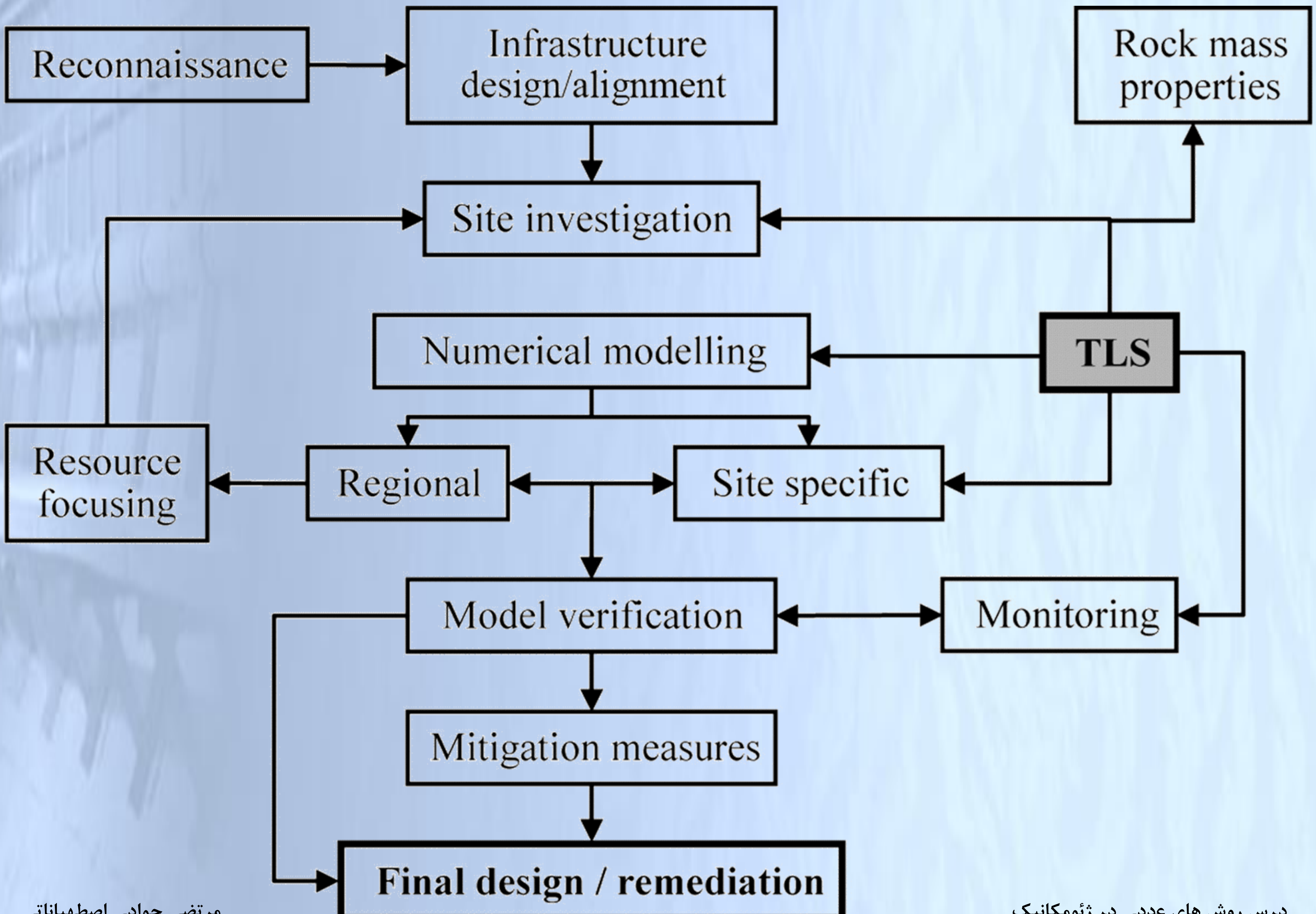
➢ خروجی روش های Probabilistic





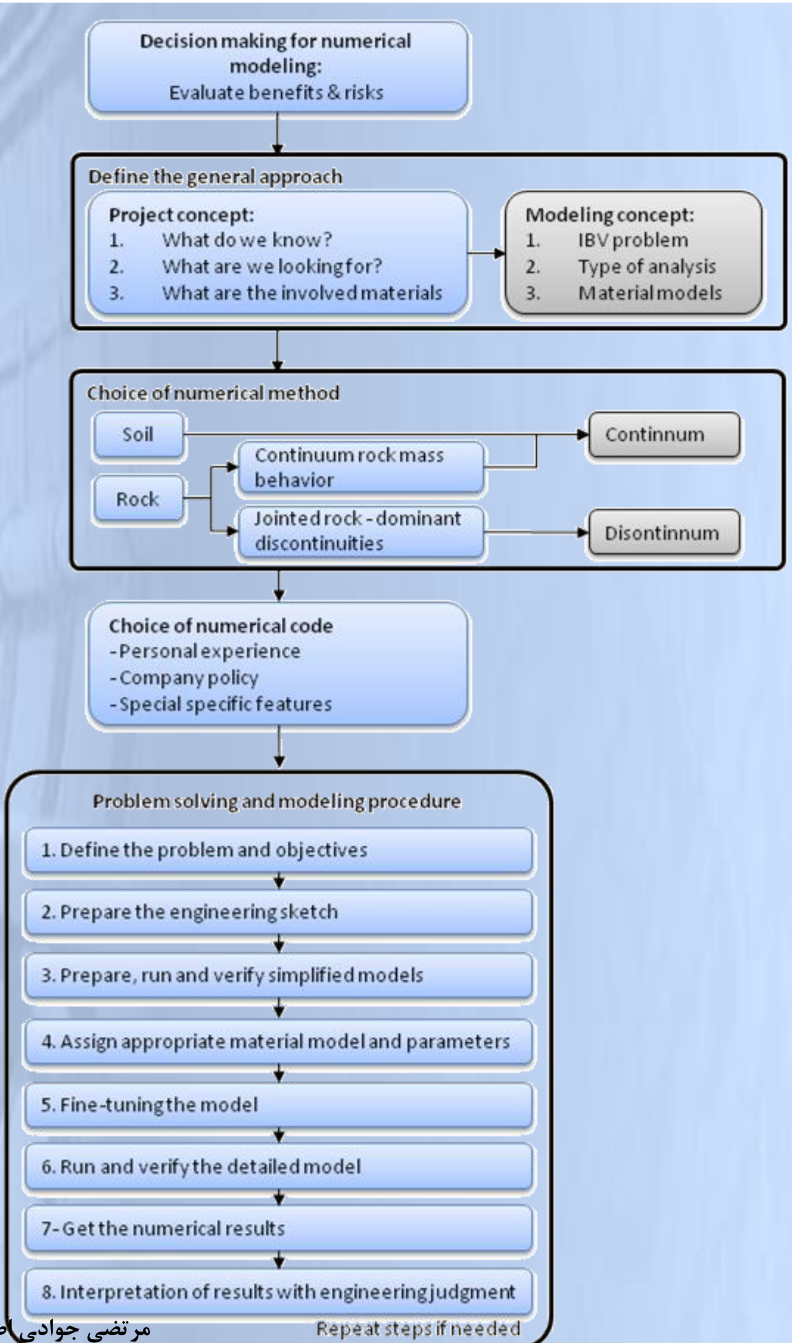
فرآیند عمومی در مدل سازی عددی

➢ جایگاه در فرآیند عمومی مهندسی



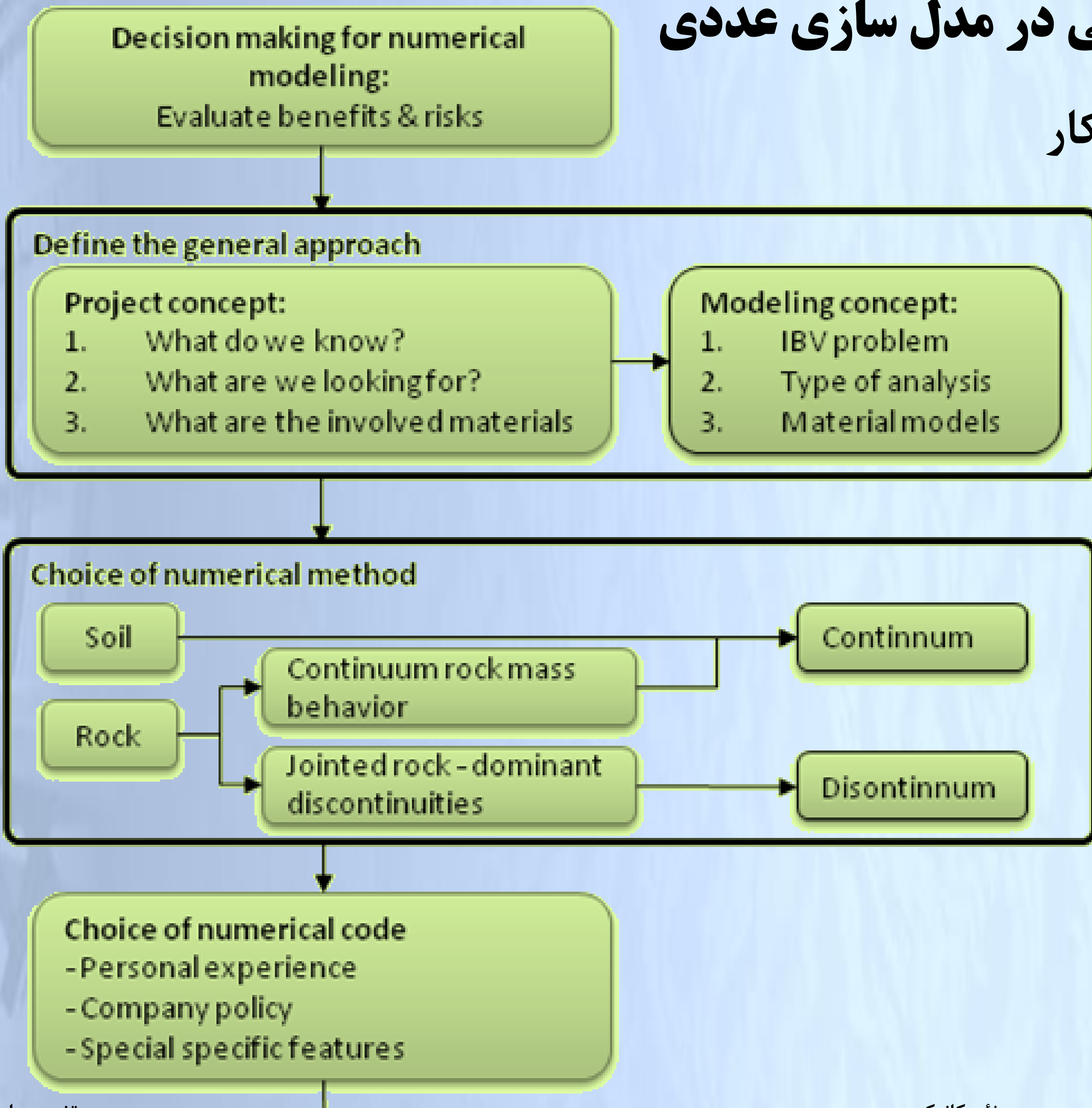
فرآیند عمومی در مدل سازی عددی

- مراحل کار



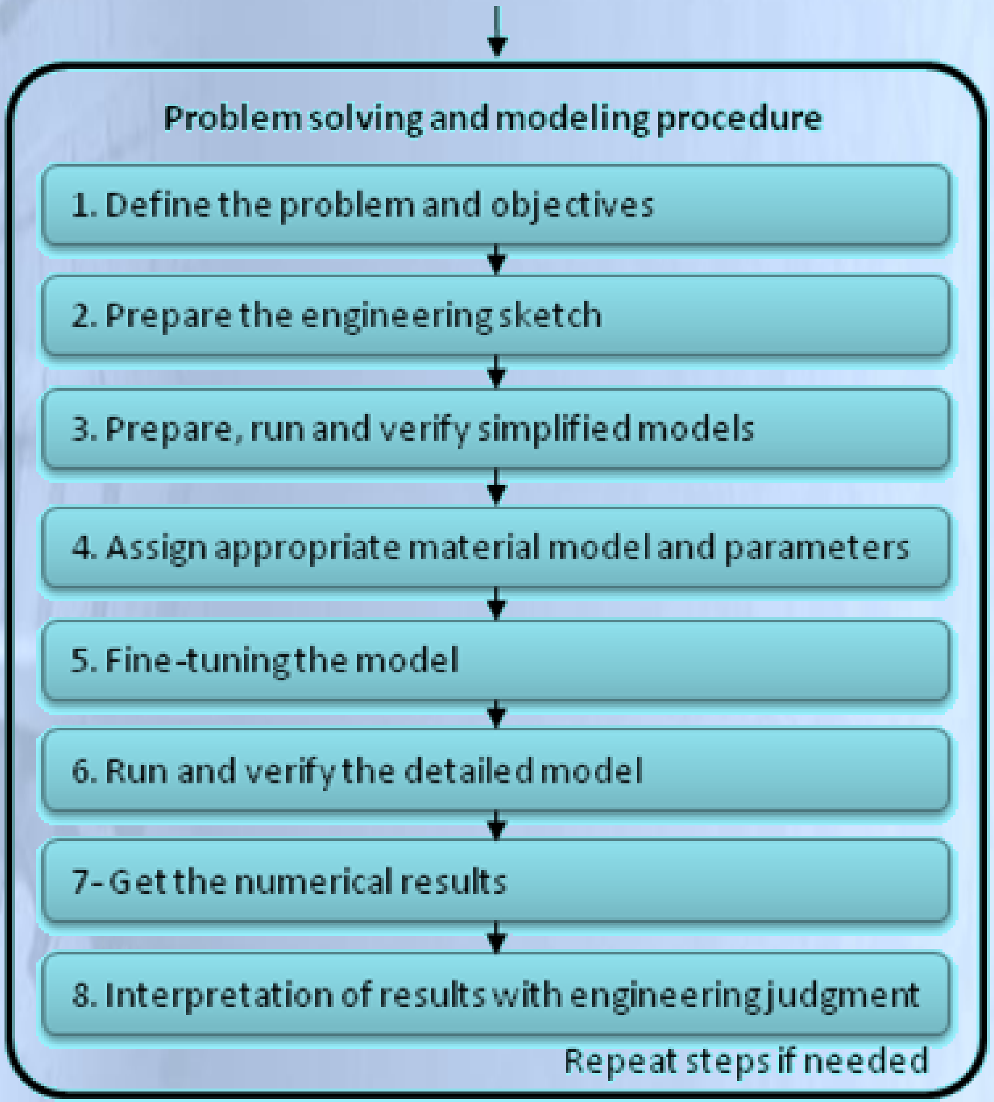
فرآیند عمومی در مدل سازی عددی

➢ مراحل کار



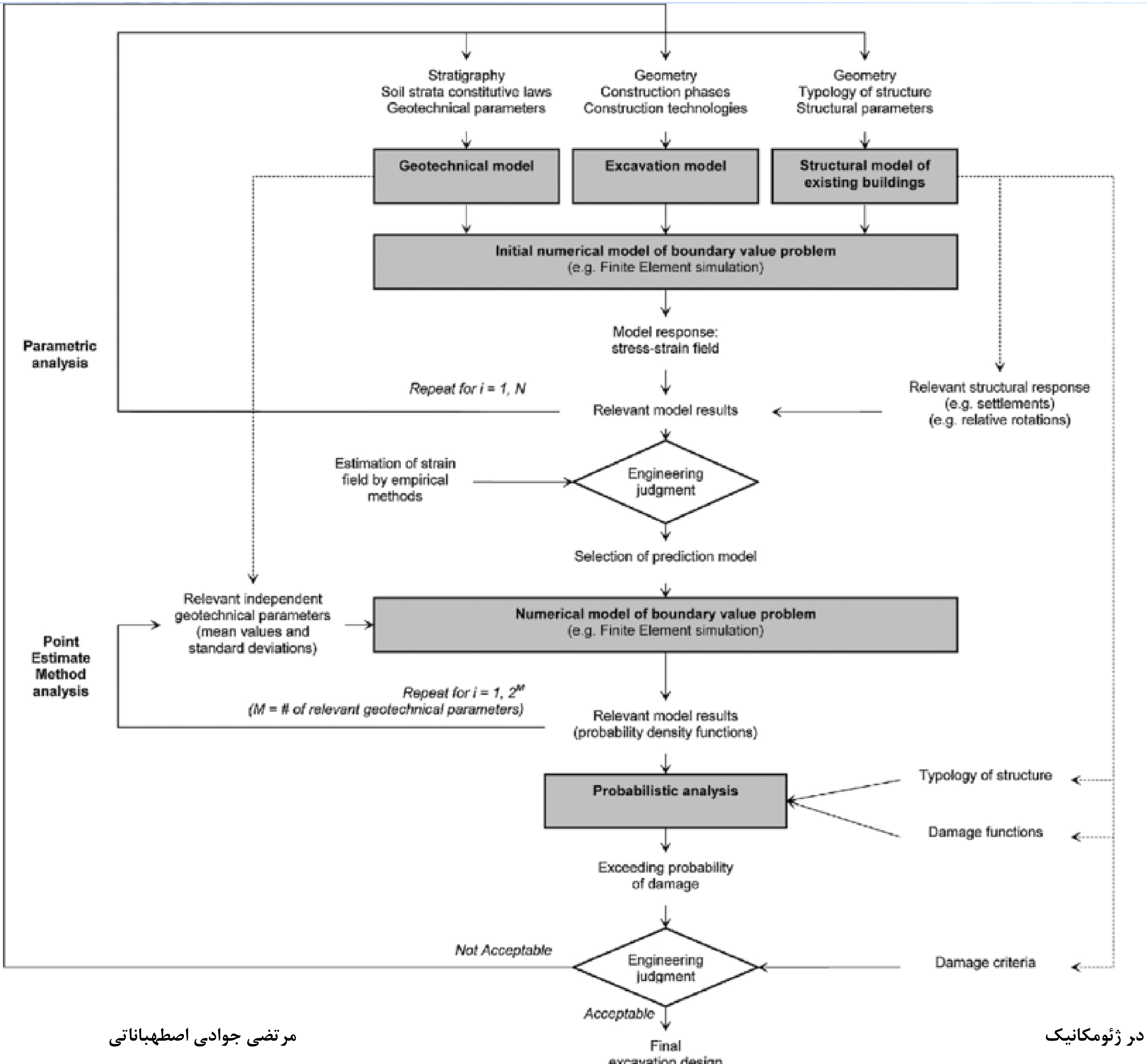
فرآیند عمومی در مدل سازی عددی

➢ مراحل کار



فرآیند عمومی د

- مراحل کار



تمرین

- از متون موجود در زمینه ژئومکانیک :

معیارهای انتخاب روش مدل سازی عددی (پیوسته و ناپیوسته)

دانشگاه صنعتی شاهرود

دانشکده معدن، نفت و ژئوفیزیک

درس روش های عددی در ژئومکانیک

مدرس

مرتضی جوادی اصطهباناتی

پاییز و زمستان ۱۳۹۶

رئوس مطالب

- معادلات دیفرانسیل
  - معادلات حرکت
  - تعادل استاتیکی

معادلات دیفرانسیل

Partial Differential Equations

- چرا معادلات دیفراسیل؟

- معادلات دیفرانسیل مرتبط با تنش

Simple problems involving homogeneous stress states have been considered so far, wherein the stress is the same throughout the component under study. Here the question of varying stress and strain fields in materials is considered. In order to solve such problems, a differential formulation is required.
To reach this goal, a number of differential equations will be derived, relating the stresses and body forces (**equations of motion**), the strains and displacements (**strain-displacement relations**) and the strains with each other (**compatibility relations**). These equations are derived from physical principles and so apply to any type of material, although the latter two are derived under the assumption of small strain.

Partial Differential Equations معادلات دیفرانسیل

The following equation is an example of a PDE:

$$a(t,x,y)\,U_t + b(t,x,y)U_x + c(t,x,y)U_{yy} = f(t,x,y) \qquad (1.1)$$

where,

- t, x, y are the *independent* variables (often time and space)
- a, b, c and f are known functions of the independent variables,
- U is the *dependent* variable and is an unknown function of the independent variables.
- partial derivatives are denoted by subscripts: $U_t = \frac{\partial U}{\partial t}, U_x = \frac{\partial U}{\partial x}, \; U_{yy} = \frac{\partial^2 U}{\partial y^2}$ etc.

The *order* of a PDE is the order of its highest derivative. A PDE is *linear* if U and all its partial derivatives occur to the first power only and there are no products involving more than one of these terms. (1.1) is second order and linear. The *dimension* of a PDE is the number of independent *spatial* variables it contains. (1.1) is 2D if x and y are spatial variables.

Partial Differential Equations معادلات دیفرانسیل

Second order linear PDEs can be formally classified into 3 generic types: elliptic, parabolic and hyperbolic. The simplest examples are:

a) Elliptic: e.g. $U_{xx} + U_{yy} = f(x, y)$.

This is Poisson's equation or Laplace's equation (when f(x,y) =0) which may be used to model the steady state temperature distribution in a plate or incompressible potential flow. Notice there is no time derivative.

b) Parabolic: e.g. $U_t = kU_{xx}$.

This is the 1D diffusion equation and can be used to model the time-dependent temperature distribution along a heated 1D bar.

c) Hyperbolic: e.g. $U_{tt} = c^2U_{xx}$.

This is the wave equation and may be used to model a vibrating guitar string or 1D supersonic flow.

d) $U_t = -cU_x$.

This first order PDE is called the advection equation. Solutions of d) also satisfy c).

e) $U_t + cU_x = kU_{xx}$.

This is the advection-diffusion equation and may be used to model transport of a pollutant in a river. The co-efficients k, c in the above PDEs quantify material properties that relate to the problem being solved e.g. k could be the coefficient of thermal conductivity in the case of a heated bar, or 1D diffusion coefficient in the case of pollutant transport; c is a wave speed, usually, in fluid flow, the speed of sound.

معادلات دیفرانسیل Partial Differential Equations

➢ راه حل و جواب معادلات دیفرانسیل

Solving a PDE means finding the unknown function U. An *analytical* (i.e. exact) solution of a PDE is a function that satisfies the PDE and also satisfies any *boundary* and/or *initial conditions* given with the PDE (more about these later). Most PDEs of interest do not have analytical solutions so a *numerical* procedure must be used to find an *approximate* solution. The approximation is made at discrete values of the independent variables and the approximation *scheme* is implemented via a computer program.

❖ جواب معادله دیفرانسیلی:

۱- جواب باید با معادله مشتقات جزئی همخوانی داشته باشد

۲- جواب باید در برای هر یک از شرایط اولیه یا مرزی درست باشد (برای نقاط با شرایط اولیه و یا مرزی، هم مقدار و هم مشتقات معادله باید برقرار باشند)

❖ جواب معادله دیفرانسیلی:

۱- حل تحلیلی

۲- حل عددی

Partial Differential Equations معادلات دیفرانسیل

➢ معادلات حرکت

Balance of forces and moments acting on any component was enforced in order to ensure that the component was in equilibrium. Here, allowance is made for stresses which vary continuously throughout a material, and force equilibrium of any portion of material is enforced.

❖ معادلات دیفرانسیلی حرکت

توزیع تنش و جابجایی (کرنش) در محیط

اصل پیوستگی تنش در محیط

تنش معادل نیرو، کرنش معادل جابجایی

du → ϵ , $[\epsilon = \frac{du}{dx}]$ → σ, [σ=Eϵ] → F, [F=σA] → D, [F=KD] → du, $[du=D_2-D_1]$

Partial Differential Equations معادلات دیفرانسیل

➢ معادلات حرکت

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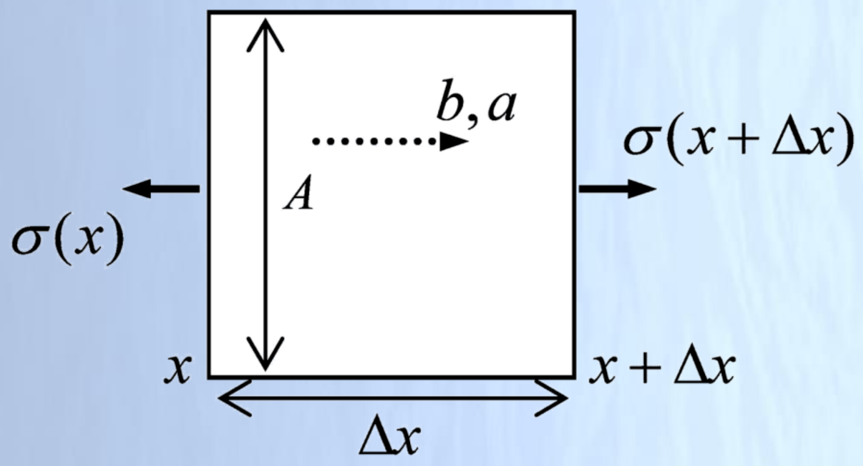
One-Dimensional Equation



The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$

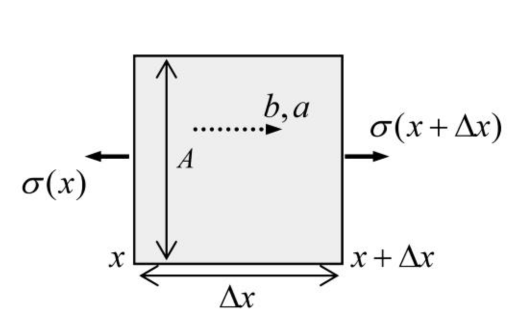
معادلات دیفرانسیل

Partial Differential Equations

➢ معادلات حرکت

One-Dimensional Equation

The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$. If the element is small, then the body force and velocity can be assumed to vary linearly over the element and the average will act at the centre of the element. Then the body force acting on the element is $Ab\Delta x$ and the inertial force is $\rho A\Delta x a$. Applying Newton's second law leads to



$$\sigma(x+\Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A$$

$$\rightarrow \frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + b = \rho a$$

so that, by the definition of the derivative, in the limit as $\Delta x \rightarrow 0$,

$$\boxed{\frac{d\sigma}{dx} + b = \rho a}$$ **1-d Equation of Motion**

which is the one-dimensional **equation of motion**. Note that this equation was derived on the basis of a physical law and must therefore be satisfied for all materials, whatever they be composed of.

The derivative $d\sigma / dx$ is the **stress gradient** – physically, it is a measure of how rapidly the stresses are changing.

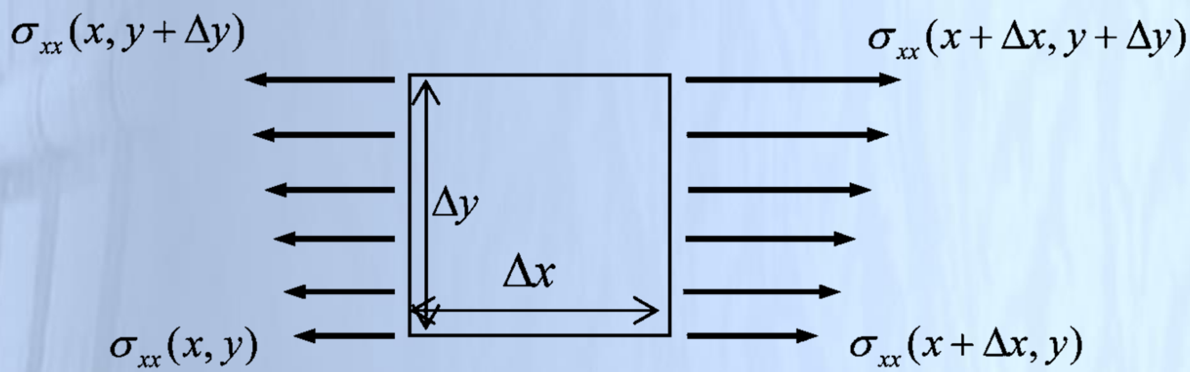
Partial Differential Equations

معادلات دیفرانسیل

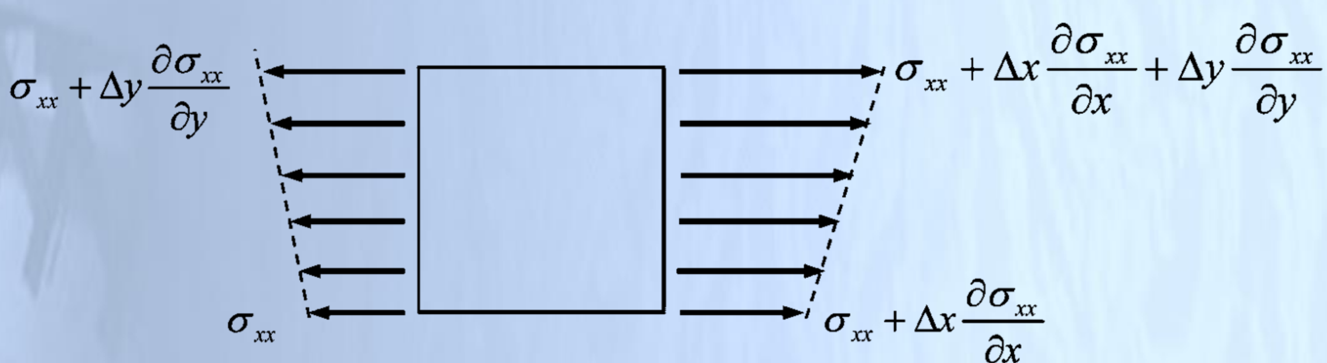
Two-Dimensional Equation

➢ معادلات حرکت

Consider now a two dimensional infinitesimal element of width and height Δ*x* and Δ*y* and unit depth (into the page). Looking at the normal stress components acting in the *x*-direction:



varying stresses acting on a differential element



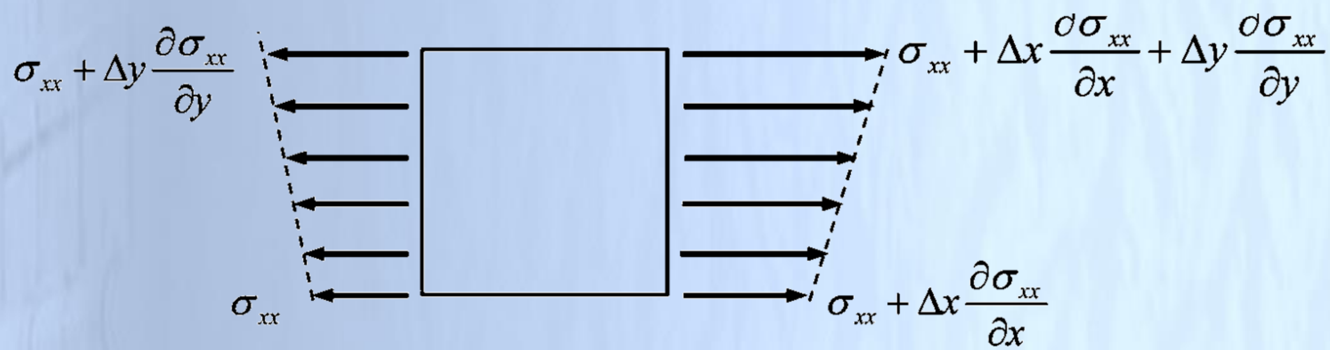
linearly varying stresses acting on a differential element

Partial Differential Equations

معادلات دیفرانسیل

➢ معادلات حرکت

Two-Dimensional Equation



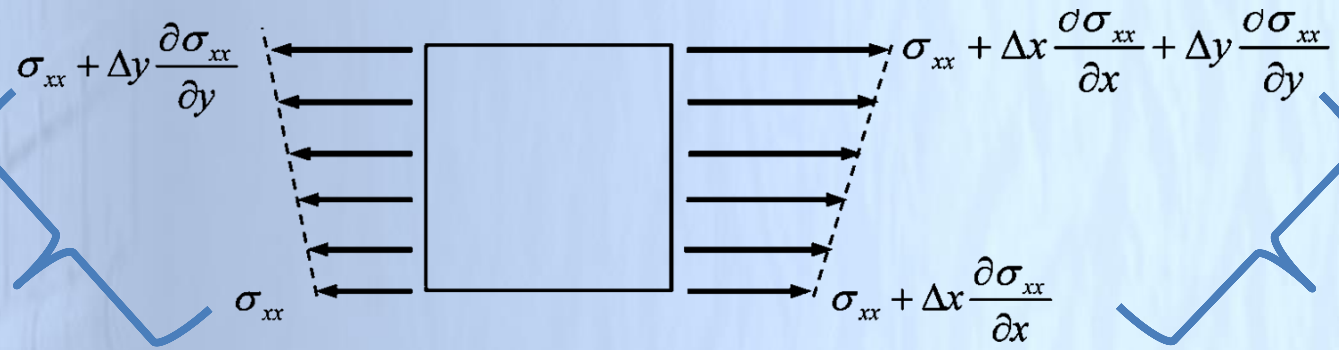
linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively:

معادلات دیفرانسیل
Partial Differential Equations

➢ معادلات حرکت

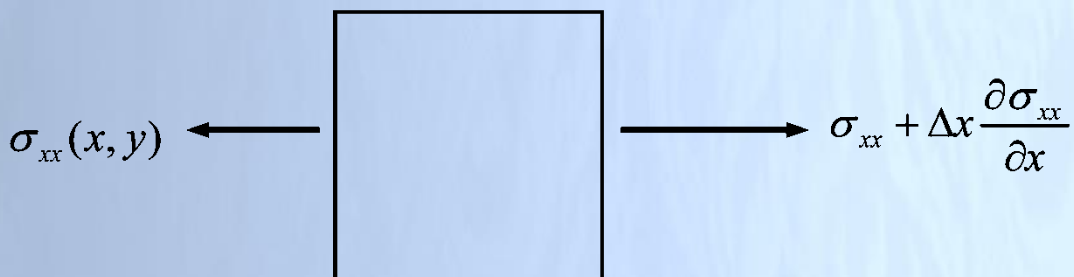
Two-Dimensional Equation



linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively:

$$\sigma_{xx}+\frac{1}{2}\Delta y\frac{\partial\sigma_{xx}}{\partial y}, \quad \sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}+\frac{1}{2}\Delta y\frac{\partial\sigma_{xx}}{\partial y}$$

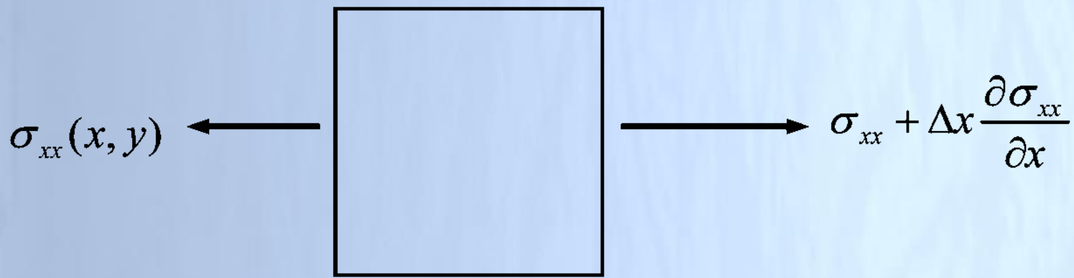


معادلات دیفرانسیل

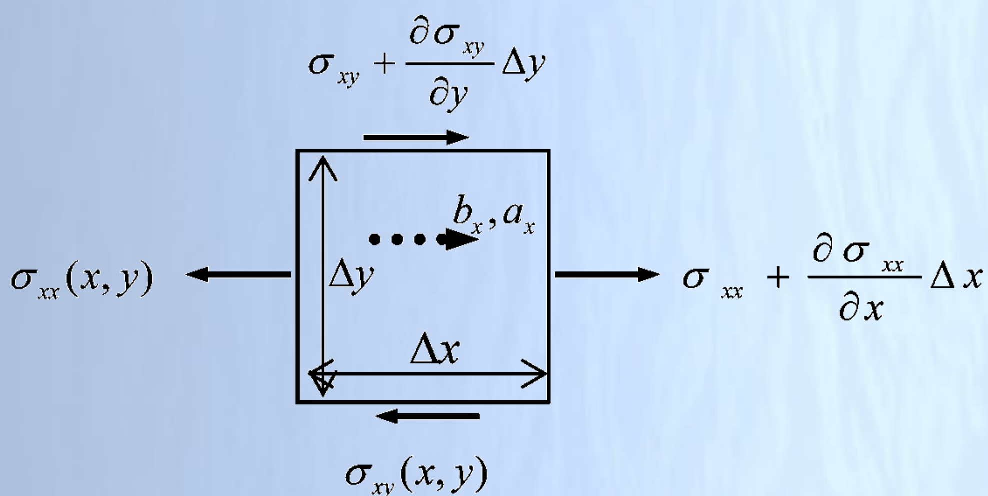
Partial Differential Equations

➢ معادلات حرکت

Two-Dimensional Equation



Carrying out the same procedure for the shear stresses contributing to a force in the *x*-direction leads to the stresses



معادلات دیفرانسیل
Partial Differential Equations

➢ معادلات حرکت
Two-Dimensional Equation

Take *ax and bx* to be the average acceleration and body force, and ρ to be the average density. Newton's law then yields

$$-\sigma_{xx}\Delta y+\left(\sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}\right)\Delta y-\sigma_{xy}\Delta x+\left(\sigma_{xy}+\Delta y\frac{\partial\sigma_{xy}}{\partial y}\right)\Delta x+b_x\Delta x\Delta y=\rho a_x\Delta x\Delta y$$

dividing through by Δ*x*Δ*y* and taking the limit, gives

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x=\rho a_x$$

A similar analysis for force components in the *y*-direction yields another equation and one then has the two-dimensional equations of motion:

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x&=\rho a_x\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+b_y&=\rho a_y\end{aligned}}$$

2-D Equations of Motion

Partial Differential Equations

معادلات دیفرانسیل

Three-Dimensional Equation

➢ معادلات حرکت

Similarly, one can consider a three-dimensional element, and one finds that

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+b_x=\rho a_x$$

$$\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+b_y=\rho a_y$$

$$\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+b_z=\rho a_z$$

3-D Equations of Motion

These three equations express force-balance in, respectively, the *x*, *y*, *z* directions

Partial Differential Equations

معادلات دیفرانسیل

➢ معادلات حرکت

Three-Dimensional Equation

Similarly, one can consider a three-dimensional element, and one finds that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = \rho a_x$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = \rho a_y$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho a_z$$

3-D Equations of Motion

If the acceleration become ZERO i.e $a_x=0$

↓

static equilibrium

تعادل استاتیکی

Partial Differential Equations معادلات دیفرانسیل

Differential equations of static equilibrium ➢ تعادل استاتیکی

Problems in solid mechanics frequently involve description of the stress distribution in a body in static equilibrium under the combined action of surface and body forces. Determination of the stress distribution must take account of the requirement that the stress field maintains static equilibrium throughout the body. This condition requires satisfaction of the equations of static equilibrium for all differential elements of the body.

❖ معادلات دیفرانسیلی تعادل استاتیکی

توزیع تنش در محیط در حالت تعادل استاتیکی تحت نیروهای خارجی

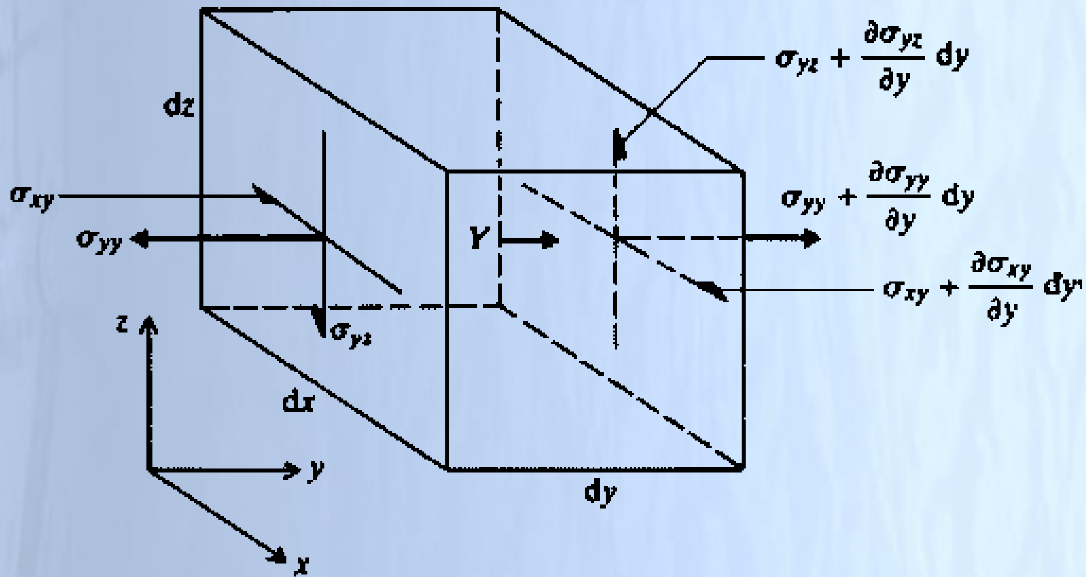
نیروهای خارجی

الزام برقراری تعادل استاتیکی در تمامی المان های توده یا محیط

du → ϵ , [$\epsilon = \frac{du}{dx}$] → σ, [σ=Eϵ] → F, [F=σA] → D, [F=KD] → du, [du=D_2-D_1]

Partial Differential Equations معادلات دیفرانسیل

Differential equations of static equilibrium ➢ تعادل استاتیکی

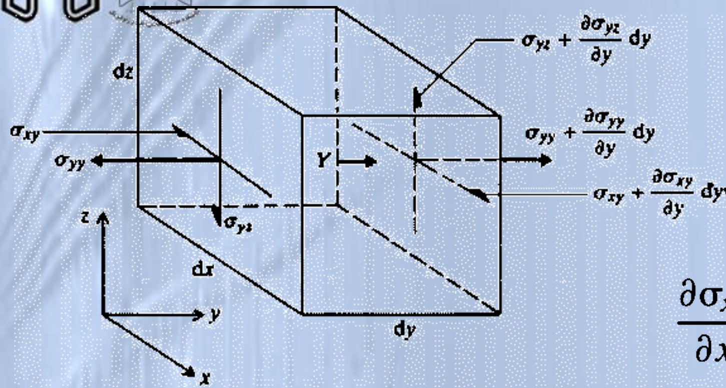


a small element of a body, in which operate body force components with magnitudes *X*, *Y*, *Z* per unit volume, directed in the positive *x*, *y*, *z* co-ordinate directions. The stress distribution in the body is described in terms of a set of stress gradients, defined by *∂σxx/∂x, ∂σxy/∂y*, etc. Considering the condition for force equilibrium of the element in the *x* direction yields the equation:

$$\frac{\partial \sigma_{xx}}{\partial x} \cdot dx \cdot dy\, dz + \frac{\partial \sigma_{xy}}{\partial y} \cdot dy \cdot dx\, dz + \frac{\partial \sigma_{zx}}{\partial z} \cdot dz \cdot dx\, dy + X\, dx\, dy\, dz = 0$$

Partial Differential Equations

معادلات دیفرانسیل



➢ تعادل استاتیکی

Differential equations of static equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} \cdot dx \cdot dy\, dz + \frac{\partial \sigma_{xy}}{\partial y} \cdot dy \cdot dx\, dz + \frac{\partial \sigma_{zx}}{\partial z} \cdot dz \cdot dx\, dy + X\, dx\, dy\, dz = 0$$

Applying the same static equilibrium requirement to the *y* and *z* directions, and eliminating the term d*x* d*y* d*z*, yields the differential equations of equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0$$

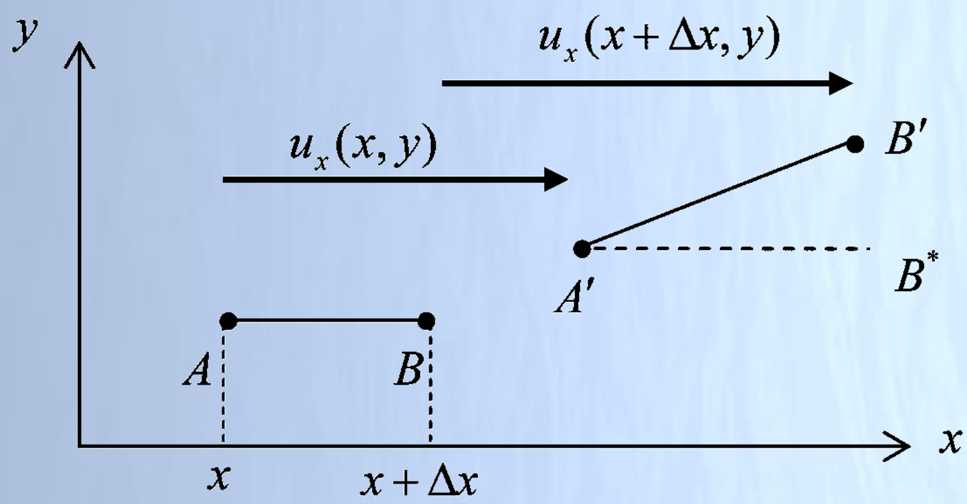
These expressions indicate that the variations of stress components in a body under load are not mutually independent. They are always involved, in one form or another, in determining the state of stress in a body.

Partial Differential Equations معادلات دیفرانسیل

Differential equations for Strain-Displacement Relations کرنش جابجایی

Normal Strain

Consider a line element of length Δ*x* emanating from position (*x*, *y*) and lying in the *x* -direction, denoted by *AB.* After deformation the line element occupies *A'B'* , having undergone a translation, extension and rotation.



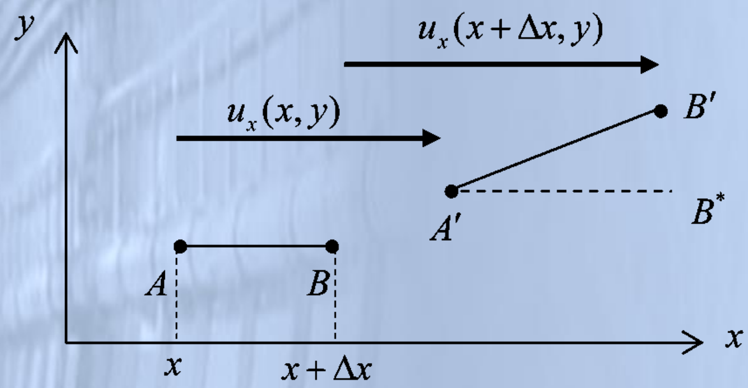
deformation of a line element

The particle that was originally at *x* has undergone a displacement *ux* (*x*, *y*) and the other end of the line element has undergone a displacement *ux*(*x*+Δ *x*, *y*).

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

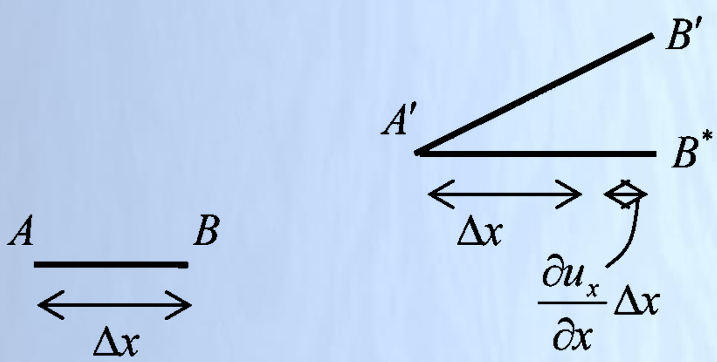
Normal Strain



$$\varepsilon_{xx} = \frac{A'B^* - AB}{AB} = \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x}$$

In the limit $\Delta x \to 0$ one has $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$

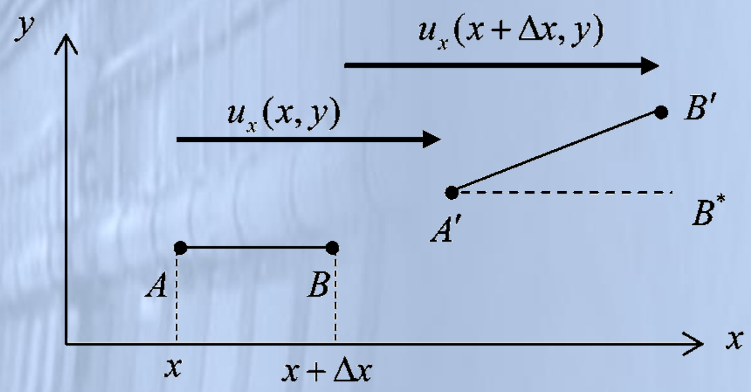
This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain *at* (*x*, *y*) . Physically, it represents the (approximate) unit change in length of a line element, as indicated in below Figure



Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Normal Strain



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In the limit $\Delta x \to 0$ one has $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$

Similarly, by considering a line element initially lying in the *y* direction, the strain in the *y*-direction can be expressed as:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

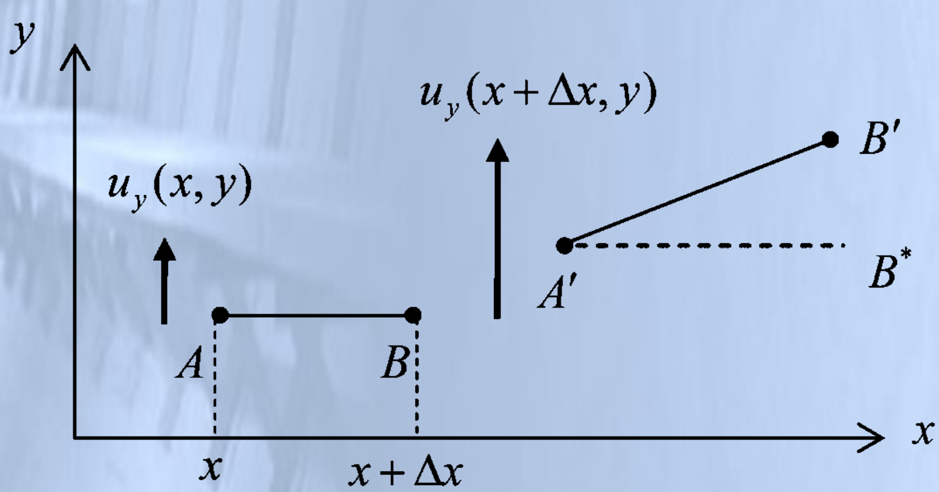
معادلات دیفرانسیل Partial Differential Equations

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Shear Strain

The particles *A* and *B* also undergo displacements in the *y* direction. In this case, one has:

$$B^*B' = \frac{\partial u_y}{\partial x}\Delta x$$



A similar relation can be derived by considering a line element initially lying in the *y*-direction:

$$\theta \approx \tan\theta = \frac{\partial u_y / \partial x}{1 + \partial u_x / \partial x} \approx \frac{\partial u_y}{\partial x}$$

provided that (i) θ is small and (ii) the displacement gradient $\partial u_x / \partial x$ is small. A similar expression for the angle λ can be derived, and hence the shear strain can be written in terms of displacement gradients.

Partial Differential Equations

معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Shear Strain

The particles *A* and *B* also undergo displacements in the *y* direction. In this case, one has:

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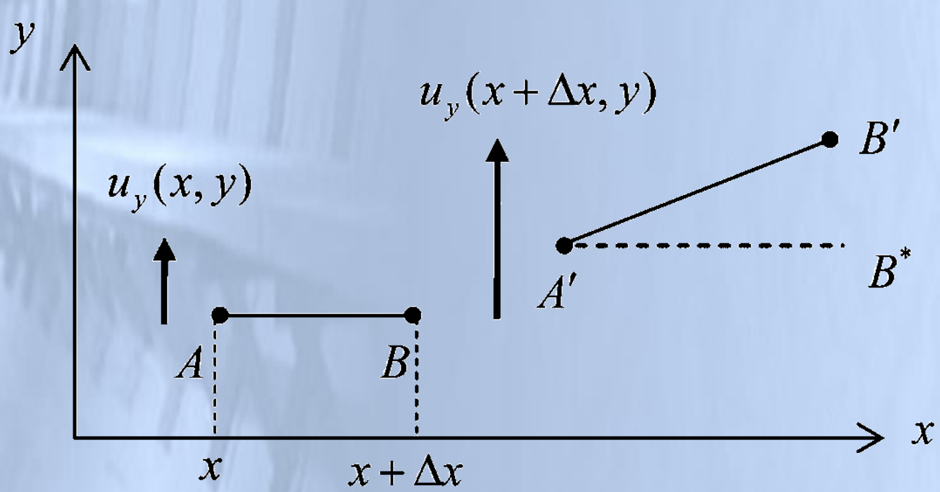

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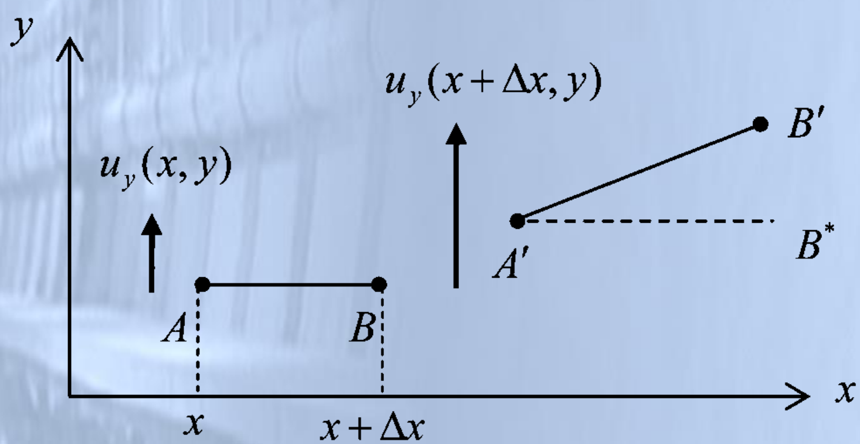
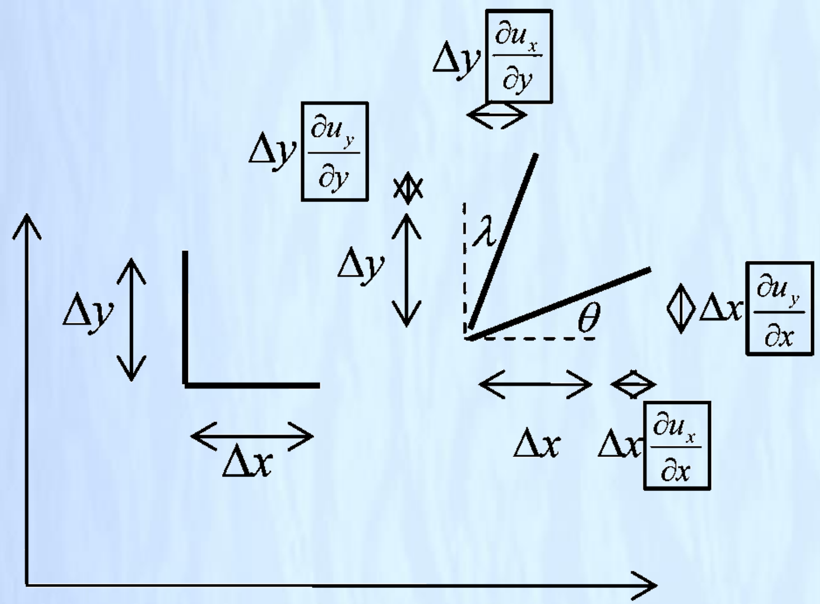
معادلات دیفرانسیل *Partial Differential Equations*

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Shear Strain

$$B^* B' = \frac{\partial u_y}{\partial x} \Delta x$$





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Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

The Small-Strain Stress-Strain Relations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

2-D Strain-Displacement relations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

3-D Stress-Strain relations

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

The Rotation

Consider an arbitrary deformation (omitting normal strains for ease of description), as shown in Fig. 1.2.6. As usual, the angles θ and λ are small, equal to their tangents, and $\theta = \partial u_y / \partial x$, $\lambda = \partial u_x / \partial y$.

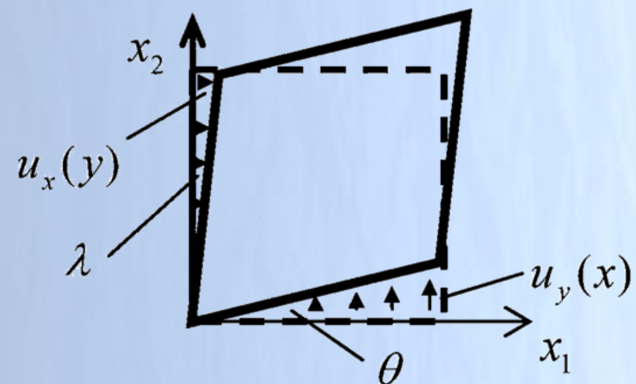


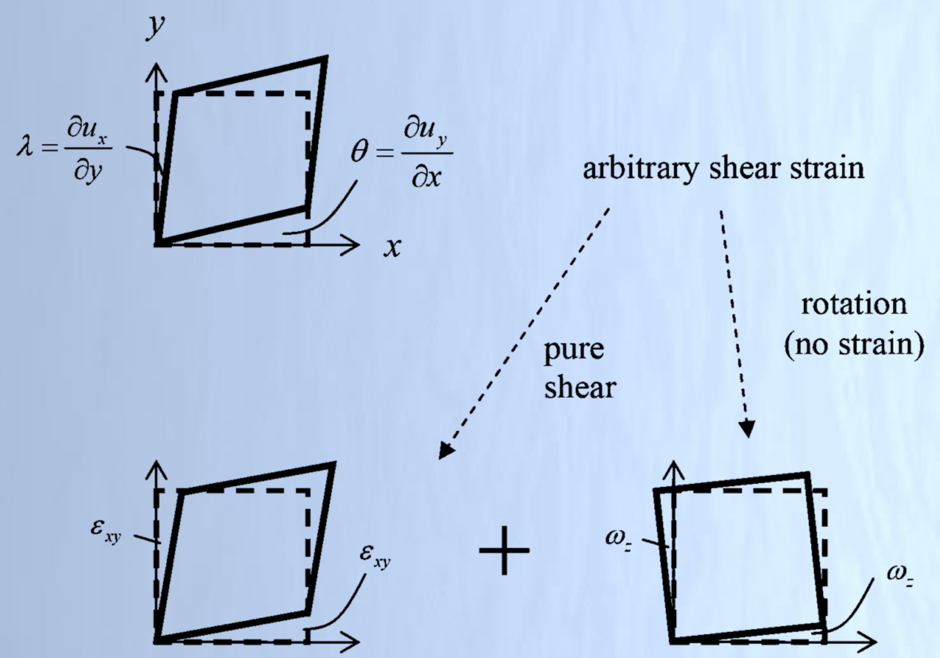
Figure 1.2.6: arbitrary deformation (shear and rotation)

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

The Rotation

Now this arbitrary deformation can be decomposed into a pure shear and a rigid rotation as depicted in Fig. 1.2.7. In the pure shear, $\theta = \lambda = \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda)$. In the rotation, the angle of rotation is then $\frac{1}{2}(\theta + \lambda)$.



$$\lambda = \frac{\partial u_x}{\partial y} \qquad \theta = \frac{\partial u_y}{\partial x}$$

$$\varepsilon_{xy} = \frac{1}{2}(\lambda + \theta) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \omega_z = \frac{1}{2}(\theta - \lambda) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

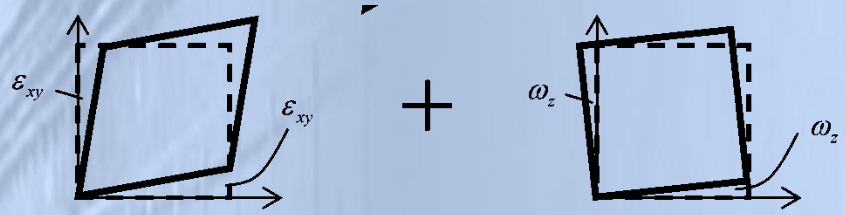
Figure 1.2.7: decomposition of a strain into a pure shear and a rotation

Partial Differential Equations

معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

The Rotation



$$\varepsilon_{xy} = \frac{1}{2}(\lambda + \theta) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \omega_z = \frac{1}{2}(\theta - \lambda) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

This leads one to define the **rotation** of a material particle, ω_z, the "z" signifying the axis about which the element is rotating:

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \qquad (1.2.10)$$

The rotation will in general vary throughout a material. When the rotation is everywhere zero, the material is said to be **irrotational**.

For a pure rotation, note that

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0, \qquad \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \qquad (1.2.11)$$

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

The three-dimensional stress-strain relations analogous to Eqns. 1.2.5 are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

3-D Stress-Strain relations

The rotations are

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)$$

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

the incremental displacements may be expressed by

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

or

$$[d\boldsymbol{\delta}] = [\mathbf{D}][d\mathbf{r}]$$

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_z & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

or

$$[d\boldsymbol{\delta}'] = [\boldsymbol{\Omega}][d\mathbf{r}]$$

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$[d\boldsymbol{\delta}''] = [\boldsymbol{\varepsilon}][d\mathbf{r}]$$

معادلات دیفرانسیل *Partial Differential Equations*

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\frac{\partial u_x}{\partial y} = \frac{1}{2}\gamma_{xy} - \Omega_z$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{2}\gamma_{xy} + \Omega_z$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \Omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \Omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}, \quad \Omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Strain compatibility equations

As seen in the previous section, the displacements can be determined from the strains through integration, to within a rigid body motion. In the two-dimensional case, there are three strain-displacement relations but only two displacement components. This implies that the **strains are not independent but are related in some way**. The relations between the strains are called **compatibility conditions**.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y}\right)$$

It follows that

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}}$$

2-D Compatibility Equation

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Strain compatibility equations

Physical Meaning of the Compatibility Condition

When all material particles in a component deform, translate and rotate, they need to meet up again very much like the pieces of a jigsaw puzzle must fit together. Fig. 1.3.1 illustrates possible deformations and rigid body motions for three line elements in a material. Compatibility ensures that they stay together after the deformation.

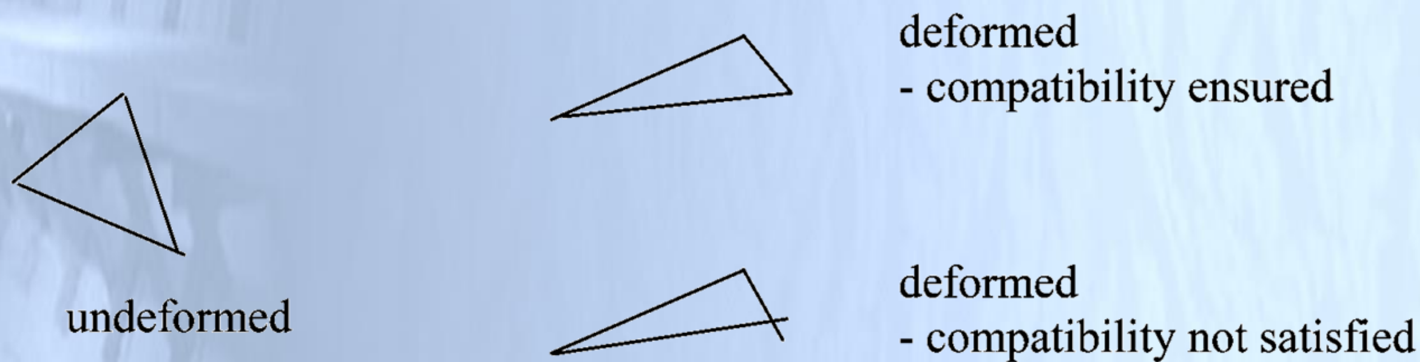


Figure 1.3.1: Deformation and Compatibility

Partial Differential Equations معادلات دیفرانسیل

➢ کرنش جابجایی Differential equations for Strain-Displacement Relations

Strain compatibility equations

There are six compatibility relations to be satisfied in the three dimensional case:

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right)$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right)$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right)$$

دانشگاه صنعتی شاهرود

دانشکده معدن، نفت و ژئوفیزیک

درس روش های عددی در ژئومکانیک

مدرس

مرتضی جوادی اصطهباناتی

پاییز و زمستان ۱۳۹۶

رئوس مطالب

- طراحی در مهندسی سنگ و ژئومکانیک
  - تفاوت مهندسی سنگ با سایر علوم
  - ابزارها و روش های مختلف طراحی
  - رفتارهای مختلف زمین و فضای زیرزمینی
- روش های عددی
  - دسته بندی بر مفاهیم مختلف
  - انتخاب نوع روش های عددی
  - مفهوم REV، عدم قطعیت، غیر یکنواختی و ناهمگنی
- فرآیند عمومی در مدل سازی عدد

Partial Differential Equations

معادلات دیفرانسیل

The following equation is an example of a PDE:

$$a(t,x,y)\,U_t + b(t,x,y)U_x + c(t,x,y)U_{yy} = f(t,x,y) \qquad (1.1)$$

where,

- t, x, y are the *independent* variables (often time and space)
- a, b, c and f are known functions of the independent variables,
- U is the *dependent* variable and is an unknown function of the independent variables.
- partial derivatives are denoted by subscripts: $U_t = \dfrac{\partial U}{\partial t}, U_x = \dfrac{\partial U}{\partial x},\ U_{yy} = \dfrac{\partial^2 U}{\partial y^2}$ etc.

The *order* of a PDE is the order of its highest derivative. A PDE is *linear* if U and all its partial derivatives occur to the first power only and there are no products involving more than one of these terms. (1.1) is second order and linear. The *dimension* of a PDE is the number of independent *spatial* variables it contains. (1.1) is 2D if x and y are spatial variables.

Partial Differential Equations معادلات دیفرانسیل

Second order linear PDEs can be formally classified into 3 generic types: elliptic, parabolic and hyperbolic. The simplest examples are:

a) Elliptic: e.g. $U_{xx} + U_{yy} = f(x, y)$.

This is Poisson's equation or Laplace's equation (when f(x,y) =0) which may be used to model the steady state temperature distribution in a plate or incompressible potential flow. Notice there is no time derivative.

b) Parabolic: e.g. $U_t = kU_{xx}$.

This is the 1D diffusion equation and can be used to model the time-dependent temperature distribution along a heated 1D bar.

c) Hyperbolic: e.g. $U_{tt} = c^2 U_{xx}$.

This is the wave equation and may be used to model a vibrating guitar string or 1D supersonic flow.

d) $U_t = -cU_x$.

This first order PDE is called the advection equation. Solutions of d) also satisfy c).

e) $U_t + cU_x = kU_{xx}$.

This is the advection-diffusion equation and may be used to model transport of a pollutant in a river. The co-efficients k, c in the above PDEs quantify material properties that relate to the problem being solved e.g. k could be the coefficient of thermal conductivity in the case of a heated bar, or 1D diffusion coefficient in the case of pollutant transport; c is a wave speed, usually, in fluid flow, the speed of sound.

معادلات دیفرانسیل Partial Differential Equations

➢ راه حل و جواب معادلات دیفرانسیل

Solving a PDE means finding the unknown function U. An *analytical* (i.e. exact) solution of a PDE is a function that satisfies the PDE and also satisfies any *boundary* and/or *initial conditions* given with the PDE (more about these later). Most PDEs of interest do not have analytical solutions so a *numerical* procedure must be used to find an *approximate* solution. The approximation is made at discrete values of the independent variables and the approximation *scheme* is implemented via a computer program.

❖ جواب معادله دیفرانسیلی:

۱- جواب باید با معادله مشتقات جزئی همخوانی داشته باشد

۲- جواب باید در برای هر یک از شرایط اولیه یا مرزی درست باشد (برای نقاط با شرایط اولیه و یا مرزی، هم مقدار و هم مشتقات معادله باید برقرار باشند)

❖ جواب معادله دیفرانسیلی:

۱- حل تحلیلی

۲- حل عددی

Partial Differential Equations

معادلات دیفرانسیل

➢ روش تفاضل محدود FDM

variables and the approximation *scheme* is implemented via a computer program. The FDM replaces all partial derivatives and other terms in the PDE by ***approximations*. **After some manipulation, a **finite difference scheme (FDS**) is created from which the approximate solution is obtained. The FDM depends fundamentally on ***Taylor's beautiful theorem***

روش تفاضل محدود Finite Difference Method

➢ تئوری تیلور Taylor's Theorem

Let U(x) have n continuous derivatives over the interval (a, b). Then for a < x_o, x_o+h < b,

$$U(x_o+h)=U(x_o)+hU_x(x_o)+h^2\frac{U_{xx}(x_o)}{2!}+\ldots+h^{n-1}\frac{U_{(n-1)}(x_o)}{(n-1)!}+O(h^n)\,, \quad (2.1)$$

تقریب مقدار یک تابع با استفاده از مقدار عددی تابع در نقطه مجاور

مقدار عددی یک تابع در نقطه (X0+h) را می توان با استفاده از تئوری تیلور و داشتن مقدار عددی مشتق تابع در نقطه مجاور (X0) تعیین نمود. مقدار مشتق تابع در نقطه (X0) همان مقدار اولیه یا شرط مرزی است و مقدار عددی تابع در نقطه (X0+h) همان مجهول است و برعکس

Finite Difference Method
روش تفاضل محدود

Taylor's Theorem
➢ تئوری تیلور

Let U(x) have n continuous derivatives over the interval (a, b). Then for $a < x_o$, $x_o+h < b$,

$$U(x_o+h)=U(x_o)+hU_x(x_o)+h^2\frac{U_{xx}(x_o)}{2!}+\ldots+h^{n-1}\frac{U_{(n-1)}(x_o)}{(n-1)!}+O(h^n)\,, \quad (2.1)$$

تقریب مقدار یک تابع با استفاده از مقدار عددی تابع در نقطه مجاور

where,

- $U_x=\frac{dU}{dx}, U_{xx}=\frac{d^2U}{dx^2},\ldots,U_{(n-1)}=\frac{d^{n-1}U}{dx^{n-1}}$.
- $U_x(x_o)$ is the derivative of U with respect to x *evaluated* at $x = x_o$.
- $O(h^n)$ is an unknown error term defined in Appendix A.

The usual interpretation of Taylor's theorem says that if we know the value of U and the values of its derivatives at point x_o then we can write down the equation (2.1) for its value at the (nearby) point x_o+h. This expression contains an unknown quantity which is written in as $O(h^n)$ and pronounced 'order h to the n'. If we discard the term $O(h^n)$ in (2.1) (i.e. *truncate* the right hand side of (2.1)) we get an *approximation* to $U(x_o+h)$. The error in this approximation is $O(h^n)$.

روش تفاضل محدود *Finite Difference Method*

➢ تقریب المان محدود با تئوری تیلور FDM by Taylor's Theorem

$$U(x_o+h)=U(x_o)+hU_x(x_o)+h^2\frac{U_{xx}(x_o)}{2!}+\ldots+h^{n-1}\frac{U_{(n-1)}(x_o)}{(n-1)!}+O(h^n)\,, \quad (2.1)$$

Truncating (2.1) after the first derivative term gives,

$$U(x_o+h)=U(x_o)+hU_x(x_o)+O(h^2) \quad (2.2)$$

جملات سوم به بعد بصورت O(h2)

Rearranging (2.2) gives,

$$U_x(x_o)=\frac{U(x_o+h)-U(x_o)}{h}+\frac{O(h^2)}{h}$$
$$=\frac{U(x_o+h)-U(x_o)}{h}+O(h) \quad (\text{by}A.3.3)$$

Neglecting the O(h) term gives,

مقدار مشتق تابع در نقطه (X0) بعنوان تابعی از مقدار تابع اصلی در نقاط (X0) و (X0+h)

$$U_x(x_o) \approx \frac{U(x_o+h)-U(x_o)}{h} \quad (2.3)$$

(2.3) is called a *first order* FD approximation to $U_x(x_o)$ since the approximation error = O(h) which depends on the *first* power of h. This approximation is called a *forward* FD approximation since we start at x_o and step forwards to the point x_o+h. h is called the *step size* (h > 0).

روش تفاضل محدود Finite Difference Method

➢ تقریب المان محدود با تئوری تیلور FDM by Taylor’s Theorem

forward FD approximation

$$U_x(x_o) \approx \frac{U(x_o+h)-U(x_o)}{h}$$

first order FD approximation

مقدار مشتق تابع در نقطه (X0) بعنوان تابعی از مقدار تابع اصلی در نقاط (X0) و (X0+h)

Finite Difference Method روش تفاضل محدود

FDM by Taylor’s Theorem ➢ تقریب المان محدود با تئوری تیلور

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_i^n}{\Delta x}$	forward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_i^n - U_{i-1}^n}{\Delta x}$	backward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$	central	second in x
$\frac{\partial^2 U}{\partial x^2} = U_{xx}$	$\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$	symmetric	second in x
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^n}{\Delta t}$	forward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^n - U_i^{n-1}}{\Delta t}$	backward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}$	central	second in t
$\frac{\partial^2 U}{\partial t^2} = U_{tt}$	$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}$	symmetric	second in t

روش تفاضل محدود
Finite Difference Method

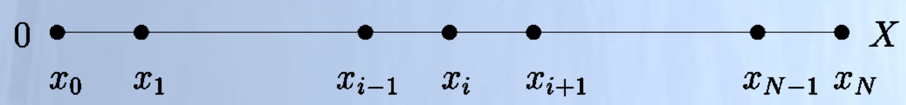
➢ اصول تفاضل محدود
FDM Principles

Finite difference method

Principle: derivatives in the partial differential equation are approximated by linear combinations of function values at the grid points

1D: $\Omega = (0, X), \quad u_i \approx u(x_i), \quad i = 0, 1, \dots, N$

grid points $x_i = i\Delta x$ mesh size $\Delta x = \frac{X}{N}$



First-order derivatives

$$\frac{\partial u}{\partial x}(\bar{x}) = \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x})}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(\bar{x}) - u(\bar{x} - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x} - \Delta x)}{2\Delta x} \qquad \text{(by definition)}$$

روش تفاضل محدود

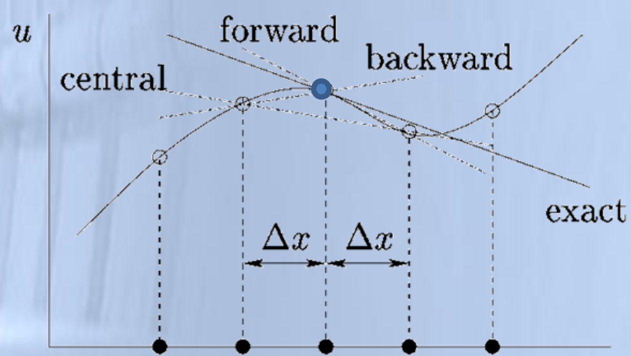
Finite Difference Method

➢ تقریب تفاضل محدود

FDM Principles

Approximation of first-order derivatives

Geometric interpretation



$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1}-u_i}{\Delta x}$$ forward difference

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_i-u_{i-1}}{\Delta x}$$ backward difference

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1}-u_{i-1}}{2\Delta x}$$ central difference

Taylor series expansion $$u(x) = \sum_{n=0}^{\infty} \frac{(x-x_i)^n}{n!} \left(\frac{\partial^n u}{\partial x^n}\right)_i, \qquad u \in C^\infty([0,X])$$

$$T_1: \quad u_{i+1} = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots$$

$$T_2: \quad u_{i-1} = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots$$

روش تفاضل محدود

Finite Difference Method

➢ تقریب تفاضل محدود

FDM Principles

Analysis of truncation errors

Accuracy of finite difference approximations

$$T_1 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1}-u_i}{\Delta x} - \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots$$

forward difference truncation error $\mathcal{O}(\Delta x)$

$$T_2 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i-u_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots$$

backward difference truncation error $\mathcal{O}(\Delta x)$

$$T_1 - T_2 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1}-u_{i-1}}{2\Delta x} - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots$$

central difference truncation error $\mathcal{O}(\Delta x)^2$

Leading truncation error

$$\epsilon_\tau = \alpha_m(\Delta x)^m + \alpha_{m+1}(\Delta x)^{m+1} + \dots \approx \alpha_m(\Delta x)^m$$

روش تفاضل محدود

Finite Difference Method

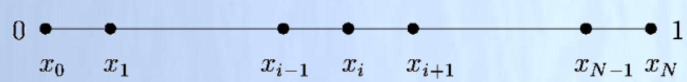
➢ مثال تفاضل محدود

Example: 1D Poisson equation

Boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

One-dimensional mesh



$$u_i \approx u(x_i), \quad f_i = f(x_i) \qquad x_i = i\Delta x, \quad \Delta x = \frac{1}{N}, \quad i = 0,1,\ldots,N$$

Central difference approximation $\qquad \mathcal{O}(\Delta x)^2$

$$\begin{cases} -\frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} = f_i, & \forall i = 1,\ldots,N-1 \\ u_0 = u_N = 0 & \text{Dirichlet boundary conditions} \end{cases}$$

SEE NEXT PAGE

Result: the original PDE is replaced by a linear system for nodal values

Finite Difference Method روش تفاضل محدود

FDM by Taylor's Theorem
➢ تقریب المان محدود با تئوری تیلور

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_i^n}{\Delta x}$	forward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_i^n - U_{i-1}^n}{\Delta x}$	backward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$	central	second in x
$\frac{\partial^2 U}{\partial x^2} = U_{xx}$	$\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$	symmetric	second in x
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^n}{\Delta t}$	forward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^n - U_i^{n-1}}{\Delta t}$	backward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}$	central	second in t
$\frac{\partial^2 U}{\partial t^2} = U_{tt}$	$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}$	symmetric	second in t

روش تفاضل محدود

Finite Difference Method

➢ مثال یک بعدی تفاضل محدود

Example: 1D Poisson equation

Linear system for the central difference scheme

$$\begin{cases} i=1 & -\frac{u_0-2u_1+u_2}{(\Delta x)^2} = f_1 \\ i=2 & -\frac{u_1-2u_2+u_3}{(\Delta x)^2} = f_2 \\ i=3 & -\frac{u_2-2u_3+u_4}{(\Delta x)^2} = f_3 \\ & \dots \\ i=N-1 & \frac{u_{N-2}-2u_{N-1}+u_N}{(\Delta x)^2} = f_{N-1} \end{cases}$$

Matrix form $\boxed{Au=F}$ $\quad A \in \mathbb{R}^{N-1\times N-1} \quad u, F \in \mathbb{R}^{N-1}$

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \dots & \\ & & & -1 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ u_{N-1} \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ f_{N-1} \end{bmatrix}$$

The matrix A is tridiagonal and symmetric positive definite $\Rightarrow$ invertible.

Finite Difference Method

روش تفاضل محدود

مثال یک بعدی تفاضل محدود

Example: 1D Poisson equation

Linear system for the central difference scheme

$$\begin{cases} i=1 & -\frac{u_0-2u_1+u_2}{(\Delta x)^2} = f_1 \\ i=2 & -\frac{u_1-2u_2+u_3}{(\Delta x)^2} = f_2 \\ i=3 & -\frac{u_2-2u_3+u_4}{(\Delta x)^2} = f_3 \\ & \dots \\ i=N-1 & \frac{u_{N-2}-2u_{N-1}+u_N}{(\Delta x)^2} = f_{N-1} \end{cases}$$

Matrix form $\boxed{Au=F}$ $\quad A \in \mathbb{R}^{N-1\times N-1} \quad u, F \in \mathbb{R}^{N-1}$

بردار با (N-1) مولفه معلوم

بردار با (N-1) مولفه مجهول

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \dots & \\ & & & -1 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ u_{N-1} \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ f_{N-1} \end{bmatrix}$$

ماتریس (N-1)*(N-1)

The matrix A is tridiagonal and symmetric positive definite $\Rightarrow$ invertible.

روش تفاضل محدود Finite Difference Method

➢ مثال یک بعدی تفاضل محدود

Example: 1D Poisson equation

Linear system for the central difference scheme

$$\begin{cases} i=1 & -\frac{u_0-2u_1+u_2}{(\Delta x)^2} = f_1 \\ i=2 & -\frac{u_1-2u_2+u_3}{(\Delta x)^2} = f_2 \\ i=3 & -\frac{u_2-2u_3+u_4}{(\Delta x)^2} = f_3 \\ & \dots \\ i=N-1 & \frac{u_{N-2}-2u_{N-1}+u_N}{(\Delta x)^2} = f_{N-1} \end{cases}$$

Matrix form $\boxed{Au=F}$ $\quad A \in \mathbb{R}^{N-1\times N-1} \quad u, F \in \mathbb{R}^{N-1}$

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \dots & \\ & & & -1 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ u_{N-1} \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ f_{N-1} \end{bmatrix}$$

The matrix A is tridiagonal and symmetric positive definite $\Rightarrow$ invertible.

با داشتن مقدار تابع *f* در نقاط (گره ها)، می توان مقدار مجهول تابع U را با حل ماتریس Au=F حل نمود.

تمرین: مثال بالا را برای تابع *f=sin(πx)* و برای مقادیر مختلف N از ۲ تا ۱۰ حل کنید

روش تفاضل محدود

Finite Difference Method

➢ مثال یک بعدی

حل مثال قبل برای شرایط مرزی دیگر

Other types of boundary conditions

Dirichlet-Neumann BC $u(0) = \frac{\partial u}{\partial x}(1) = 0$

$$u_0 = 0, \quad \frac{u_{N+1} - u_{N-1}}{2\Delta x} = 0 \quad \Rightarrow \quad u_{N+1} = u_{N-1}$$

central difference

Extra equation for the last node

$$-\frac{u_{N-1} - 2u_N + u_{N+1}}{(\Delta x)^2} = f_N \quad \longrightarrow \quad \frac{-u_{N-1} + u_N}{(\Delta x)^2} = \frac{1}{2} f_N$$

Extended linear system $Au = F$ $\quad A \in \mathbb{R}^{N\times N} \quad u, F \in \mathbb{R}^N$

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ u_{N-1} \\ u_N \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ f_{N-1} \\ \frac{1}{2} f_N \end{bmatrix}$$

The matrix A remains tridiagonal and symmetric positive definite.

Finite Difference Method

روش تفاضل محدود

➢ مثال یک بعدی

حل مثال قبل برای سایر شرایط مرزی

Other types of boundary conditions

Non-homogeneous Dirichlet BC $u(0) = g_0$ only F changes

$$u_0 = g_0 \quad \Rightarrow \quad \frac{2u_1 - u_2}{(\Delta x)^2} = f_1 + \frac{g_0}{(\Delta x)^2} \quad \text{first equation}$$

Non-homogeneous Neumann BC $\frac{\partial u}{\partial x}(1) = g_1$ only F changes

$$\frac{u_{N+1} - u_{N-1}}{2\Delta x} = g_1 \quad \Rightarrow \quad u_{N+1} = u_{N-1} + 2\Delta x g_1$$

$$-\frac{u_{N-1} - 2u_N + u_{N+1}}{(\Delta x)^2} = f_N \quad \longrightarrow \quad \frac{-u_{N-1} + u_N}{(\Delta x)^2} = \frac{1}{2} f_N + \frac{g_1}{\Delta x}$$

Non-homogeneous Robin BC $\frac{\partial u}{\partial x}(1) + \alpha u(1) = g_2$ A and F change

$$\frac{u_{N+1} - u_{N-1}}{2\Delta x} + \alpha u_N = g_2 \quad \Rightarrow \quad u_{N+1} = u_{N-1} - 2\Delta x \alpha u_N + 2\Delta x g_2$$

$$-\frac{u_{N-1} - 2u_N + u_{N+1}}{(\Delta x)^2} = f_N \quad \longrightarrow \quad \frac{-u_{N-1} + (1 + \alpha\Delta x) u_N}{(\Delta x)^2} = \frac{1}{2} f_N + \frac{g_2}{\Delta x}$$

Finite Difference Method روش تفاضل محدود

➢ مثال یک بعدی (وابسته به زمان)

We construct a simple FDS to find an approximate solution of a simple PDE. This PDE will be studied in more detail in Chapter 4. For now it suffices to generate a simple FDS to provide motivation for further study. The 1D linear advection equation is,

هدف:

$$U_t + v\,U_x = 0, \qquad (2.12a)$$

where the independent variables are t (time) and x (space). x is restricted to the finite interval [p, q] which is called the *computational domain*. v is a constant and the dependent variable, U = U(t, x). In addition to the PDE, we need *initial conditions* for U. Let the initial conditions be,

شرایط اولیه:

$$U(0, x) = f(x), \quad p \le x \le q. \qquad (2.12b)$$

i.e. the initial value of U is given for every x value in the computational domain by a *known* function f(x).

A *solution* to (2.12a, 2.12b) is a function U = U(t, x) which satisfies the PDE (2.12a) at all points x in the computational domain and all times t *and* the *initial conditions* (2.12b). U(t, x), the exact solution of (2.12a,b), is defined at an *infinite* number of values of the independent variables t and x. We will create a FDS to approximate U at a *finite* set of values of the independent variables. The approximate values of U on this finite set will be denoted by u. We proceed in stages.

روش تفاضل محدود *Finite Difference Method*

➢ مثال یک بعدی (وابسته به زمان)

The computational domain (Figure 2.1) contains an infinite number of x values so first we must replace them by a finite set. This process is called spatial discretisation.

هندسه مسئله:

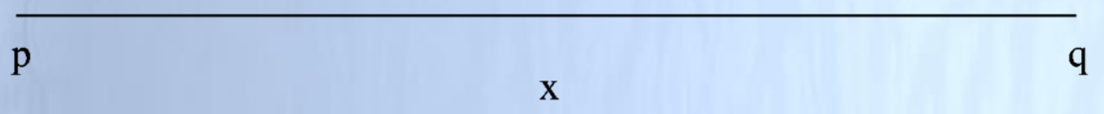


Figure 2.1: 1D computational domain.

For simplicity the computational domain is replaced by a grid of N *equally spaced* grid points. Starting with the first grid point at x = p and ending with the last grid point at x = q, the constant grid spacing, Δx, is,

گسسته سازی دامنه هندسی

$$\Delta x = \frac{(q-p)}{(N-1)} \qquad (2.13a)$$

N گره

روش تفاضل محدود *Finite Difference Method*

➢ مثال یک بعدی (وابسته به زمان)

The values of x in the discretised computational domain are indexed by subscripts to give,

$$x_1 = p, x_2 = p + \Delta x, \dots , x_i = p + (i-1)\,\Delta x, \dots , x_N = p + (N-1)\,\Delta x = q. \qquad (2.13b)$$

مختصات (موقعیت) گره ها

Since the grid spacing is constant,

$$x_{i+1} = x_i + \Delta x \qquad (2.13c)$$

The discretised computational domain is shown in Figure 2.2:

گسسته سازی دامنه هندسی (مش بندی)

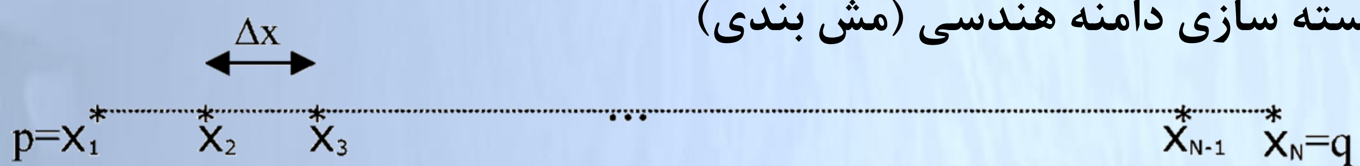


Figure 2.2 Discretised computational domain.

روش تفاضل محدود Finite Difference Method

➢ مثال یک بعدی (وابسته به زمان)

$U_t + v\, U_x = 0,$

Fixing t at $t = t_n$ we approximate the spatial partial derivative, U_x, in (2.12a) at each point (t_n, x_i) using the forward difference formula from the toolkit in Table 2.1 to give,

گسسته سازی ترم مکانی از معادله دیفرانسیل (مشتق مکانی)

$$U_x(t_n, x_i) \approx \frac{U_{i+1}^n - U_i^n}{\Delta x} \qquad (2.14)$$

مرحله اول گسسته سازی معادلات

Replacing U_x in (2.1a) by its approximation (2.14) gives,

جاگذاری ترم گسسته در معادله اصلی

$$U_t + v\frac{U_{i+1}^n - U_i^n}{\Delta x} = 0 \qquad (2.15)$$

(2.15) is said to be in *semi-discrete* form since only the spatial derivative has been discretised.

Note: The *grid* is also called the *mesh* and the operation of discretising the computational domain is called *gridding* or *meshing*.

Finite Difference Method

روش تفاضل محدود

➢ مثال یک بعدی (وابسته به زمان)

Fixing x at $x = x_i$ we approximate the temporal partial derivative, U_t, in (2.12a) at each point (t_n, x_i) using the first order forward difference formula from the toolkit in Table 2.1 (where Δt is the spacing between time levels) to give,

گسسته سازی ترم زمانی از معادله دیفرانسیل (مشتق زمان)

$$U_t \approx \frac{U_i^{n+1} - U_i^n}{\Delta t} \qquad (2.16)$$

On substituting (2.16) for U_t, 2.15 becomes,

جاگذاری ترم گسسته در معادله اصلی

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + v\frac{U_{i+1}^n - U_i^n}{\Delta x} = 0 \qquad (2.17a)$$

which rearranges to give,

معادله گسسته شده

$$U_i^{n+1} = U_i^n - \frac{v\Delta t}{\Delta x}\left(U_{i+1}^n - U_i^n\right) \qquad (2.17b)$$

روش تفاضل محدود *Finite Difference Method*

➢ مثال یک بعدی (وابسته به زمان)

مقدار U در نقطه iام و تایم استپ n+1

$$U_i^{n+1} = U_i^n - \frac{v\Delta t}{\Delta x}\left(U_{i+1}^n - U_i^n\right)$$

مقدار U در نقطه iام در تایم استپ n

$$U_i^n = U(t_n, x_i)$$

مقدار U در نقطه i+1ام در تایم استپ n

$$u_i^{n+1} = u_i^n - \frac{v\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

Uمقدار واقعی (دقیق)
u مقدار تقریبی (حاصل از حل عددی)

روش تفاضل محدود *Finite Difference Method*

➢ مثال یک بعدی (وابسته به زمان)

$$U_i^{n+1} = U_i^n - \frac{v\Delta t}{\Delta x}\left(U_{i+1}^n - U_i^n\right)$$

برای هر تایم استپ، یک مجموعه از معادلات برای کل گره ها و بر اساس مقادیر گره ها در تایم استپ قبل نوشته می شود.

برای حل، لازم است مقدار تابع u در گره ها و در زمان t=0 و یا تایم استپ اول بعنوان شرایط مرزی و یا اولیه مشخص باشد (یا با حل عددی معادلات مستقل از زمان و یا شرایط مرزی و یا شرایط اولیه)

تعداد N گره داخلی (مجهول)
تعداد m تایم استپ
تعداد N*m مجهول

گزینه اول: معادلات N مجهولی در تایم استپ های مختلف بطور جداگانه حل شوند. یعنی m بار معادله N مجهولی حل شود {روش صریح Implicit}

گزینه دوم: تعداد N*m معادله نوشته شده (بطور مثال جواب یک ماتریس N*m باشد) و تمام معادلات با هم حل شوند {روش ضمنی Explicit}

Finite Difference Method

روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس

Elliptic PDEs form a class of PDEs that may be used to model *steady state* problems (i.e. the dependent variable remains constant over time). Solutions of elliptic PDEs are over closed regions on which boundary values are given in some way. These boundary values determine the solution of the PDE in the interior of the region. The two most widely used elliptic PDEs are Laplace's equation and Poisson's equation.

In 2D, Laplace's equation is:

هدف:

$$U_{xx}+U_{yy}=0 \qquad (3.1)$$

Laplace's equation may be used to model a wide range of phenomena including steady state groundwater flow and temperature distribution over a region. Additionally Laplace's equation can describe 'potential flow' which can be used in a simplified description of water flow amongst other things.

Finite Difference Method روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

$U_{xx}+U_{yy}=0$ حل عددی معادله لاپلاس

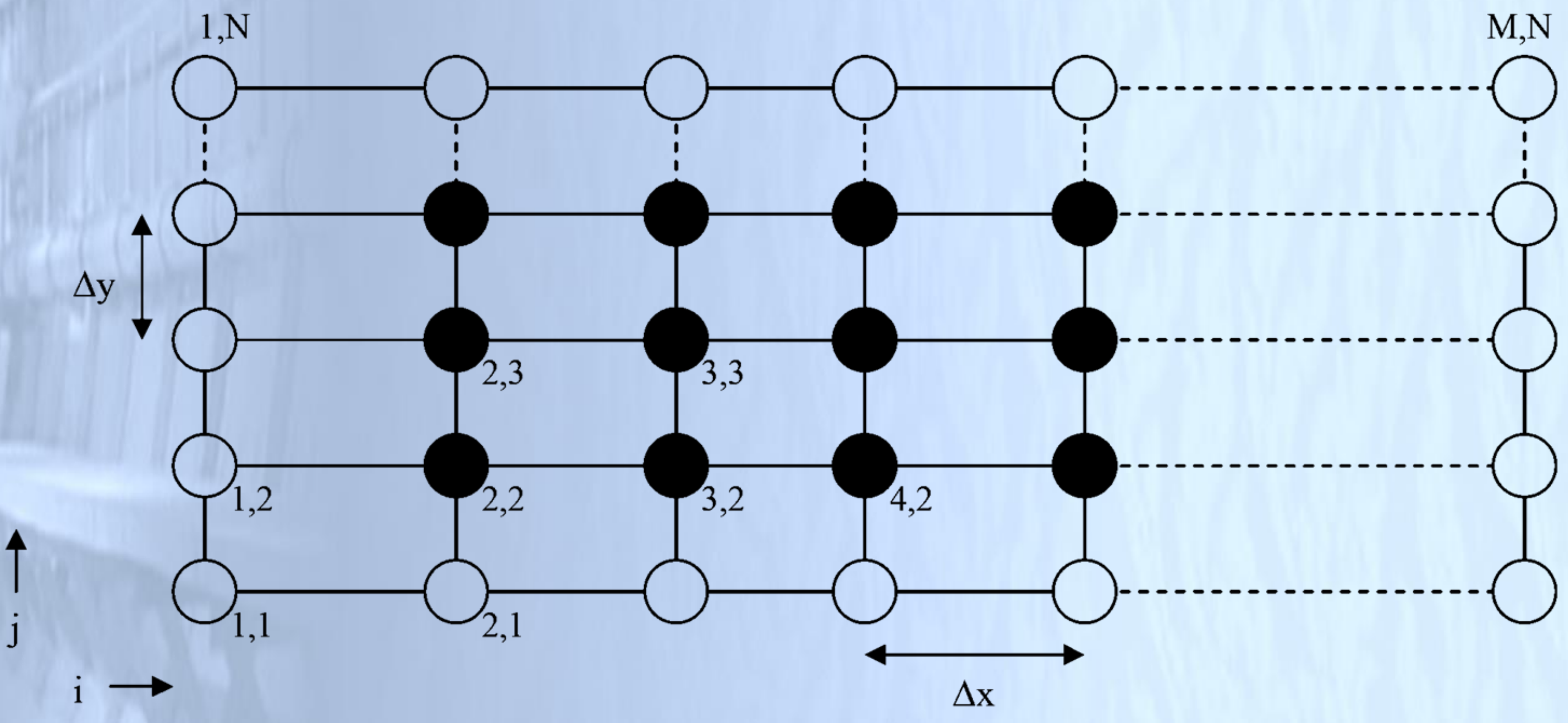


Figure 3.1: Computational grid showing interior grid points (black) and boundary grid points (white)

گسسته سازی دامنه به گره ها (مش بندی)

Finite Difference Method

روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

$U_{xx}+U_{yy}=0$ حل عددی معادله لاپلاس

Each partial derivative in Equation (3.1) is replaced by a symmetric FD approximation from our tool kit (Table 2.1) to give,

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}=0 \qquad (3.3a)$$

گسسته سازی

Letting $b = \Delta x/\Delta y$, (3.3a) can be rewritten to give,

نسبت ابعاد المان ها

$$u_{i,j}=\frac{u_{i+1,j}+u_{i-1,j}+b^2u_{i,j+1}+b^2u_{i,j-1}}{2(1+b^2)} \qquad (3.3b)$$

گسسته سازی

Equation (3.3b) shows that $u_{i,j}$ depends on its 4 surrounding values. This is called a 5-point stencil. Sometimes 'compass notation' is used and (3.3b) becomes,

بر اساس گره های مجاور

$$u_o=\frac{u_E+u_W+b^2u_N+b^2u_S}{2(1+b^2)} \qquad (3.3c)$$

where o denotes the current grid point and subscripts N, S, E and W denote its north, south, east and west neighbours respectively.

روش تفاضل محدود

Finite Difference Method

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

تعداد کل گره ها M*N

تعداد گره های داخلی: (M-2)*(N-2)

تعداد مجهول: (M-2)*(N-2)

گره های سفید: معلوم

گره های سیاه: مجهول

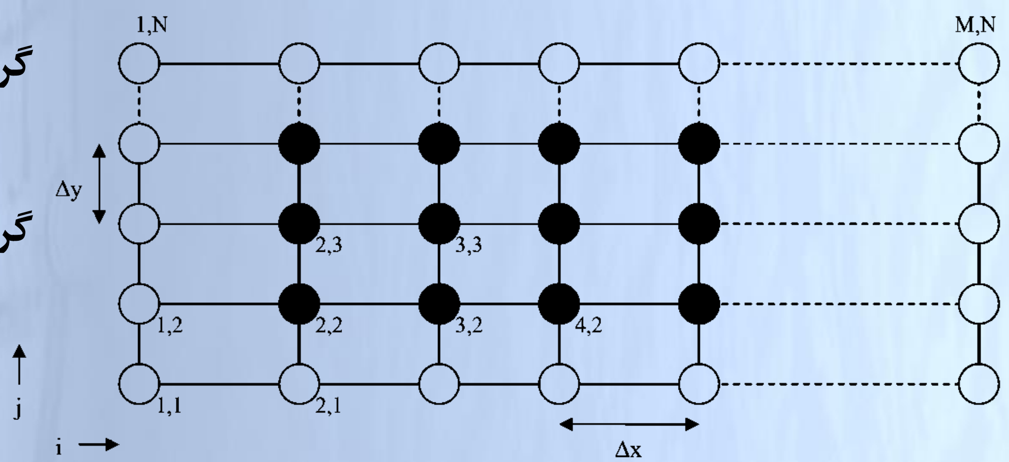


Figure 3.1: Computational grid showing interior grid points (black) and boundary grid points (white)

Notes:

1. Using the indexing system in Figure 3.1 the *unknown* value of u nearest to the bottom left hand corner of the computational domain is $u_{2,2}$ and the *unknown* value nearest to the top right hand corner of the computational domain is $u_{M-1,N-1}$.

2. In an MxN grid there will be (M-2)x(N-2) unknown interior values of $u_{i,j}$ which may be a very large number.

3. Assuming that the boundary values of u are known then (3.3b) gives a system of (M-2)x(N-2) linear equations for $u_{i,j}$ in (M-2)x(N-2) unknowns.

روش تفاضل محدود *Finite Difference Method*

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$u_{i,j}=\frac{u_{i+1,j}+u_{i-1,j}+b^2u_{i,j+1}+b^2u_{i,j-1}}{2(1+b^2)}$$

There are two basic methods of solving for $u_{i,j}$. Both methods set up a system of linear equations as follows. Letting $c = 1/(2(1+b^2))$,

$d = b^2/(2(1+b^2))$, rearranging Equation (3.3b) gives,

$$u_{i,j}=cu_{i-1,j}+du_{i,j-1}+du_{i,j+1}+cu_{i+1,j} \qquad (3.4)$$

$c = 1/(2(1+b^2))$ $\qquad$ $d = b^2/(2(1+b^2))$

$$u_{i,j}=cu_{i-1,j}+du_{i,j-1}+du_{i,j+1}+cu_{i+1,j}$$

Finite Difference Method روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$u_{i,j}=cu_{i-1,j}+du_{i,j-1}+du_{i,j+1}+cu_{i+1,j}$$

Evaluating Equation (3.4) at successive grid points starting at 2,2 and sweeping along the *rows first* gives,

$$\begin{aligned}
u_{2,2} &= cu_{1,2} + du_{2,1} + du_{2,3} + cu_{3,2} \\
u_{3,2} &= cu_{2,2} + du_{3,1} + du_{3,3} + cu_{4,2} \\
&\vdots \\
u_{M-1,2} &= cu_{M-2,2} + du_{M-1,1} + du_{M-1,3} + cu_{M,2} \qquad (3.5) \\
u_{2,3} &= cu_{1,3} + du_{2,2} + du_{2,4} + cu_{3,3} \\
&\vdots \\
u_{M-1,N-1} &= cu_{M-2,N-1} + du_{M-1,N-2} + du_{M-1,N} + cu_{M,N-1}
\end{aligned}$$

روش تفاضل محدود *Finite Difference Method*

➢ مثال دو بعدی تفاضل محدود

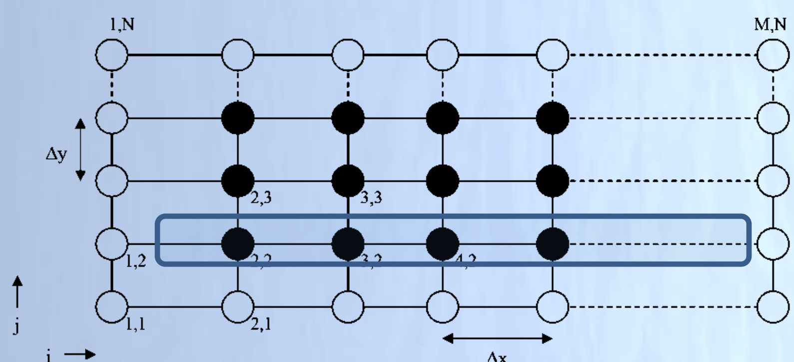
حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$u_{i,j}=cu_{i-1,j}+du_{i,j-1}+du_{i,j+1}+cu_{i+1,j}$$

برای سطر اول گره های مجهول

تعداد M-2 معادله

$$\left\{\begin{array}{lllll} u_{2,2} & =cu_{1,2} & +du_{2,1} & +du_{2,3} & +cu_{3,2} \\ u_{3,2} & =cu_{2,2} & +du_{3,1} & +du_{3,3} & +cu_{4,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{M-1,2} & =cu_{M-2,2} & +du_{M-1,1} & +du_{M-1,3} & +cu_{M,2} \end{array}\right\}$$



روش تفاضل محدود

Finite Difference Method

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$u_{i,j}=cu_{i-1,j}+du_{i,j-1}+du_{i,j+1}+cu_{i+1,j}$$

برای کل سطر های گره های مجهول

برای گره های داخلی (مجهول)

$$\left\{\begin{array}{lllll}
u_{2,2} & =cu_{1,2} & +du_{2,1} & +du_{2,3} & +cu_{3,2} \\
u_{3,2} & =cu_{2,2} & +du_{3,1} & +du_{3,3} & +cu_{4,2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
u_{M-1,2} & =cu_{M-2,2} & +du_{M-1,1} & +du_{M-1,3} & +cu_{M,2} \\
u_{2,3} & =cu_{1,3} & +du_{2,2} & +du_{2,4} & +cu_{3,3} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
u_{M-1,N-1} & =cu_{M-2,N-1} & +du_{M-1,N-2} & +du_{M-1,N} & +cu_{M,N-1}
\end{array}\right\}$$

تعداد (M-2) در (N-2) معادله

Finite Difference Method روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

$U_{xx}+U_{yy}=0$ حل عددی معادله لاپلاس

$$
\begin{array}{llllll}
u_{2,2} & =cu_{1,2} & +du_{2,1} & +du_{2,3} & +cu_{3,2} & \\
u_{3,2} & =cu_{2,2} & +du_{3,1} & +du_{3,3} & +cu_{4,2} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
u_{M-1,2} & =cu_{M-2,2} & +du_{M-1,1} & +du_{M-1,3} & +cu_{M,2} & (3.5) \\
u_{2,3} & =cu_{1,3} & +du_{2,2} & +du_{2,4} & +cu_{3,3} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
u_{M-1,N-1} & =cu_{M-2,N-1} & +du_{M-1,N-2} & +du_{M-1,N} & +cu_{M,N-1} &
\end{array}
$$

The boundary values $u_{1,2}$, $u_{2,1}$, $u_{M,2}$, $u_{1,3}$, etc. are *known*. In each equation the known u values are moved to the left hand side and the unknown u values moved to the right hand side (and positioned to preserve the ordering). For example in the first equation $u_{1,2}$ and $u_{2,1}$ are known (being boundary values) so the equation becomes,

$$-cu_{1,2}-du_{2,1}=-u_{2,2}+du_{2,3}+cu_{3,2}$$

The result is that Equations (3.5) can be written as a single matrix equation,

$$\underline{d} = A\,\underline{u} \qquad (3.6a)$$

روش تفاضل محدود *Finite Difference Method*

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$\underline{d} = A\,\underline{u} \qquad (3.6a)$$

where,

$\underline{d}$ is an (M-2)(N-2) by 1 matrix of known constants,

A is a (M-2)(N-2) by (M-2)(N-2) matrix of known coefficients and

$\underline{u}$ is a (M-2)(N-2) by 1 matrix of unknowns and

$\underline{u} = (u_{2,2}\ u_{3,2} \ldots u_{(M-1),2}\ u_{2,3}\ u_{3,3} \ldots u_{(M-1),3} \ldots \ldots u_{(M-1),(N-1)})^T.$

The solution to (3.6a) may be written symbolically as,

$$\underline{u} = A^{-1}\,\underline{d} \qquad (3.6b)$$

As A may be very large we must study efficient ways of finding $\underline{u}$.

Finite Difference Method روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

حل عددی معادله لاپلاس $U_{xx}+U_{yy}=0$

$$\underline{u}= A^{-1}\,\underline{d}$$

Solution Techniques:

Direct Solution Method
Iterative Solution Methods
Jacobi Iteration
Gauss-Seidel Iteration
Successive Over Relaxation (SoR) Method
Line SoR
……

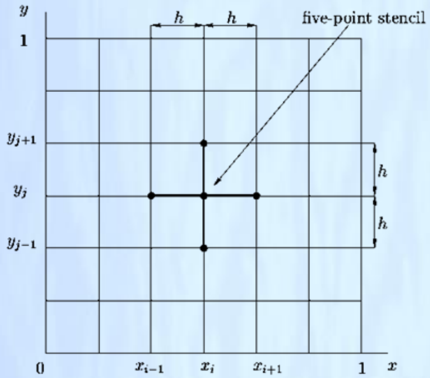
روش تفاضل محدود Finite Difference Method

➢ مثال دو بعدی تفاضل محدود

Example: 2D Poisson equation

Boundary value problem

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f & \text{in } \Omega = (0,1)\times(0,1) \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$



Uniform mesh: $\Delta x = \Delta y = h, \quad N = \frac{1}{h}$

$$u_{i,j} \approx u(x_i, y_j), \quad f_{i,j} = f(x_i, y_j), \quad (x_i, y_j) = (ih, jh), \quad i,j = 0,1,\ldots,N$$

Central difference approximation $\mathcal{O}(h^2)$

SEE NEXT PAGE

$$\begin{cases} -\frac{u_{i-1,j}+u_{i,j-1}-4u_{i,j}+u_{i+1,j}+u_{i,j+1}}{h^2} = f_{i,j}, & \forall i,j = 1,\ldots,N-1 \\ u_{i,0} = u_{i,N} = u_{0,j} = u_{N,j} = 0 & \forall i,j = 0,1,\ldots,N \end{cases}$$

Finite Difference Method روش تفاضل محدود

FDM by Taylor's Theorem ➢ تقریب المان محدود با تئوری تیلور

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_i^n}{\Delta x}$	forward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_i^n - U_{i-1}^n}{\Delta x}$	backward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$	central	second in x
$\frac{\partial^2 U}{\partial x^2} = U_{xx}$	$\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$	symmetric	second in x
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^n}{\Delta t}$	forward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^n - U_i^{n-1}}{\Delta t}$	backward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}$	central	second in t
$\frac{\partial^2 U}{\partial t^2} = U_{tt}$	$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}$	symmetric	second in t

Finite Difference Method روش تفاضل محدود

مثال دو بعدی تفاضل محدود

Example: 2D Poisson equation

Linear system $\boxed{Au = F}$ $\quad A \in \mathbb{R}^{(N-1)^2 \times (N-1)^2} \quad u, F \in \mathbb{R}^{(N-1)^2}$

row-by-row $\quad u = [u_{1,1} \dots u_{N-1,1}\ u_{1,2} \dots u_{N-1,2}\ u_{1,3} \dots u_{N-1,N-1}]^T$

node numbering $\quad F = [f_{1,1} \dots f_{N-1,1}\ f_{1,2} \dots f_{N-1,2}\ f_{1,3} \dots f_{N-1,N-1}]^T$

$$A = \begin{bmatrix} B & -I & & & \\ -I & B & -I & & \\ & \dots & \dots & & \\ & & -I & B & -I \\ & & & -I & B \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \dots & \dots & & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

The matrix A is sparse, block-tridiagonal (for the above numbering) and SPD.

$$\text{cond}_2(A) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} = \mathcal{O}(h^{-2})$$

Caution: convergence of iterative solvers deteriorates as the mesh is refined

روش تفاضل محدود Finite Difference Method

➢ مثال دو بعدی تفاضل محدود

Example: 2D Poisson equation

Linear system $\boxed{Au = F}$ $\quad A \in \mathbb{R}^{(N-1)^2\times(N-1)^2} \quad u, F \in \mathbb{R}^{(N-1)^2}$

row-by-row node numbering

$$u = [u_{1,1} \dots u_{N-1,1}\ u_{1,2} \dots u_{N-1,2}\ u_{1,3} \dots u_{N-1,N-1}]^T$$

$$F = [f_{1,1} \dots f_{N-1,1}\ f_{1,2} \dots f_{N-1,2}\ f_{1,3} \dots f_{N-1,N-1}]^T$$

$$A = \begin{bmatrix} B & -I & & & \\ -I & B & -I & & \\ & \dots & \dots & & \\ & & -I & B & -I \\ & & & -I & B \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \dots & \dots & & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}$$

ماتریس (N-1)*(N-1)

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

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Finite Difference Method

روش تفاضل محدود

➢ مثال دو بعدی تفاضل محدود

Example: 2D Poisson equation

Linear system $\boxed{Au = F}$ $\quad A \in \mathbb{R}^{(N-1)^2 \times (N-1)^2} \quad u, F \in \mathbb{R}^{(N-1)^2}$

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$$u = [u_{1,1} \dots u_{N-1,1}\ u_{1,2} \dots u_{N-1,2}\ u_{1,3} \dots u_{N-1,N-1}]^T$$

$$F = [f_{1,1} \dots f_{N-1,1}\ f_{1,2} \dots f_{N-1,2}\ f_{1,3} \dots f_{N-1,N-1}]^T$$

$$A = \begin{bmatrix} B & -I & & & \\ -I & B & -I & & \\ & \dots & \dots & & \\ & & -I & B & -I \\ & & & -I & B \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \dots & \dots & & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}$$

ماتریس
(N-1)^2*(N-1)^2

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

The matrix A is sparse, block-tridiagonal (for the above numbering) and SPD.

$$\text{cond}_2(A) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} = \mathcal{O}(h^{-2})$$

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