1. Find the Fourier series of the periodic function $f(x)=\begin{cases}0 & , & -5<x<0\\ 3 & , & 0<x<5\end{cases}$

Ans. $f(x)\approx\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3\left(1-\cos(n\pi)\right)}{n\pi}\sin\left(\frac{n\pi}{5}x\right)$

2. Find the Fourier series of the periodic function $f(x)=x^2 \quad , \quad 0<x<2\pi$

Ans. $f(x)\approx\frac{4}{3}\pi^2+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos(nx)-\frac{4\pi}{n}\sin(nx)\right]$

3. Find the Fourier cosine series of $f(x)=\sin(x) \quad , \quad 0<x<\pi$

Ans. $f(x)\approx\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1+\cos(n\pi)\right]}{n^2-1}\cos(nx)$

4. Solve the boundary-value problem $\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2} \quad ; \quad \begin{matrix}u(0,t)=0 & , & u(2,t)=0\\ u(x,0)=x & & \end{matrix}$

Ans. $u(x,t)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}\sin\left(\frac{n\pi}{2}x\right)\exp\left(\frac{-3n^2\pi^2}{4}t\right)$

5. Find the Fourier sine and cosine series of the periodic function $f(x)=\begin{cases}x & , & 0<x<4\\ 8-x & , & 4<x<8\end{cases}$

Ans. $f(x)\approx\begin{cases}\text{sine series} & , \quad \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{8}x\right)\\ \text{cosine series} & , \quad \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{2\cos(n\pi/2)-\cos(n\pi)-1}{n^2}\right]\cos\left(\frac{n\pi}{8}x\right)\end{cases}$

6. Solve the boundary-value problem $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \quad ; \quad \begin{matrix}u(0,t)=0 & , & u(6,t)=0\\ u(x,0)=\begin{cases}1 & , & 0<x<3\\ 0 & , & 3<x<6\end{cases} & & \end{matrix}$

Ans. $u(x,t)=2\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/3)}{n\pi}\sin\left(\frac{n\pi}{6}x\right)\exp\left(\frac{-n^2\pi^2}{36}t\right)$

7. Solve the equation $\frac{\partial^2 y}{\partial t^2}=9\frac{\partial^2 y}{\partial x^2}$ subject to $\begin{Bmatrix} y(0,t)=0 & , & y(2,t)=0 \\ y(x,0)=0.05x(2-x) & , & y_t(x,0)=0 \end{Bmatrix}$.

Ans. $y(x,t)=\frac{1.6}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left(\frac{2n-1}{2}\pi x\right)\cos\left(\frac{3(2n-1)}{2}\pi t\right)$

8. Using Fourier integral, show that $\int_0^{\infty}\frac{\cos(\alpha x)}{\alpha^2+1}d\alpha=\frac{\pi}{2}e^{-x} \quad , \quad x\ge 0$.

9. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with respect to $u(0,t)=0 \quad , \quad u(x,0)=\begin{Bmatrix} 1 & , & 0<x<1 \\ 0 & , & x\ge 1 \end{Bmatrix}$. Suppose that $u(x,t)$ is bounded for $x>0$ and $t>0$.

Ans. $u(x,t)=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos(\lambda)}{\lambda}\sin(\lambda x)\exp\left(-\lambda^2 t\right)dx$

10. Prove that
$$\sin(x+iy)=\sin(x)\cosh(y)+i\cos(x)\sinh(y)$$
$$\cos(x+iy)=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$$

11. Show that $w=f(z)=\bar{z}$ has no derivative at any point.

12. If the partial derivatives of first and second order of the real and imaginary parts, $u(x,y)$ and $v(x,y)$, of an analytic function $w=f(z)$ are continuous, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

Note: The logarithm (as an inverse of the exponential) is a multi-valued function of z. The multi-valuedness is introduced by the polar angle θ:

$$\ln(z)=\ln\left(re^{i\theta}\right)=\ln(r)+i\theta$$

Then the angle θ may be given its principal value θ_0, $-\pi<\theta_0\le\pi$, which leads to the *principal value* of $\ln(z)$:

$$\mathrm{Ln}(z)=\ln(r)+i\theta_0 \quad ; \quad -\pi<\theta_0\le\pi$$

But many other values of θ will still give the same z in $z=re^{i\theta}$, namely:

$$\theta=\theta_0+2k\pi \quad ; \quad k=0,\pm1,\pm2,\dots$$

and each of these values of θ gives rise, in turn, to a value of $\ln(z)$, namely:

$$\ln(z)=\ln(r)+i\left(\theta_0+2k\pi\right) \quad ; \quad k=0,\pm1,\pm2,\ldots$$

so that all of the infinitely many different values of $\ln(z)$ differ from the principal value by an integral multiple of $2\pi i$.

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13. Find the principal value of $\ln(z)$ for $z=\dfrac{1+i}{1-i}$.

Ans. $\text{P.V.}\equiv i\ \pi/2$

14. If $u(x,y)$ and $v(x,y)$ are the real and imaginary parts of an analytic function of $z=x+iy$, show that the family of curves $\mathrm{u}=\text{constant}$ is orthogonal to the family $\mathrm{v}=\text{constant}$ at every point of intersection where $f'(z)\neq 0$.

15. Evaluate $I=\int_{1+i}^{2+4i} z^2 dz$ on the path $\left\{x=t \quad , \quad y=t^2\right\}$, where, $1\le t\le 2$.

Ans. $I=-\dfrac{86}{3}-6i$

16. Prove that $\oint_C \dfrac{dz}{(z-a)^n}=\begin{cases}2\pi i & , \quad n=1\\ 0 & , \quad n=2,3,4,\ldots\end{cases}$, in which, the simple path C encloses $z=a$.

17. Evaluate $I=\oint_{|z-1|=3} \dfrac{e^z}{z(z+1)}dz$.

Ans. $I=2\pi i\left(1-e^{-1}\right)$

Note: **Taylor series**

Theorem: Let $f(z)$ be analytic everywhere inside a circle C_0 with center at z_0 and radius r_0. Then at each point z inside C_0:

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$
$$+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\cdots$$

that is, the power series here converges to $f(z)$ when $|z-z_0|<r_0$.

This is the expansion of $f(z)$ into a Taylor series about the point z_0.

Laurent series

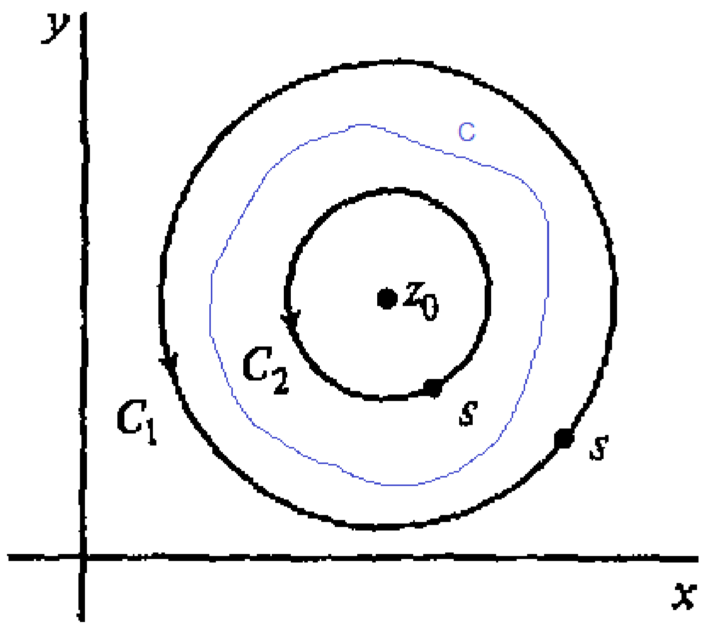
Let C_1 and C_2 be two concentric circles centered at a point Z_0 and with radii r_1 and r_2, respectively, where, $r_2 < r_1$.

Theorem: If $f(z)$ is analytic on C_1 and C_2 and throughout the annular domain between those two circles, then at each point z in that domain $f(z)$ is represented by the expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where, $\left\{ \begin{array}{ll} a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(s)}{(s - z_0)^{n+1}} ds & n = 0, 1, 2, \ldots \\ b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(s)}{(s - z_0)^{-n+1}} ds & n = 1, 2, \ldots \end{array} \right\}$. The simple closed contour C is between C_1 and C_2; each path of integration being taken counterclockwise.

The series here is called a Laurent series.



If $f(z)$ is analytic at every point inside and on C_1 (or C) except at the point z_0 itself, the radius r_2 may be taken arbitrarily small. The above expansion is then valid when $0 < |z - z_0| < r_1$. If $f(z)$ is analytic at all points inside and on C_1 (or C), the function $f(z)/(z - z_0)^{-n+1}$ is analytic inside and on C_2 because $-n + 1 \le 0$.

18. Find the Laurent series of $f(z)=\dfrac{1}{z(z+2)^3}$ at $z=-2$.

Ans. $f(z)=-\dfrac{1}{2(z+2)^3}-\dfrac{1}{4(z+2)^2}-\dfrac{1}{8(z+2)}-\dfrac{1}{16}-\dfrac{1}{32}(z+2)-\cdots$ converges on $0<|z+2|<2$.

19. Evaluate the integral $I=\int_C \dfrac{5z-2}{z(z-1)}dz$, where, $C:|z|=2$.

Ans. $I=10\pi i$

20. Find the residue at $z=5\pi$ of the function $f(z)=\cot(z)$.

Ans. $\operatorname{Res}\{f(z)\}_{z=5\pi}=\lim\limits_{z\to 5\pi}(z-5\pi)\dfrac{\cos(z)}{\sin(z)}=\left(\lim\limits_{z\to 5\pi}\dfrac{z-5\pi}{\sin(z)}\right)\left(\lim\limits_{z\to 5\pi}\cos(z)\right)=\left(\lim\limits_{z\to 5\pi}\dfrac{1}{\cos(z)}\right)(-1)=1$

21. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2(x^2+2x+2)}dx=\frac{7}{50}\pi$

22. Evaluate $I=\oint_C \dfrac{e^z+z}{(z-1)^4}dz$, in which, $C:|z|=2$

Ans. $I=\pi ie/3$

23. Show that $\displaystyle\int_0^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx=\frac{\pi}{4}$.

24. Establish the following integration formulas:

- $\displaystyle\int_0^{\infty}\frac{\cos(2x)}{(x^2+4)^2}dx=\frac{5\pi}{32}\exp(-4)$
- $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin(ax)}{x^4+4}dx=\frac{\pi}{2}e^{-a}\sin(a) \qquad ; \quad a>0$

25. Expand the function $w=f(z)=\dfrac{1}{z(z-1)}$ in a Laurent series about $z=0$ and $z=1$.

Ans. $f(z)=\begin{cases}-\dfrac{1}{z}-1-z-z^2-\cdots & ; \quad 0<|z|<1\\[2mm] \dfrac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots & ; \quad 0<|z-1|<1\end{cases}$

Note: All the points of the z-plane at which an analytic function does not have a unique derivative are said to be *singular points*. They are the points at which the function ceases to be analytic. If we concern ourselves only with single-valued functions of the complex variable $w = f(z)$, then $f(z)$ may have two types of singularities:

1. **Poles, or non-essential singular points**
2. **Essential singular points**

If the Laurent series of $f(z)$ about z_0 (the singular point of $f(z)$) has only a finite number of powers of $(z - z_0)$ with negative exponents, then $f(z)$ is said to have a pole at $z = z_0$; and the largest of the negative exponents (m) is called the order of the pole z_0. In this case $f(z)$ has a pole of the mth order at the point z_0. On the other hand, if the Laurent series of $f(z)$ about z_0 has an infinite number of negative powers of $(z - z_0)$, the point $z = z_0$ is said to be an essential singular point, and $f(z)$ is said to have an essential singularity at $z = z_0$.

26. Show that $f(z) = \dfrac{1}{(z-2)^2 (z-5)^3 (z-1)}$ has a pole of the third order at $z = 5$.
27. Show that $f(z) = \exp(1/z)$ has an essential singularity at $z = 0$.
28. Prove that $I = \displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)}{a + b\cos(\theta)} d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right) \quad ; \quad a > b > 0$
29. Show that $\displaystyle\int_0^{\infty} \frac{x^4}{x^6 + 1} dx = \frac{\pi}{3}$
30. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.
31. Apply the definition of derivative directly to prove that $f'(z) = -1/z^2$ when $f(z) = 1/z$, provided $z \neq 0$.

Note: Satisfaction of the Cauchy-Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient for the existence of the derivative of $f(z)$ at that point. But with certain continuity conditions, we have the following useful theorem.

Theorem: Let the function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε neighborhood of the point $z_0 = x_0 + i y_0$. Suppose that the first partial derivatives of the functions u and v with respect to x and y exist in that neighborhood and are continuous at (x_0, y_0). Then if those partial derivatives satisfy the Cauchy-Riemann equations at (x_0, y_0), the derivative $f'(z_0)$ exists. It should be noted that the Cauchy-Riemann equations are $u_x = v_y \quad \& \quad u_y = -v_x$.

32. The function $u(x,y)=y^3-3x^2y$ is harmonic throughout the entire xy plane. Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

Ans. $v(x,y)=x^3-3xy^2+c$

33. The transient one-dimensional heat conduction in a slab is expressed by the equation $u_t(x,t)=\alpha u_{xx}(x,t)$, where α is the thermal diffusivity of the slab. (a) If the slab is initially at temperature zero throughout $(u(x,0)=0)$, and the face $x=0$ is kept at that temperature $(u(0,t)=0)$, while the face $x=\pi$ is kept at a constant temperature u_0 when $t>0$, determine the temperature distribution $u(x,t)$. (b) Suppose that the face $x=0$ is kept at temperature zero $(u(0,t)=0)$ and that the face $x=\pi$ is insulated $(u_x(\pi,t)=0)$. Also, let the initial temperatures be $u(x,0)=f(x)$.

Ans. (a) $u(x,t)=\dfrac{u_0}{\pi}\left[x+2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin(nx)\exp(-n^2\alpha t)\right]$

(b) $u(x,t)=\sum_{n=1}^{\infty}\left\{\dfrac{2}{\pi}\int_0^{\pi}f(x)\sin\left(\dfrac{2n-1}{2}x\right)dx\right\}\sin\left(\dfrac{2n-1}{2}x\right)\exp\left[-\dfrac{(2n-1)^2\alpha}{4}t\right]$

34. Solve the boundary-value problem $\dfrac{\partial T}{\partial t}=\dfrac{\partial^2 T}{\partial x^2}$; $\left\{\begin{matrix} T(0,t)=1 & T(\pi,t)=3 \\ T(x,0)=2 & \end{matrix}\right\}$.

Ans. $T(x,t)=1+\dfrac{2x}{\pi}+\sum_{n=1}^{\infty}\dfrac{4\cos(n\pi)}{n\pi}\sin(nx)\exp(-n^2t)$

35. Show that $u(x,t)=\sin(2x)e^{-8t}$ is a solution to the boundary value problem $u_t=2u_{xx}$ $\quad u(0,t)=u(\pi,t)=0 \quad u(x,0)=\sin(2x)$

36. Show that $v=F(y-3x)$, where F is an arbitrary differentiable function, is a general solution of the equation $\dfrac{\partial v}{\partial x}+3\dfrac{\partial v}{\partial y}=0$. Also, find the particular solution which satisfies the condition $v(0,y)=4\sin(y)$.

Ans. $v(x,y)=F(y-3x)=4\sin(y-3x)$

37. Show that $y(x,t)=F(2x+5t)+G(2x-5t)$ is a general solution of $4\frac{\partial^2 y}{\partial t^2}=25\frac{\partial^2 y}{\partial x^2}$. Also, find a particular solution satisfying the conditions $y(0,t)=y(\pi,t)=0 \quad y(x,0)=\sin(2x) \quad y_t(x,0)=0$.

Ans. $y(x,t)=\frac{1}{2}\sin(2x+5t)+\frac{1}{2}\sin(2x-5t)=\sin(2x)\cos(5t)$

38. Solve the boundary value problem $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2} \quad ; \quad u(0,t)=10 \quad u(3,t)=40 \quad u(x,0)=25$.

Ans. $u(x,t)=10(x+1)+\frac{30}{\pi}\sum_{m=1}^{\infty}\frac{\cos(m\pi)-1}{m}\sin\left(\frac{m\pi}{3}x\right)\exp\left(-\frac{2m^2\pi^2}{9}t\right)$

39. A circular plate of unit radius, whose faces are insulated, has half of its boundary kept at constant temperature u_1 and the other half at constant temperature u_2. Find the steady-state temperature of the plate. In polar coordinates (r,ϕ), the partial differential equation for steady-state heat flow is $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}=0$. The boundary conditions are $u(1,\phi)=\begin{cases} u_1 & , \quad 0<\phi<\pi \\ u_2 & , \quad \pi<\phi<2\pi \end{cases}$.

Ans. $u(r,\phi)=\frac{u_1+u_2}{2}+\sum_{m=1}^{\infty}\frac{(u_1-u_2)\left[1-\cos(m\pi)\right]}{m\pi}r^m\sin(m\phi)$

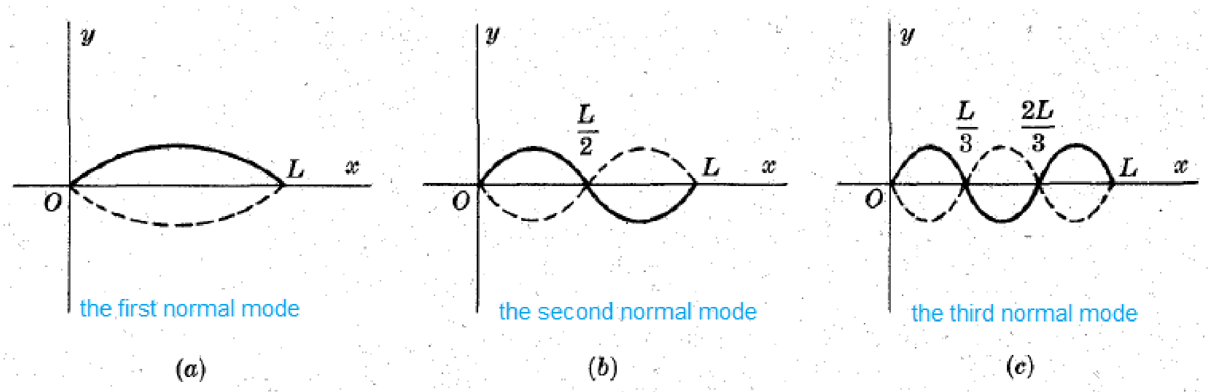
40. A string of length L is stretched between points $(0,0)$ and $(L,0)$ on the x-axis. At time $t=0$, it has a shape given by $f(x)$; and it is released from rest. Find the displacement of the string at any later time. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \quad , \quad 0<x<L \quad t>0$ where, $y(x,t)$ is the displacement from x-axis at time t. Since the ends of the string are fixed at $x=0$ and $x=L$, the boundary conditions are $y(0,t)=y(L,t)=0 \quad , \quad t>0$.

Ans. $y(x,t)=\sum_{m=1}^{\infty}\left\{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\right\}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m\pi a}{L}t\right)$

- The terms in this series represent the *natural or normal modes of vibration*. The frequency of the *m*th normal mode is obtained from the term involving $\cos\left(\frac{m\pi a}{L}t\right)$ and is given by:

$$2\pi f_m = \frac{m\pi a}{L} \quad \Rightarrow \quad f_m = \frac{ma}{2L}$$

Since all the frequencies are integer multiples of the lowest frequency, f_1, the vibrations of the string will yield a musical tone, as in the case of a violin or piano string. The first three normal modes are illustrated in the following figure.



As time increases, the shapes of these modes vary from curves shown solid to curves shown dashed and then back again; the time for a complete cycle is the period and the reciprocal of this period is the frequency. The mode (a) is called the fundamental mode or first harmonic, while (b) and (c) are called the second and third harmonic (or first and second overtone), respectively.

41. Find the steady-state temperature distribution $T(x,y)$ in the uniform slab of metal governed by the Laplace equation $\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0\right)$. The temperatures of the boundaries are $\begin{cases} T(x,0)=T(x,a)=0 \quad ; \quad 0<x<\infty \\ T(0,y)=f(y) \end{cases}$; where, $f(y)$ is a bounded function. State any additional condition that must be imposed on $T(x,y)$ for the solution to be physically possible.

Ans. $T(x,y)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)\exp\left(\frac{-n\pi}{a}x\right)$, where, $C_n=\frac{2}{a}\int_0^a f(y)\sin\left(\frac{n\pi}{a}y\right)dy \qquad n=1,2,\ldots$

42. Find the characteristics of the hyperbolic equation $x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}=0$. Reduce the equation to canonical form and solve it, if possible.

Ans. $c_1=y \quad , \quad c_2=xy \quad ; \quad u(x,y)=f(y)+g(xy)$

43. Find the Laurent series representing the function $f(z)=\frac{z}{(z^2+1)(z+2)}$ in the annulus $0<|z-i|<2$.

Ans. $\frac{2-i}{10}(z-i)^{-1}+\frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2+i}{(2i)^{n+1}}-\frac{4}{(2+i)^{n+1}}\right](z-i)^n$

44. Evaluate $\int_C (z+2\bar{z})dz$, where C is a path from $z=0$ to $z=1+2i$, consisting of the line segment from $z=0$ to $z=1$ followed by the line segment from $z=1$ to $z=1+2i$.

Ans. $\frac{7}{2}-6i$

45. Derive Taylor series expansion of $f(z)=e^z$ about $z=1$.

Ans. $f(z)=e^z=\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}f^{(n)}(1)=e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad ; \quad |z-1|<\infty$

46. Obtain expansions of $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}$ in Maclaurin series which are valid in $|z|<1$.

Ans. $f(z)=-1+\sum_{n=1}^{\infty}(-1)^{n+1}\left(1+4^{-n}\right)z^n \qquad ; \quad |z|<1$

47. Find Maclaurin series for $f(z)=\dfrac{z}{e^{z}-1}$, if $0<|z|<2\pi$.

Ans. $1-\dfrac{z}{2}+\dfrac{z^2}{12}-\dfrac{z^4}{720}+\cdots$

48. Expand $f(z)=\dfrac{4z+3}{z(z-3)(z+2)}$ in Laurent series about $z=0$ in the domain $|z|>3$.

Ans. $\dfrac{4}{z^2}+\dfrac{8}{z^3}+\dfrac{31}{z^4}+\cdots$

49. Prove that $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\cos(\theta)+c\sin(\theta)}=\frac{2\pi}{\sqrt{a^2-b^2-c^2}}$, if $a^2>b^2+c^2$.

50. Prove that $\displaystyle\int_0^{2\pi}\frac{\cos(3\theta)}{5-4\cos(\theta)}d\theta=\frac{\pi}{12}$

51. Prove that $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x(x^2+px+q)}=\frac{-\pi p}{q\sqrt{4q-p^2}}$ $\quad where \quad p^2<4q$

52. Prove that $\displaystyle\int_0^{\infty}\frac{x^2}{(x^2+9)(x^2+4)^2}dx=\frac{\pi}{200}$

53. Prove that $\displaystyle\int_0^{\infty}\frac{\cos(mx)}{a^2+x^2}dx=\frac{\pi}{2a}\exp(-ma) \qquad , \quad m>0$

54. Prove that $\displaystyle\int_0^{\pi}\frac{1+2\cos(\theta)}{5+4\cos(\theta)}d\theta=0$

55. Find the residue of $f(z)=\dfrac{\cot(\pi z)}{(z-a)^2}$ at (a) $z=a$, and at (b) $z=n$ $\quad (\text{an integer or zero})$.

Ans.
$(a) \qquad -\pi\operatorname{cosec}^2(\pi a)$

$(b) \qquad \dfrac{1}{\pi(n-a)^2}$