1. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}+u=0 \quad , \quad \begin{Bmatrix} u(0,y,z)=0 & , & u(a,y,z)=0 \\ u(x,0,z)=0 & , & u(x,b,z)=0 \\ u(x,y,0)=\alpha & , & u(x,y,c)=\beta \end{Bmatrix}$. (Hint: The problem is assumed linear in z)

Ans.

$$u(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{mn}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(E_{mn}z+F_{mn}\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

$$\text{where,}\quad E_{mn}=\begin{cases}0 & ; \ n,m\equiv even \\ \dfrac{16(\alpha-\beta)}{nm\pi^2 c} & ; \ n,m\equiv odd\end{cases} \quad \& \quad F_{mn}=\begin{cases}0 & ; \ n,m\equiv even \\ \dfrac{16\alpha}{nm\pi^2} & ; \ n,m\equiv odd\end{cases}$$

2. $\dfrac{1}{\alpha}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} \quad , \quad \left\{u(x,0)=f(x) \quad , \quad u(0,t)=0 \quad ; \quad \dfrac{\partial u}{\partial x}(\ell,t)=0\right\}$

Ans.

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left[\frac{\pi}{2\ell}(2n-1)x\right]\exp\left[-\alpha\frac{n^2(2n-1)^2}{(2\ell)^2}t\right]$$

$$\text{where,}\qquad B_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\left[\frac{\pi}{2\ell}(2n-1)x\right]dx$$

3. $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}-g \quad , \quad \begin{Bmatrix} u(0,t)=0 \quad ; \quad u(\pi,t)=0, & u(x,0)=f(x) \\ & u_t(x,0)=0 \end{Bmatrix}$, and g is constant.

Ans.

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos(nct)+B_n\sin(nct)-2g\frac{1-\cos(n\pi)}{c^2n^3\pi}\right]\sin(nx)$$

$$\text{where,}\quad A_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx+2g\frac{1-\cos(n\pi)}{c^2n^3\pi} \quad \& \quad B_n=0$$

4. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0 \quad , \quad \begin{Bmatrix} \dfrac{\partial u}{\partial \theta}\Big|_{\theta=0}=0 & , & \dfrac{\partial u}{\partial \theta}\Big|_{\theta=\pi}=0 \\ u(a,\theta)=f(\theta) & , & \left|u(0,\theta)\right|<\infty \end{Bmatrix}$, Note: As described by the equation and its boundary conditions, the geometry is a semi-circle with $0\le r\le a$ and $0\le\theta\le\pi$.

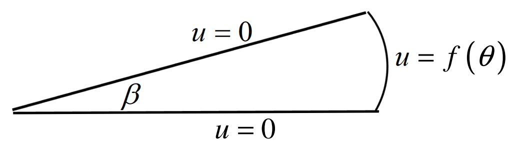
Ans. $u(r,\theta)=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}f(\theta)\,d\theta+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\left(\dfrac{r}{a}\right)^n\cos(n\theta)\int_0^{\pi}f(\theta)\cos(n\theta)\,d\theta$

5. $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad , \quad \left\{\begin{matrix} u(r,0)=0 & ; & u(r,\pi)=0 \\ u(1,\theta)=0 & ; & u(b,\theta)=u_0 \end{matrix}\right\}$

Ans. $u(r,\theta)=\frac{4u_0}{\pi}\sum_{k=0}^{\infty}\frac{r^{2k+1}-r^{-(2k+1)}}{b^{2k+1}-b^{-(2k+1)}}\frac{\sin\left[(2k+1)\theta\right]}{2k+1}$

6. $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0;$

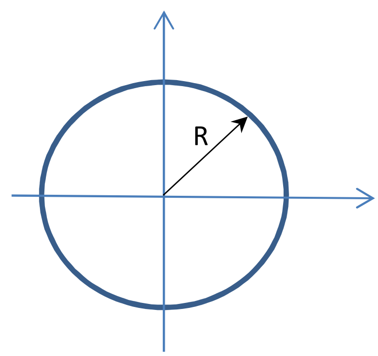
Note: Please see the figure (a sector of a circle with radius a).



Ans. $u(r,\theta)=\frac{2}{\beta}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{\frac{n\pi}{\beta}}\left[\int_0^{\beta} f(\eta)\sin\left(\frac{n\pi}{\beta}\eta\right)d\eta\right]\sin\left(\frac{n\pi}{\beta}\theta\right)$

7. $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad , \quad \left\{\frac{\partial u}{\partial r}(R,\theta)=\theta\right\}$

Note: A circle with radius R.



Ans. $u(r,\theta)=\pi-2\sum_{n=0}^{\infty}\frac{r^n}{n^2R^{n-1}}\sin(n\theta)$

8. $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2} \quad , \quad \left\{\begin{matrix} \frac{\partial u}{\partial x}\Big|_{x=0}=0 & ; & \frac{\partial u}{\partial x}\Big|_{x=\ell}=0 \\ u(x,0)=\sin\left(\frac{\pi}{\ell}x\right) & & \end{matrix}\right\}$

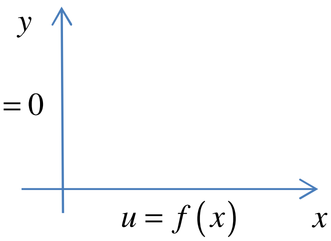
Ans. $u(x,t)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos\left(\frac{k\pi}{\ell}x\right)\exp\left(-\alpha\left(\frac{k\pi}{\ell}\right)^2 t\right)$

9. $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}+A\sin(\omega t) \quad , \quad \{u(0,t)=0 \;\; ; \;\; u(\ell,t)=0\}$. Note: The steady-state solution is obtained using sine Fourier transform.

Ans. $u(x,t)=\sum_{n=1}^{\infty}\left[A_n\sin(\lambda_n t)+B_n\cos(\lambda_n t)+2A\frac{1-\cos(n\pi)}{n\pi(\lambda_n^2-\omega^2)}\right]\sin\left(\frac{n\pi}{\ell}x\right)$ where, $\lambda_n=\frac{n\pi}{\ell}c$

10. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0 \quad , \quad \begin{cases} u(0,y)=0 \quad ; \quad u(x,0)=f(x) \\ u \text{ is bounded as } x\to\infty \text{ and } y\to\infty. \end{cases}$

Note: As shown in the Fig., the region is semi-infinite $(x>0,\, y>0)$.



Ans. $u(x,y)=\dfrac{1}{\pi}\displaystyle\int_0^\infty y f(v)\left[\frac{y}{y^2+(x-v)^2}-\frac{y}{y^2+(x+v)^2}\right]dv$

11. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0 \quad , \quad \begin{cases} u(0,y)=0 \quad ; \quad u(1,y)=0 \\ u \text{ is bounded as } y\to\infty. \quad u(x,0)=100 \end{cases}$

Ans. $u(x,y)=\displaystyle\sum_{k=0}^{\infty}\frac{400}{\pi(2k+1)}\sin\left[(2k+1)\pi x\right]\exp\left[-(2k+1)\pi y\right]$

12. $\dfrac{\partial u}{\partial t}=\kappa\dfrac{\partial^2 u}{\partial x^2}+H(x,t) \quad , \quad \begin{cases} u(0,t)=0 \quad ; \quad u(L,t)=0 \\ u(x,0)=U(x) \end{cases}$

Ans.

$$u(x,t)=\sum_{n=1}^{\infty}a_n\exp\left[-\left(\frac{n\pi}{L}\right)^2\kappa t\right]\sin\left(\frac{n\pi}{L}x\right)+\sum_{n=1}^{\infty}\left[\int_0^t\exp\left(-\left(\frac{n\pi}{L}\right)^2\kappa(t-s)\right)H_n(s)\,ds\right]\sin\left(\frac{n\pi}{L}x\right)$$

where, $a_n=\dfrac{2}{L}\displaystyle\int_0^L U(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad ; \quad n=1,2,3,\ldots$