Chapter 7.
Time-Varying Fields and Maxwell's Equations

Electrostatic & Time Varying Fields

- Electrostatic fields

$$\nabla \times \mathbf{E} = 0, \ \nabla \bullet \mathbf{D} = \rho_{\upsilon}$$

$$\nabla \bullet \mathbf{B} = 0, \ \nabla \times \mathbf{H} = \mathbf{J}$$

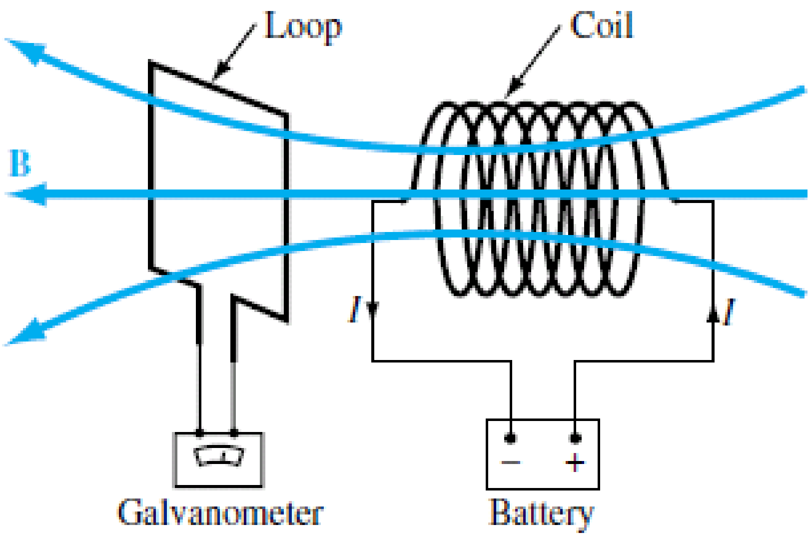
$$\mathbf{D} = \varepsilon \mathbf{E}$$

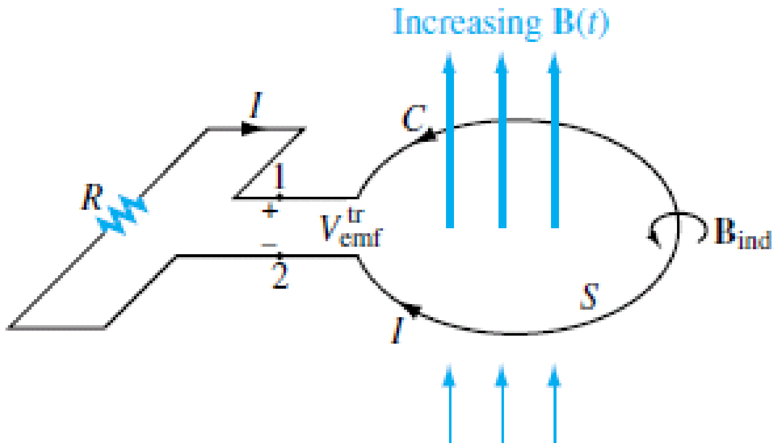
$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

- In the electrostatic model, electric field and magnetic fields are not related each other.

Faraday's law

- A major advance in EM theory was made by M. Faraday in 1831 who discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed.





electromotive force (emf): $V = -\frac{d\Phi}{dt}$ (V)

Faraday's law

- Fundamental postulate for electromagnetic induction is

$$V = -\frac{d\Phi}{dt} \rightarrow V = \oint_C \mathbf{E} \bullet dl \rightarrow \oint_C \mathbf{E} \bullet dl = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} \Rightarrow \boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

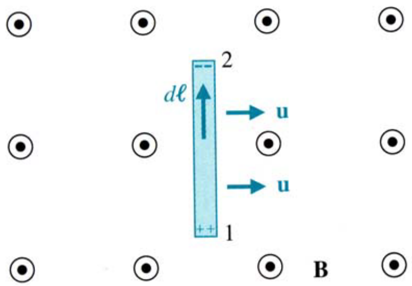
- The electric field intensity in a region of time-varying magnetic flux density is therefore non conservative and cannot be expressed as the negative gradient of a scalar potential.

- The negative sign is an assertion that the induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux → Lentz's law

$$V = -\frac{d\Phi}{dt} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

7-2.3. A moving conductor in a static magnetic field

- Charge separation by magnetic force



$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$

- *To an observer moving with the conductor, there is no apparent motion and the magnetic force can be interpreted as an inducted electric field acting along the conductor and producing a voltage.*

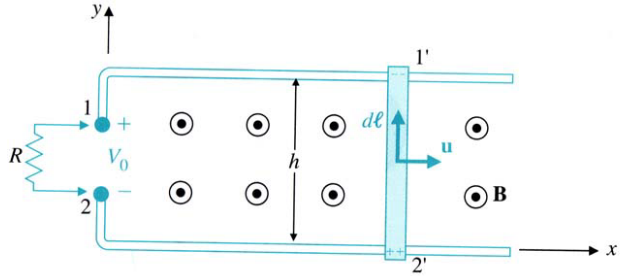
$$V_{21} = \int_1^2 (\mathbf{u} \times \mathbf{B}) \bullet dl$$

- Around a circuit, *motional emf* or *flux cutting emf*

$$V_{21} = \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl$$

A moving conductor in a static magnetic field

- Example 7-2



(a) Open voltage V_0 ?
(b) Electric power in R
(c) Mechanical power required to move the sliding bar

$$(a)\ V_0 = V_1 - V_2 = \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl = \int_2^1 (\mathbf{a}_x u \times \mathbf{a}_z B_0) \bullet (\mathbf{a}_y dl) = -uB_0 h$$

$$(b)\ I = \frac{V_0}{R} = \frac{uB_0 h}{R} \rightarrow P_e = I^2 R = \frac{(uB_0 h)^2}{R} \quad (\mathrm{W})$$

$$(c)\ P_M = \mathbf{F}_M \bullet \mathbf{u}, \quad \mathbf{F}_M = \text{mechanical force to counteract the magnetic force } \mathbf{F}_{\mathrm{m}}$$

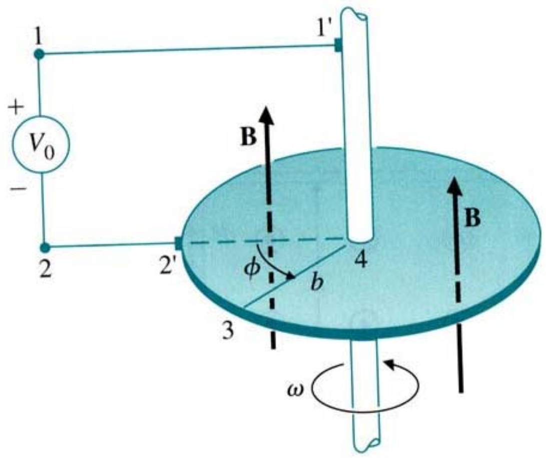
$$\mathbf{F}_{mag} = I \int_2^1 dl \times \mathbf{B} = -\mathbf{a}_x I B_0 h \quad (\mathrm{N}) \quad \rightarrow \mathbf{F}_M = -\mathbf{F}_{mag}$$

$$I = \frac{uB_0 h}{R} \rightarrow \mathbf{F}_M = \mathbf{a}_x \frac{u (B_0 h)^2}{R} \rightarrow P_M = \frac{u^2 (B_0 h)^2}{R} \quad (\mathrm{W})$$

$$\Rightarrow\ P_e = P_M$$

A moving conductor in a static magnetic field

- Example 7-3. Faraday disk generator



$$V_0 = \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl = \int_3^4 \left[\left(\mathbf{a}_\phi r\omega \times B_0 \mathbf{a}_z \right) \bullet \left(\mathbf{a}_r dr \right) \right]$$

$$= \omega B_0 \int_b^0 r dr = -\frac{\omega B_0 b^2}{2} \quad (\mathrm{V})$$

Magnetic force & electric force

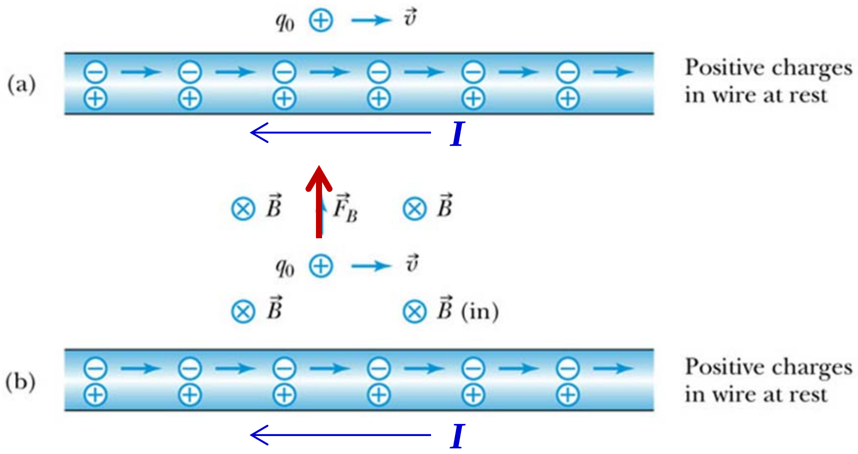
$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

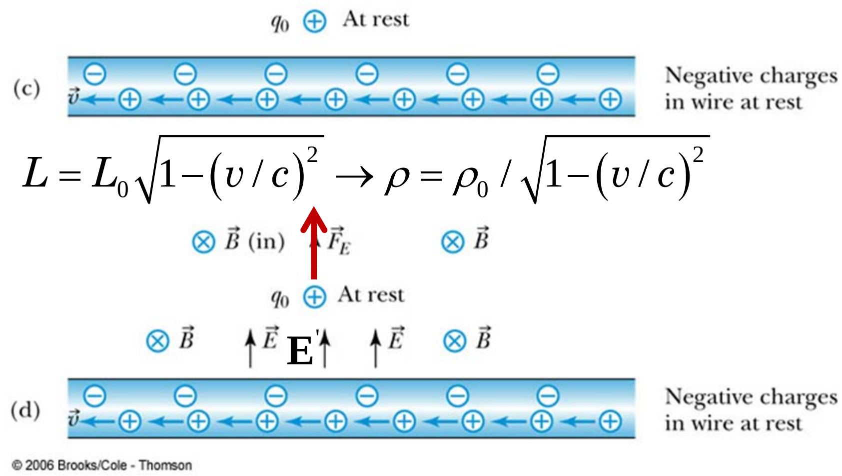
When a charge q_0 moves parallel to the current on a wire,
the magnetic force on q_0 is equivalent to the electric force on q_0.

At the rest frame on wire

At the moving frame on charge

$$\mathbf{F}_B = q_0 \mathbf{v} \times \mathbf{B} \longleftrightarrow \mathbf{F}_E = q_0 \mathbf{E}'$$





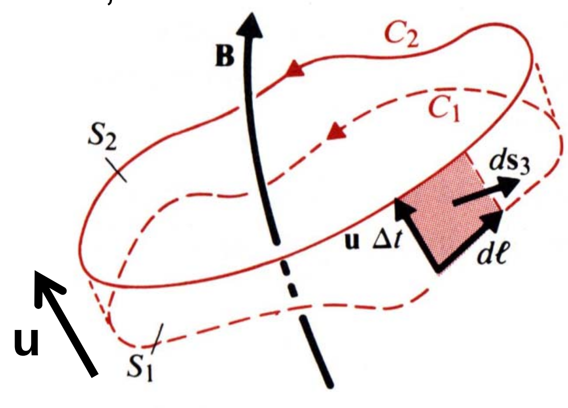
$$L = L_0\sqrt{1-(v/c)^2} \rightarrow \rho = \rho_0 / \sqrt{1-(v/c)^2}$$

➔ To observer moving with q_0 under E and B fields, there is no apparent motion.
But, the force on q_0 can be interpreted as caused by an electric field, E'.

7-2.4. A moving circuit in a time-varying magnetic field

➔ To observer moving with q_0 under E and B fields, there is no apparent motion. But, the force on q_0 can be interpreted as caused by an electric field, E'.

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$



Now, consider a conducting circuit with contour C and surface S moves with a velocity ***u*** under static **E** and **B** fields.

Changing in magnetic flux due to the circuit movement produces an emf, V:

$$-\frac{d\Phi}{dt} = V_B$$

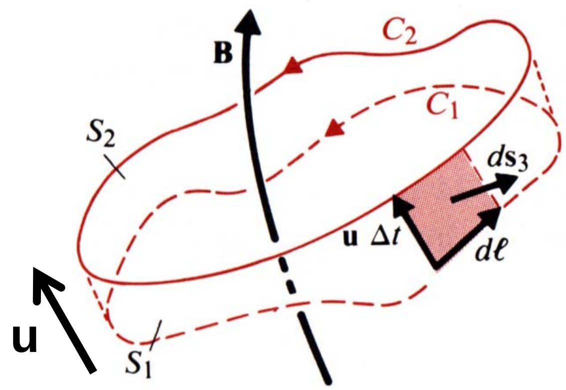
On the other hand, the moving circuit experiences an emf, V', due to E':

$$\oint_C \mathbf{E}' \cdot dl = V_{E'}$$

➔ Is it true that $V_B = V_{E'}$

$V_B = V_{E'}$??

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$



From the Faraday's law of $\oint_C \mathbf{E} \bullet dl = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s}$,

by replacing $\mathbf{E}$ with $\mathbf{E} = \mathbf{E}' - \mathbf{u} \times \mathbf{B}$,

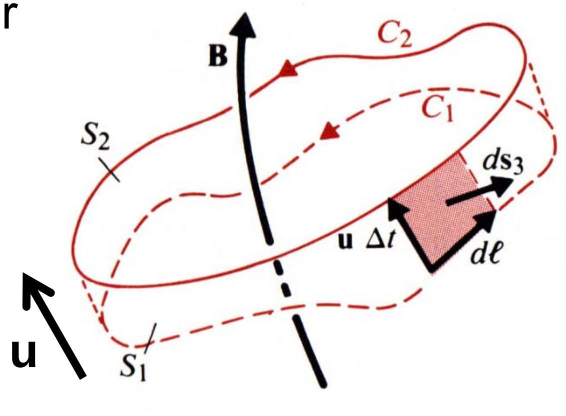
$$\oint_C \mathbf{E}' \bullet dl = -\underbrace{\oint_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s}}_{\text{time variation at rest}} + \underbrace{\oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl}_{\text{motional emf}} \qquad (\mathrm{V})$$

Note that $V_{E'} = \oint_C \mathbf{E}' \bullet dl$ $\qquad V_B = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \bullet d\mathbf{s}$

Therefore, we need to prove that $\frac{d}{dt}\int_S \mathbf{B} \bullet d\mathbf{s} = \oint_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl$

$V_B = V_{E'}$??

- Time-rate of change of magnetic flux through the contour



$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B}\bullet d\mathbf{s} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\bullet d\mathbf{s}_2 - \int_{S_1}\mathbf{B}(t)\bullet d\mathbf{s}_1\right]$$

$$\mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t}\Delta t + \text{H.O.T.},$$

$$\frac{d}{dt}\int_S \mathbf{B}\bullet d\mathbf{s} = \int_S \frac{\partial \mathbf{B}(t)}{\partial t}\bullet d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}\bullet d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\bullet d\mathbf{s}_1 + \text{H.O.T.}\right]$$

- In going from C_1 to C_2, the circuit covers a region bounded by S_1, S_2, and S_3

$$\int_V \nabla\bullet\mathbf{B}\, d\upsilon = 0 = \int_{S_2}\mathbf{B}\bullet d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\bullet d\mathbf{s}_1 + \int_{S_3}\mathbf{B}\bullet d\mathbf{s}_3$$

$$d\mathbf{s}_3 = d\mathbf{l}\times\mathbf{u}\Delta t \rightarrow \int_{S_2}\mathbf{B}\bullet d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\bullet d\mathbf{s}_1 = -\Delta t\oint_C(\mathbf{u}\times\mathbf{B})\bullet d\mathbf{l}$$

$$emf = V' = \oint_C \mathbf{E}'\bullet d\mathbf{l}$$

$$\rightarrow \frac{d}{dt}\int_S \mathbf{B}\bullet d\mathbf{s} = \int_S \frac{\partial \mathbf{B}(t)}{\partial t}\bullet d\mathbf{s} - \oint_C(\mathbf{u}\times\mathbf{B})\bullet d\mathbf{l}$$

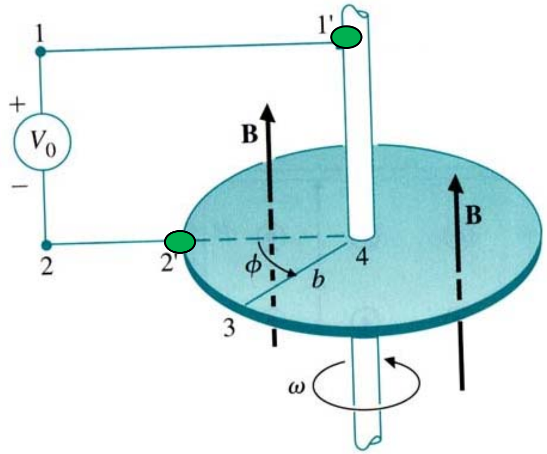
- Therefore, the *emf* induced in the moving circuit C is equivalent to the emf induced by the change in magnetic flux

$$V_{E'} = V_B \quad\Rightarrow\quad V = -\frac{d\Phi}{dt}$$

: same form as not in motion.

A moving circuit in a time-varying magnetic field

- Example 7.3
  - Determine the open-circuit voltage of the Faraday disk generator



$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = B_0 \int_0^b \int_0^{\omega t} r\,dr\,d\phi = B_0 (\omega t) \frac{b^2}{2}$$

$$V_B = -\frac{d\Phi}{dt} = -\frac{\omega B_0 b^2}{2}$$

Compare!

$$V_{E'} = \oint_C \mathbf{E}' \cdot dl = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot dl$$

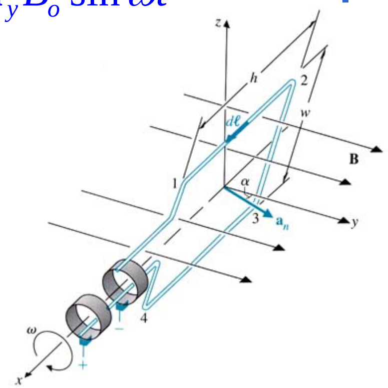
$$V_{E'} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot dl = \int_3^4 \left[(\mathbf{a}_\phi r\omega \times B_0 \mathbf{a}_z) \cdot (\mathbf{a}_r dr) \right]$$

$$= \omega B_0 \int_b^0 r\,dr = -\frac{\omega B_0 b^2}{2}$$

$$\Rightarrow V_B = V_{E'}$$

$$V_{E'} = \oint_C \mathbf{E}' \bullet dl = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \bullet ds + \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl \qquad V_B = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \bullet ds$$

(Example 7-4) Find the induced emf in the rotating loop under $\mathbf{B}(t) = \mathbf{a}_y B_o \sin \omega t$



(a) when the loop is at rest with an angle α.

$$\Phi = \int \mathbf{B} \bullet ds = (\mathbf{a}_y B_o \sin \omega t) \bullet (\mathbf{a}_n hw) = B_0 hw \sin \omega t \cos \alpha$$

$$V_a = -\frac{d\Phi}{dt} = -B_0 S \omega \cos \omega t \cos \alpha, \ \ (S = hw : \text{the area of the loop})$$

(b) When the loop rotates with an angular velocity ω about the *x*-axis

$$V_a' = \oint_C (\mathbf{u} \times \mathbf{B}) \bullet dl = \int_2^1 [(a_n \frac{w}{2}\omega) \times (a_y B_0 \sin \omega t)] \cdot a_x dx + \int_4^3 [(-a_n \frac{w}{2}\omega) \times (a_y B_0 \sin \omega t)] \cdot a_x dx$$

$$= 2(\frac{w}{2}\omega B_0 \sin \omega t \sin \omega t)h = B_0 S \omega \sin \omega t \sin \alpha \quad (\alpha = \omega t)$$

$$\Rightarrow V_{E'} = V_a + V_a' = B_0 S \omega (\cos^2 \omega t - \sin^2 \omega t) = -B_0 S \omega \cos 2\omega t$$

Compare!

$$\Phi(t) = \mathbf{B}(t) \bullet [\mathbf{a}_n(t) S] = B_0 S \sin \omega t \cos \alpha = B_0 S \sin \omega t \cos \omega t = \frac{1}{2} B_0 S \sin(2\omega t)$$

$$\Rightarrow V_B = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{1}{2} B_0 S \sin(2\omega t)\right) = -B_0 S \omega \cos 2\omega t \quad \Rightarrow \quad V_B = V_{E'}$$

7-3. How Maxwell fixed Ampere's law?

- We now have the following collection of two curl eqns. and two divergence eqns.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \bullet \mathbf{D} = \rho_\upsilon, \qquad \nabla \bullet \mathbf{B} = 0$$

- Charge conservation law → the equation of continuity

$$\nabla \bullet \mathbf{J} = -\frac{\partial \rho_\upsilon}{\partial t}$$

- The set of four equations is now consistent with the equation of continuity?

Taking the divergence of $\nabla \times \mathbf{H} = \mathbf{J}$,

$\nabla \bullet (\nabla \times \mathbf{H}) = \nabla \bullet \mathbf{J} = 0 \leftarrow \nabla \bullet (\nabla \times \mathbf{H}) = 0$ from the vector null identity

$\nabla \bullet \mathbf{J}$ does not vanish in a time-varying situation

and this equation is, in general, not true.

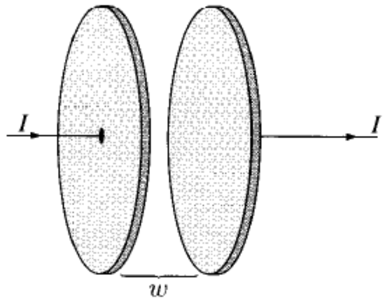
Maxwell's equations

- A term $\partial \rho_v / \partial t$ must be added to the equation.

$$\nabla \bullet (\nabla \times \mathbf{H}) = 0 = \nabla \bullet \mathbf{J} + \frac{\partial \rho_\upsilon}{\partial t} = \nabla \bullet \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \leftarrow \nabla \bullet \mathbf{D} = \rho_\upsilon$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times B = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$



- The additional term $\partial \mathbf{D}/\partial t$ means that a time-varying electric field will give rise to a magnetic field, even in the absence of a free current flow ($\mathbf{J}=0$).

- $\partial \mathbf{D}/\partial t$ is called displacement current (density).

Maxwell's equations

- Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \bullet \mathbf{D} = \rho_\upsilon$$

$$\nabla \bullet \mathbf{B} = 0$$

Continuity equation

$$\nabla \bullet \mathbf{J} = -\frac{\partial \rho_\upsilon}{\partial t}$$

Lorentz's force

$$F = q(E + \upsilon \times B)$$

- These four equations, together with the equation of continuity and Lorentz's force equation form the foundation of electromagnetic theory. These equations can be used to explain and predict *all* macroscopic electromagnetic phenomena.

- The four Maxwell's equations are not all independent
  - *The two divergence equations can be derived from the two curl equations by making use of the equation of continuity*

Maxwell's equations

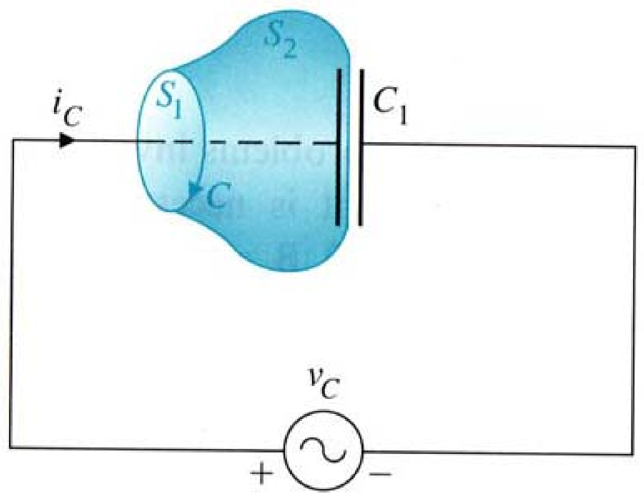
- Integral form & differential form of Maxwell's equations

Differential form	Integral form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \bullet dl = -\int_S \frac{\partial B}{\partial t} \cdot dS = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \bullet dl = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \bullet d\mathbf{s}$	Ampere's law
$\nabla \bullet \mathbf{D} = \rho_\upsilon$	$\oint_C \mathbf{D} \bullet d\mathbf{s} = Q$	Gauss's law
$\nabla \bullet \mathbf{B} = 0$	$\oint_S \mathbf{B} \bullet d\mathbf{s} = 0$	No isolated magnetic charge

Maxwell's equations

- Example 7-5
  - Verify that the displacement current = conduction current in the wire.

i_C: conduction current in the connecting wire



$\upsilon_C = V_0 \sin \omega t$

$$i_C = C_1 \frac{d\upsilon_C}{dt} = C_1 V_0 \omega \cos \omega t$$

$$C_1 = \frac{\varepsilon A}{d}$$

$$E = \frac{\upsilon_C}{d}\text{: uniform between the plates}$$

$$\rightarrow D = \varepsilon E = \varepsilon \frac{V_0}{d} \sin \omega t$$

$$i_D = \int_A \frac{\partial \mathbf{D}}{\partial t} \bullet d\mathbf{s} = \left(\varepsilon \frac{V_0}{d} \omega \cos \omega t \right) A = C_1 V_0 \omega \cos \omega t = i_C$$

Maxwell's equations

- Example 7-5
  - Determine the magnetic field intensity at a distance *r* from the wire

Two typical open surfaces with rim C may be chosen:

(a) a planar disk surface S_1, (b) a curved surface S_2

For the surface S_1, $\mathbf{D} = 0$

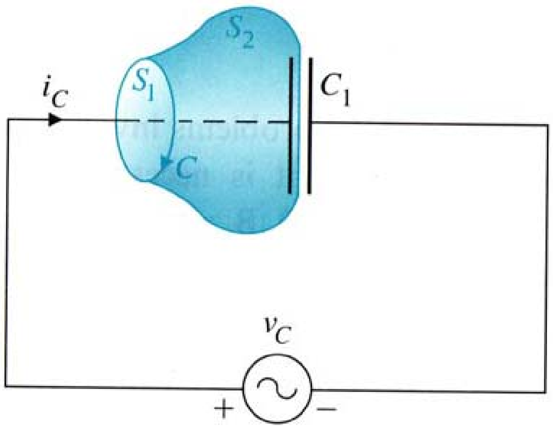
$$\oint_C \mathbf{H} \bullet dl = 2\pi r H_\phi = \int_{S_1} \mathbf{J} \bullet d\mathbf{s} = i_C = C_1 V_0 \omega \cos \omega t \rightarrow H_\phi = \frac{C_1 V_0}{2\pi r} \omega \cos \omega t$$

Since the surface S_2 passes through the dielectric medium,

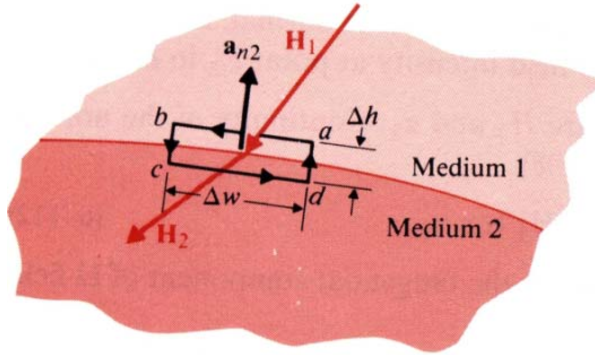
no conducting current flows through $S_2 \rightarrow displacement\ current$

$$2\pi r H_\phi = i_D = \int_A \frac{\partial \mathbf{D}}{\partial t} \bullet d\mathbf{s} = i_C$$

$$\rightarrow H_\phi = \frac{C_1 V_0}{2\pi r} \omega \cos \omega t$$



7-5. Electromagnetic Boundary Conditions



$$\oint_C \mathbf{E} \bullet dl = -\int_S \frac{\partial B}{\partial t} \cdot dS \;\rightarrow\; 0 \;\; when \; \Delta h \rightarrow 0$$

$$\Rightarrow E_{1t} = E_{2t}$$

$$\oint_C \mathbf{H} \bullet dl = \int_S \left(J + \frac{\partial \mathbf{D}}{\partial t} \right) \bullet \mathbf{ds} \; ; \quad \int_S \left(\frac{\partial \mathbf{D}}{\partial t} \right) \bullet \mathbf{ds} \rightarrow \; 0 \; when \; \Delta h \rightarrow 0$$

$$\Rightarrow a_{n2} \times \left(H_1 - H_2 \right) = J_s \quad \left(H_{1t} - H_{2t} = J_{sn} \right)$$

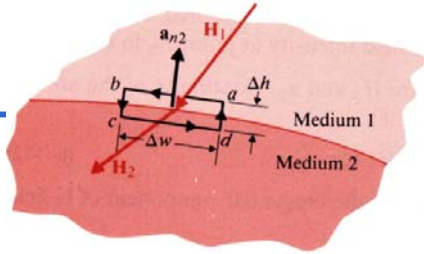
$$\oint_s \mathbf{D} \cdot dS = \int_V \rho d\upsilon \;\rightarrow\; \mathbf{a}_{n_2} \bullet \left(\mathbf{D}_1 - \mathbf{D}_2 \right) = \rho_s \;\; when \; \Delta h \rightarrow 0$$

$$\Rightarrow D_{1n} - D_{2n} = \rho_s$$

$$\oint_s B \cdot dS = 0$$

$$\Rightarrow B_{1n} - B_{2n} = 0 \quad \left(\mu_1 H_{1n} = \mu_2 H_{2n} \right)$$

Electromagnetic Boundary Conditions



Both static and time-varying electromagnetic fields satisfy the same boundary conditions:

$$E_{1t} = E_{2t}$$

➔ *The tangential component of an E field is continuous across an interface.*

$$H_{1t} - H_{2t} = J_{sn}$$

➔ *The tangential component of an H field is discontinuous across an interface where a surface current exists.*

$$B_{1n} = B_{2n}$$

➔ *The normal component of an B field is continuous across an interface.*

$$D_{1n} - D_{2n} = \rho_s$$

➔ *The normal component of an D field is discontinuous across an interface where a surface charge exists.*

Boundary conditions at an interface between two lossless linear media

Between two lossless media (ε, μ) with $\sigma = 0$, and $\rho_s = 0$, $\mathbf{J}_s = 0$

$$\left(\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{H} = \frac{\mathbf{B}}{\mu}\right) \Rightarrow$$

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

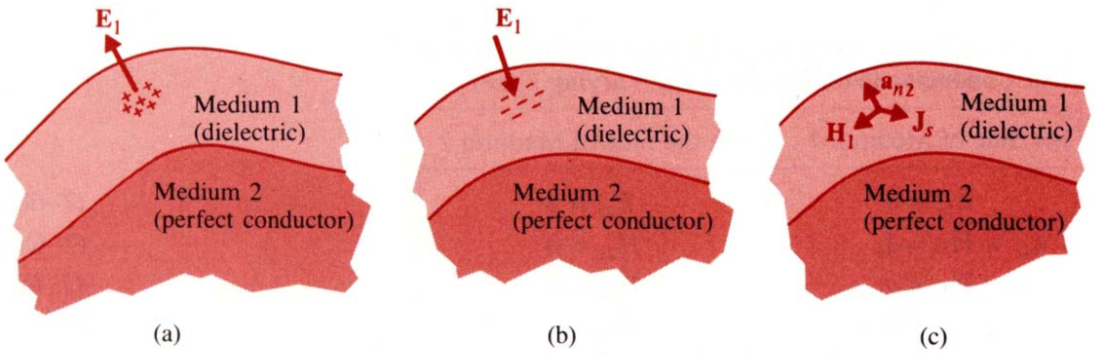
$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Boundary conditions at an interface between dielectric and perfect conductor

In a perfect conductor (σ → *infinite*, for example, supercondiuctors),

$$\rightarrow E_2 = 0 = D_2,\ H_2 = 0 = B_2$$

Medium 1(dielectric)	Medium2 (perfect metal)
$E_{1t} = 0$	$E_{2t} = 0$
$H_{1t} = J_s$	$H_{2t} = 0$
$D_{1n} = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$



Boundary conditions

- Table

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

7-4. Potential functions

- Vector potential $\mathbf{B} = \nabla \times \mathbf{A} \quad (\mathrm{T}) \quad (\leftarrow \nabla \cdot \mathbf{B} = 0)$

- Electric field for the time-varying case.

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\rightarrow \quad \underbrace{\mathbf{E} = -\nabla V}_{} \underbrace{- \frac{\partial \mathbf{A}}{\partial t}}_{} \quad (\mathrm{V/m})$$

Due to charge distribution ρ

Due to time-varying current **J**

Wave equation for vector potential A

From $\mu\mathbf{H} = \mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{D} = \varepsilon\mathbf{E} = \varepsilon\left(-\nabla V - \frac{\partial\mathbf{A}}{\partial t}\right),$

$$\nabla\times\mathbf{H} = \left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J} + \mu\varepsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial\mathbf{A}}{\partial t}\right)$$

$$\nabla(\nabla\bullet\mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \nabla\left(\mu\varepsilon\frac{\partial V}{\partial t}\right) - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} \quad \text{or}$$

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} + \nabla\left[\nabla\bullet\mathbf{A} + \mu\varepsilon\frac{\partial V}{\partial t}\right]$$

$\left[\nabla\bullet\mathbf{A} + \mu\varepsilon\frac{\partial V}{\partial t}\right] \rightsquigarrow$ **0 (Lorenz condition, or gauge)**

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

(# Show that the Lorentz condition is consistent with the equation of continuity. Prob. P.7-12)

⇨ Non-homogeneous wave equation for vector potential ***A***

⇨ traveling wave with a velocity of $\frac{1}{\sqrt{\varepsilon\mu}}$

Wave equations for scalar potential *V*

From $E = -\nabla V - \frac{\partial A}{\partial t}$ $\; and \; \nabla \bullet E = \frac{\rho_v}{\varepsilon}$,

$$\nabla \bullet \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \bullet \mathbf{A}) = -\frac{\rho_v}{\varepsilon} \leftarrow \nabla \bullet \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

⇨ for scalar potential ***V***

$$\left(\nabla \bullet \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0 \right)$$

➔ The Lorentz condition uncouples the wave equations for A and for V.

➔ The wave equations reduce to Poisson’s equations in static cases.

Gauge freedom

- Electric & magnetic field $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},\ \mathbf{B} = \nabla \times \mathbf{A}$

- Gauge transformation

If $\mathbf{A} \to \mathbf{A} + \nabla\psi$, B remains unchanged.

$$\to \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} - \nabla\frac{\partial \psi}{\partial t} = -\nabla\left(V + \frac{\partial \psi}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t}$$

$\to$ Thus, if V is further changed to $V \to V - \frac{\partial \psi}{\partial t}$, $\mathbf{E}$ also remains same.

➔ Gauge invariance

E & B fields are unchanged if we take any function ψ(x,t) on simultaneously **A** and *V* via:

$$\mathbf{A} \to \mathbf{A} + \nabla\psi$$
$$V \to V - \frac{\partial \psi}{\partial t}$$

⇨

$$\nabla \bullet \mathbf{A} + \mu\varepsilon\frac{\partial V}{\partial t} = 0$$

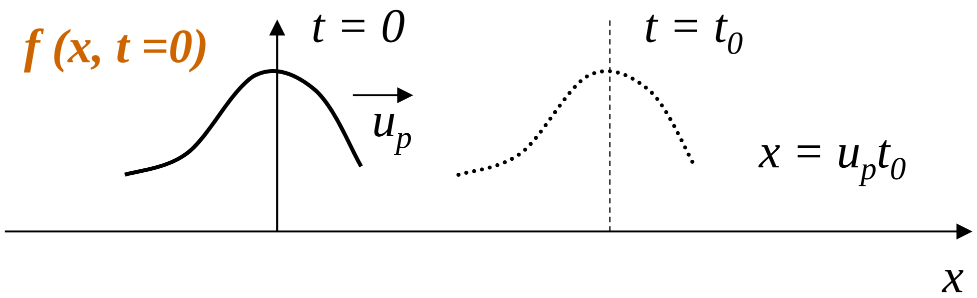
⇨

$$\nabla^2\psi - \mu\varepsilon\frac{\partial^2 \psi}{\partial t^2} = 0$$

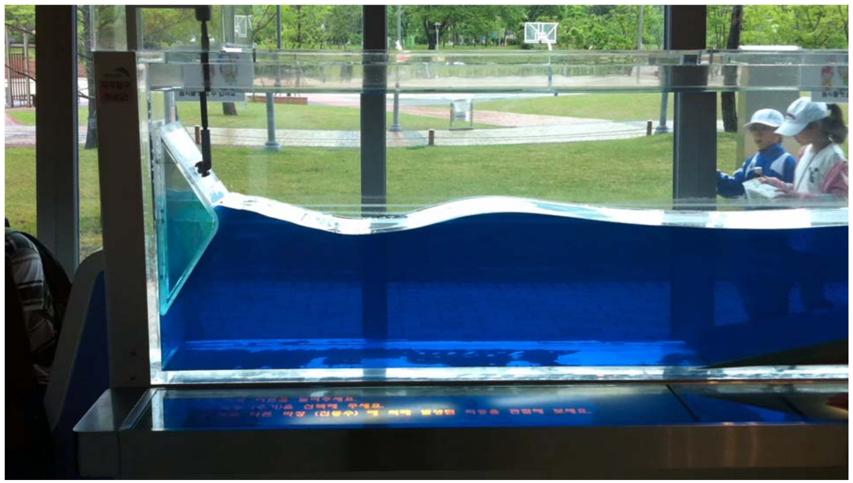
➔ The Lorentz condition can be converted to a wave equation.

7-6. Solution of wave equations

- The mathematical form of waves

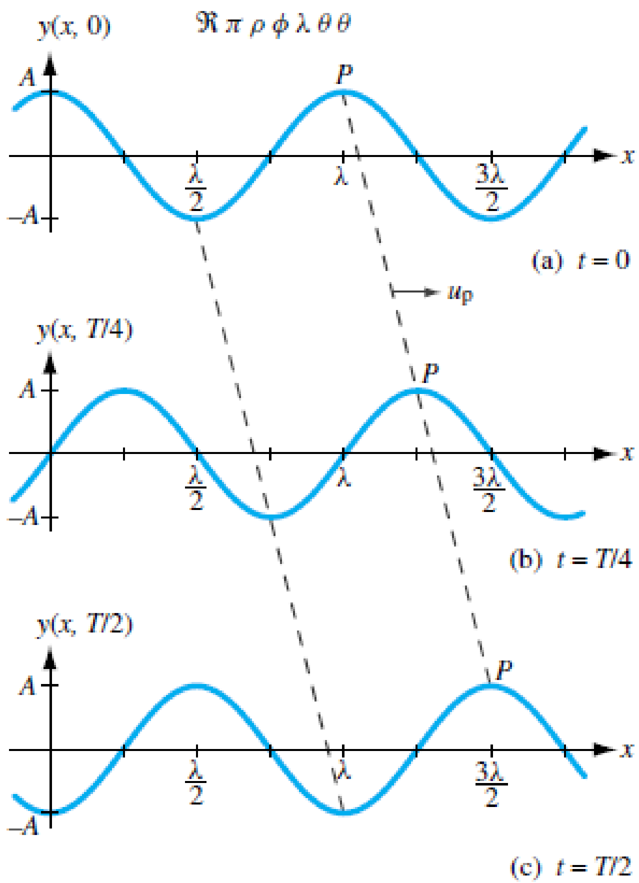


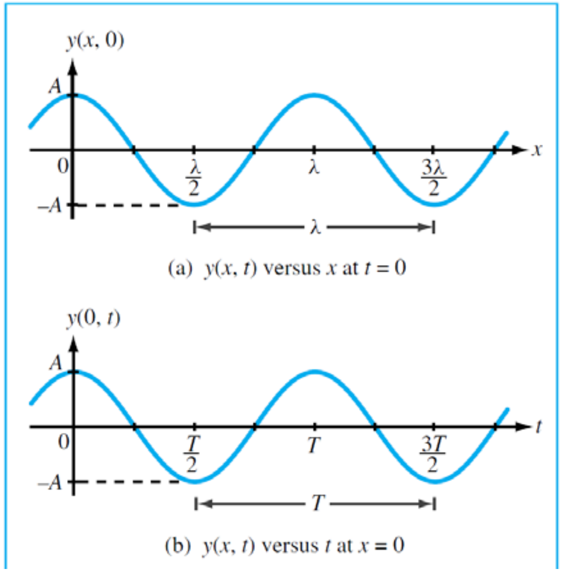
$$f(x,t) = f(x \pm u_p t) \rightarrow \frac{\partial^2 f}{\partial x^2} - \frac{1}{u_p^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad \text{: wave equation}$$



Wave equation

- Simple wave
  - http://navercast.naver.com/science/physics/1376





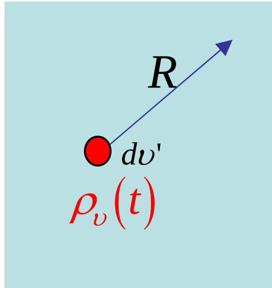
$$y(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right) = A\cos(kx - \omega t)$$

$$\omega = \frac{2\pi}{T} = 2\pi f, \quad k = \frac{2\pi}{\lambda}, \quad u_p = \frac{\omega}{k}$$

Solution of wave equations from potentials

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_\upsilon}{\varepsilon}$$

: Nonhomogeneous wave equation for scalar electric potential



First consider a point charge at time t, $\rho(t)\Delta v'$, located at a origin.

Except at the origin, ***V*(R)** satisfies the following homogeneous equation ($\rho = 0$):

Since $\nabla^2 V = \frac{1}{R^2}\left(R^2 \frac{\partial V}{\partial R}\right)$ for spherical symmetry $V(R,\theta,\phi) = V(R)$

$$\Rightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

Introducing a new varible, $V(R,t) = \frac{1}{R} U(R,t)$

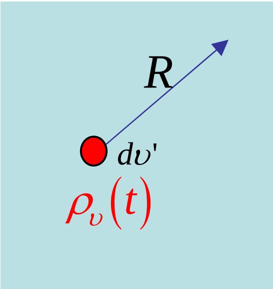
$$\Rightarrow \frac{\partial^2 U}{\partial R^2} - \mu\varepsilon \frac{\partial^2 U}{\partial t^2} = 0 \rightarrow U(R,t) = U\left(t - \frac{R}{u_p}\right) \text{ or } U(R - u_p t), u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

$\Rightarrow$ Thus, we can write in a form of $V(R,t) = V\left(t - \frac{R}{u_p}\right)$

Solution of wave equations

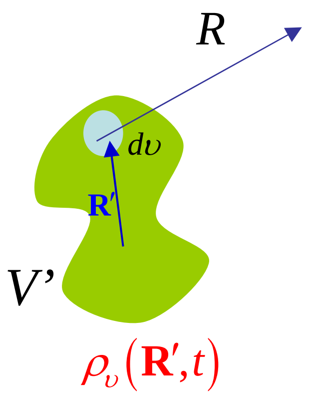
The potential at R for a point charge $\rho_\upsilon(t)\Delta\upsilon$ is,

$$\Delta V(R) = \frac{\rho_\upsilon(t)\Delta\upsilon'}{4\pi\varepsilon R}$$



$$V(R,t) = V\left(t - \frac{R}{u_p}\right) \rightarrow \Delta V\left(t - R/u_p\right) = \frac{\rho_\upsilon\left(t - R/u_p\right)\Delta\upsilon'}{4\pi\varepsilon R}$$

Now consider a charge distribution over a volume *V'* .



$$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho_\upsilon\left(R', t - R/u_p\right)}{R} d\upsilon' \quad \text{(V)}$$

$$\mathbf{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}\left(R', t - R/u_p\right)}{R} d\upsilon' \quad \text{(Wb/m)}$$

➔ **The potentials at a distance R from the source at time t depend on the values of ρ and J at an earlier time (t- R/u) ➔ Retarded in time**

➔ **Time-varying charges and currents generate retarded scalar potential, retarded vector potential.**

Source free wave equations

Maxwell's equations in source-free non-conducting media (ε, μ, σ=0).

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \nabla \bullet \mathbf{E} = 0, \quad \nabla \bullet \mathbf{H} = 0$$

- Homogeneous wave equation for **E** & **H**.

$$\nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \qquad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

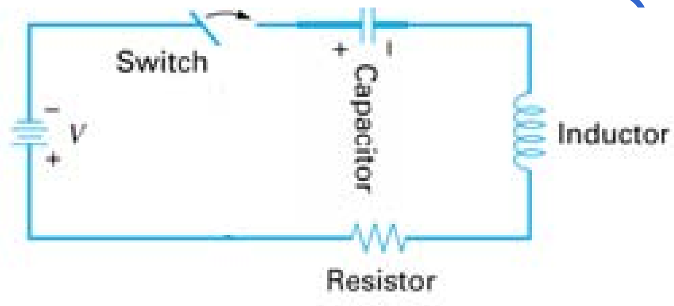
$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \mathbf{E} - \frac{1}{u_p^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

In an entirely similar way, $\nabla^2 \mathbf{H} - \frac{1}{u_p^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$

Review –The use of Phasors

Consider a RLC circuit



$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=V(t),\quad i(t)=I_0\cos(\omega t+\phi)$$

Phasor method (exponential representation)

$$V(t)=V_0\cos\omega t=\mathrm{Re}\left[\left(V_0e^{j0}\right)e^{j\omega t}\right]=\mathrm{Re}\left(V_se^{j\omega t}\right)$$

$$i(t)=\mathrm{Re}\left[\left(I_0e^{j\phi}\right)e^{j\omega t}\right]=\mathrm{Re}\left(I_se^{j\omega t}\right)$$

$$V_s=V_0e^{j0}$$
$$I_s=I_0e^{j\phi}$$

➔ (Scalar) **phasors** that contain amplitude and phase information but are independent of time t.

If we use phasors in the RLC circuit, $\frac{di}{dt}=\mathrm{Re}\left(j\omega I_se^{j\omega t}\right),\int idt=\mathrm{Re}\left(\frac{I_s}{j\omega}e^{j\omega t}\right)$

$$\rightarrow\left[R+j\left(\omega L-\frac{1}{\omega C}\right)\right]I_s=V_s$$

$$\rightarrow i(t)=\mathrm{Re}\left[V_s/\left\{R+j\left(\omega L-\frac{1}{\omega C}\right)\right\}\right]e^{i\omega t}$$

Time-harmonic Maxwell's & wave equations

- Vector phasors.

$$\mathbf{E}(x,y,z,t)=\mathrm{Re}\left[\mathbf{E}(x,y,z)e^{j\omega t}\right]$$

$$\mathbf{H}(x,y,z,t)=\mathrm{Re}\left[\mathbf{H}(x,y,z)e^{j\omega t}\right]$$

- Time-harmonic (cos ωt) Maxwell's equations in terms of vector phasors

$$\nabla\bullet\mathbf{D}=\rho_\upsilon$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\bullet\mathbf{B}=0$$

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

⇒

$$\nabla\bullet\mathbf{E}=\frac{\rho_\upsilon}{\varepsilon}$$

$$\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}$$

$$\nabla\bullet\mathbf{H}=0$$

$$\nabla\times\mathbf{H}=\mathbf{J}+j\omega\varepsilon\mathbf{E}$$

Time-harmonic Maxwell's & wave equations

Time-harmonic wave equations (nonhomogeneous ***Helmholtz's equations***)

$$\nabla^2 V(R,t) - \mu\varepsilon \frac{\partial^2 V(R,t)}{\partial t^2} = -\frac{\rho_\upsilon(R,t)}{\varepsilon}$$

$$\rightarrow \nabla^2 V(R) - \mu\varepsilon \left(j\omega\right)^2 V(R) = -\frac{\rho_\upsilon(R)}{\varepsilon}$$

$$k = \text{wave number} = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u_p} = \frac{2\pi}{\lambda}$$

$$\nabla^2 V(R) + k^2 V(R) = -\frac{\rho_\upsilon}{\varepsilon}$$

$$\nabla^2 \mathbf{A}(R) + k^2 \mathbf{A}(R) = -\mu\mathbf{J}$$

$$\left(\leftarrow \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \right)$$

Time-harmonic retarded potential

- Phasor form of retarded scalar potential

$$V(R)=\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho_\upsilon(R')e^{-jkR}}{R}d\upsilon' \qquad (\mathrm{V})$$

$$\cos(\omega t-kx)$$
$$\rightarrow e^{j(\omega t-kx)}=e^{j\omega t}e^{-kx}$$

- Phasor form of retarded vector potential.

$$\mathbf{A}(R)=\frac{\mu}{4\pi}\int_{V'}\frac{\mathbf{J}(R')e^{-jkR}}{R}d\upsilon' \quad (\mathrm{Wb/m})$$

- When $kR=2\pi\frac{R}{\lambda}<<1$

$$e^{-jkR}=1-jkR+\frac{k^2R^2}{2}+\ldots\approx 1$$

$V(R),\mathbf{A}(R)\rightarrow$ static expressions(Eq. 3-39 & Eq. 5-22)

Time-harmonic retarded potential

- Example: Find the magnetic field intensity **H** and the value of β when $\varepsilon = 9\varepsilon_0$

$$\mathbf{E}(z,t) = \mathbf{a}_y 5\cos\left(10^9 t - \beta z\right) \quad (\text{V/m})$$

$$\omega = 10^9 \ (1/\text{s})$$

$$\mathbf{E}(z) = \mathbf{a}_y 5e^{-j\beta z}$$

$$\mathbf{H}(z) = -\frac{1}{j\omega\mu_0}\nabla\times\mathbf{E}$$

$$\mathbf{H}(z) = -\frac{1}{j\omega\mu_0}\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 5e^{-j\beta z} & 0 \end{vmatrix}$$

$$= -\frac{1}{j\omega\mu_0}\left(-\mathbf{a}_x\frac{\partial}{\partial z}5e^{-j\beta z}\right) = \left(-\frac{\beta}{\omega\mu_0}5e^{-j\beta z}\right)\mathbf{a}_x = H_x(z)\mathbf{a}_x$$

$$\mathbf{E}(z) = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\varepsilon}\left(\mathbf{a}_y\frac{\partial}{\partial z}H_x\right) = \mathbf{a}_y\left(\frac{\beta^2}{\omega^2\mu_0\varepsilon}\right)5e^{-j\beta z}$$

$$\rightarrow \beta = \omega\sqrt{\mu_0\varepsilon} = 3\omega\sqrt{\mu_0\varepsilon_0} = \frac{3\omega}{c} = 10 \quad (\text{rad/m})$$

$$\mathbf{H}(z) = -\mathbf{a}_x\frac{\beta}{\omega\mu_0}5e^{-j\beta z} = -\mathbf{a}_x(0.0398)e^{-j10z}$$

$$\mathbf{H}(z,t) = -\mathbf{a}_x\frac{\beta}{\omega\mu_0}5e^{-j\beta z} = -\mathbf{a}_x(0.0398)\cos\left(10^9 t - 10z\right)$$

The EM Waves in lossy media

- If a medium is conducting ($\sigma \neq 0$), a current $\mathbf{J}=\sigma\mathbf{E}$ will flow

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \longrightarrow \nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} = j\omega\varepsilon_c\mathbf{E},$$

$$\varepsilon_c\ (\text{complex permittivity} = \varepsilon' - j\varepsilon'') = \varepsilon - j\frac{\sigma}{\omega}$$

- When an external time-varying electric field is applied to material bodies, small displacements of bound charges result, giving rise to a volume density of polarization. This polarization vector will vary with the same frequency as that of the applied field.
- As the frequency increases, the inertia of the charged particles tends to prevent the particle displacements from keeping in phase with the field changes, leading to a frictional damping mechanism that causes power loss.
- This phenomenon of out of phase polarization can be characterized by a complex electric susceptibility and hence a complex permittivity.
- Loss tangent, δ_c

The EM Waves in lossy media

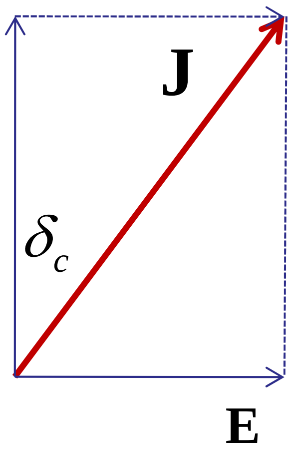
- Loss tangent, δ_c

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon_c \mathbf{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E}$$

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

$$\tan\delta_c = \frac{\varepsilon''}{\varepsilon'} \approx \frac{\sigma}{\omega\varepsilon}, \quad \delta_c : \text{loss angle}$$



- Good conductor if $\frac{\sigma}{\omega\varepsilon} >> 1$

- Good insulator if $\frac{\sigma}{\omega\varepsilon} << 1$

- Moist ground : loss tangent ~ 1.8×10^{4}@1kHz, 1.8×10^{-3} @10GHz

The electromagnetic spectrum

- Spectrum of electromagnetic waves