DERIVATIVES

1. Using the definition, find the derivative of $w = f(z) = z^3 - 2z$ at the point where (*a*) $z = z_0$, (*b*) $z = -1$.

(*a*) By definition, the derivative at $z = z_0$ is

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \{z_0^3 - 2z_0\}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2\,\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z} \\ &= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0\,\Delta z + (\Delta z)^2 - 2 = 3z_0^2 - 2 \end{aligned}$$

In general, $f'(z) = 3z^2 - 2$ for all z.

(*b*) From (*a*), or directly, we find that if $z_0 = -1$ then $f'(-1) = 3(-1)^2 - 2 = 1$.

2. Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

By definition, $$\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\,\Delta y$ approaches zero.

Then $$\begin{aligned} \frac{d}{dz}\bar{z} &= \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x + iy + \Delta x + i\,\Delta y} - \overline{x + iy}}{\Delta x + i\,\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\,\Delta y - (x - iy)}{\Delta x + i\,\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} \end{aligned}$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$.

If $\Delta x = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$.

Then since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e. $f(z) = \bar{z}$ is *non-analytic* anywhere.

3. If $w = f(z) = \frac{1 + z}{1 - z}$, find (*a*) $\frac{dw}{dz}$ and (*b*) determine where $f(z)$ is non-analytic.

(*a*) ***Method 1,*** using the definition.

$$\begin{aligned} \frac{dw}{dz} &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\dfrac{1 + (z + \Delta z)}{1 - (z + \Delta z)} - \dfrac{1 + z}{1 - z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{2}{(1 - z - \Delta z)(1 - z)} = \frac{2}{(1 - z)^2} \end{aligned}$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

Method 2, using differentiation rules.

By the quotient rule [see Problem 10(*c*)] we have if $z \neq 1$,

$$\frac{d}{dz}\left(\frac{1 + z}{1 - z}\right) = \frac{(1 - z)\frac{d}{dz}(1 + z) - (1 + z)\frac{d}{dz}(1 - z)}{(1 - z)^2} = \frac{(1 - z)(1) - (1 + z)(-1)}{(1 - z)^2} = \frac{2}{(1 - z)^2}$$

(*b*) The function $f(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a *singular point* of $f(z)$.

6. **If $f(z) = u + iv$ is analytic in a region $\mathcal{R}$, prove that u and v are harmonic in $\mathcal{R}$ if they have continuous second partial derivatives in $\mathcal{R}$.**

If $f(z)$ is analytic in $\mathcal{R}$ then the Cauchy-Riemann equations (1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and (2) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are satisfied in $\mathcal{R}$. Assuming u and v have continuous second partial derivatives, we can differentiate both sides of (1) with respect to x and (2) with respect to y to obtain (3) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y}$ and (4) $\frac{\partial^2 v}{\partial y\, \partial x} = -\frac{\partial^2 u}{\partial y^2}$ from which $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, i.e. u is harmonic.

Similarly, by differentiating both sides of (1) with respect to y and (2) with respect to x, we find $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and v is harmonic.

It will be shown later (Chapter 5) that if $f(z)$ is analytic in $\mathcal{R}$, all its derivatives exist and are continuous in $\mathcal{R}$. Hence the above assumptions will not be necessary.

7. (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

(a) $\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y \qquad (1)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \qquad (2)$$

Adding (1) and (2) yields $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

(b) From the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y \qquad (3)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x} \cos y - xe^{-x} \cos y - ye^{-x} \sin y \qquad (4)$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x} \cos y + xe^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x) \\ &= ye^{-x} \sin y + xe^{-x} \cos y + F(x) \qquad (5) \end{aligned}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x} \sin y - xe^{-x} \cos y + e^{-x} \cos y + F'(x) = -ye^{-x} \sin y - xe^{-x} \cos y - ye^{-x} \sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then from (5),

$$v = e^{-x}(y \sin y + x \cos y) + c$$

For another method, see Problem 40.

11. Prove that (a) $\dfrac{d}{dz}e^z = e^z$, (b) $\dfrac{d}{dz}e^{az} = ae^{az}$ where a is any constant.

(a) By definition, $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = u + iv$ or $u = e^x \cos y$, $v = e^x \sin y$.

Since $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = e^x \sin y = -\dfrac{\partial u}{\partial y}$, the Cauchy-Riemann equations are satisfied. Then by Problem 5 the required derivative exists and is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x \cos y + ie^x \sin y = e^z$$

(b) Let $w = e^\zeta$ where $\zeta = az$. Then by part (a) and Problem 39,

$$\frac{d}{dz}e^{az} = \frac{d}{dz}e^\zeta = \frac{d}{d\zeta}e^\zeta \cdot \frac{d\zeta}{dz} = e^\zeta \cdot a = ae^{az}$$

We can also proceed as in part (a).

13. Prove that $\dfrac{d}{dz}z^{1/2} = \dfrac{1}{2z^{1/2}}$, realizing that $z^{1/2}$ is a multiple-valued function.

A function must be single-valued in order to have a derivative. Thus since $z^{1/2}$ is multiple-valued (in this case two-valued) we must restrict ourselves to one branch of this function at a time.

Case 1.

Let us first consider that branch of $w = z^{1/2}$ for which $w = 1$ where $z = 1$. In this case, $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and so} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

Case 2.

Next we consider that branch of $w = z^{1/2}$ for which $w = -1$ where $z = 1$. In this case too, we have $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

In both cases we have $\dfrac{d}{dz}z^{1/2} = \dfrac{1}{2z^{1/2}}$. Note that the derivative does not exist at the branch point $z = 0$. In general a function does not have a derivative, i.e. is not analytic, at a branch point. Thus branch points are singular points.

14. Prove that $\dfrac{d}{dz}\ln z = \dfrac{1}{z}$.

Let $w = \ln z$. Then $z = e^w$ and $dz/dw = e^w = z$. Hence

$$\frac{d}{dz}\ln z = \frac{dw}{dz} = \frac{1}{dz/dw} = \frac{1}{z}$$

Note that the result is valid regardless of the particular branch of $\ln z$. Also observe that the derivative does not exist at the branch point $z = 0$, illustrating further the remark at the end of Problem 13.

15. Prove that $\dfrac{d}{dz}\ln f(z) = \dfrac{f'(z)}{f(z)}$.

Let $w = \ln \zeta$ where $\zeta = f(z)$. Then

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} = \frac{1}{\zeta}\cdot\frac{d\zeta}{dz} = \frac{f'(z)}{f(z)}$$

1. Evaluate $\displaystyle\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (*a*) the parabola $x = 2t$, $y = t^2 + 3$; (*b*) straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$; (*c*) a straight line from $(0,3)$ to $(2,4)$.

(*a*) The points $(0,3)$ and $(2,4)$ on the parabola correspond to $t = 0$ and $t = 1$ respectively. Then the given integral equals

$$\int_{t=0}^{1} \{2(t^2+3) + (2t)^2\}\,2\,dt + \{3(2t) - (t^2+3)\}\,2t\,dt \quad = \quad \int_0^1 (24t^2 + 12 - 2t^3 - 6t)\,dt \quad = \quad 33/2$$

(*b*) Along the straight line from $(0,3)$ to $(2,3)$, $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 \quad = \quad \int_{x=0}^{2} (6 + x^2)\,dx \quad = \quad 44/3$$

Along the straight line from $(2,3)$ to $(2,4)$, $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy \quad = \quad \int_{y=3}^{4} (6 - y)\,dy \quad = \quad 5/2$$

Then the required value $= 44/3 + 5/2 = 103/6$.

(*c*) An equation for the line joining $(0,3)$ and $(2,4)$ is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then the line integral equals

$$\int_{y=3}^{4} \{2y + (2y-6)^2\}\,2\,dy + \{3(2y-6) - y\}\,dy \quad = \quad \int_3^4 (8y^2 - 39y + 54)\,dy \quad = \quad 97/6$$

The result can also be obtained by using $y = \frac{1}{2}(x + 6)$.

2. Evaluate $\displaystyle\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by (*a*) $z = t^2 + it$, (*b*) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

(*a*) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$ respectively. Then the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\,d(t^2 + it) \quad = \quad \int_0^2 (t^2 - it)(2t + i)\,dt \quad = \quad \int_0^2 (2t^3 - it^2 + t)\,dt \quad = \quad 10 - 8i/3$$

Another Method. The given integral equals

$$\int_C (x - iy)(dx + i\,dy) \quad = \quad \int_C x\,dx + y\,dy \; + \; i\int_C x\,dy - y\,dx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) \; + \; i\int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$

$$= \quad \int_0^2 (2t^3 + t)\,dt \; + \; i\int_0^2 (-t^2)\,dt \quad = \quad 10 - 8i/3$$

(*b*) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) \quad = \quad \int_C x\,dx + y\,dy \; + \; i\int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from $(0,0)$ to $(0,2)$ for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy \; + \; i\int_{y=0}^{2} (0)(dy) - y(0) \quad = \quad \int_{y=0}^{2} y\,dy \quad = \quad 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2 \cdot 0 \; + \; i\int_{x=0}^{4} x \cdot 0 - 2\,dx \quad = \quad \int_0^4 x\,dx \; + \; i\int_0^4 -2\,dx \quad = \quad 8 - 8i$$

Then the required value $= 2 + (8 - 8i) = 10 - 8i$.

21. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z=a$ is (*a*) outside C, (*b*) inside C.

(*a*) If a is outside C, then $f(z) = 1/(z-a)$ is analytic everywhere inside and on C. Hence by Cauchy's theorem, $\oint_C \frac{dz}{z-a} = 0$.

(*b*) Suppose a is inside C and let Γ be a circle of radius ϵ with center at $z=a$ so that Γ is inside C [this can be done since $z=a$ is an interior point].

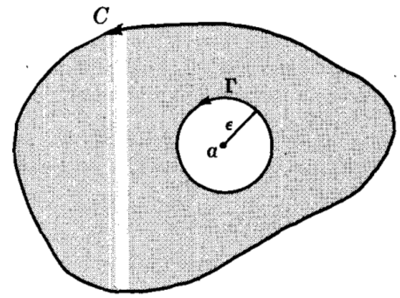

Fig. 4-20

By Problem 20,

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \qquad (1)$$

Now on Γ, $|z-a| = \epsilon$ or $z-a = \epsilon e^{i\theta}$, i.e. $z = a + \epsilon e^{i\theta}$, $0 \leqq \theta < 2\pi$. Thus since $dz = i\epsilon e^{i\theta}\, d\theta$, the right side of (*1*) becomes

$$\int_{\theta=0}^{2\pi} \frac{i\epsilon\, e^{i\theta}\, d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

which is the required value.

22. Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z=a$ is inside the simple closed curve C.

As in Problem 21,
$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n}$$
$$= \int_0^{2\pi} \frac{i\epsilon\, e^{i\theta}\, d\theta}{\epsilon^n e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\, d\theta$$
$$= \frac{i}{\epsilon^{n-1}} \frac{e^{(1-n)i\theta}}{(1-n)i}\bigg|_0^{2\pi} = \frac{1}{(1-n)\,\epsilon^{n-1}}\left[e^{2(1-n)\pi i} - 1\right] = 0$$

where $n \neq 1$.

23. If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points $(1,1)$ and $(2,3)$, find the value of

$$\int_C (12z^2 - 4iz)\,dz$$

Method 1. By Problem 17, the integral is independent of the path joining $(1,1)$ and $(2,3)$. Hence any path can be chosen. In particular let us choose the straight line paths from $(1,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$.

Case 1. Along the path from $(1,1)$ to $(2,1)$, $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$. Then the integral equals

$$\int_{x=1}^{2} \{12(x+i)^2 - 4i(x+i)\}\,dx \quad = \quad \{4(x+i)^3 - 2i(x+i)^2\}\Big|_1^2 \quad = \quad 20 + 30i$$

Case 2. Along the path from $(2,1)$ to $(2,3)$, $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy$, $dz = i\,dy$. Then the integral equals

$$\int_{y=1}^{3} \{12(2+iy)^2 - 4i(2+iy)\}i\,dy \quad = \quad \{4(2+iy)^3 - 2i(2+iy)^2\}\Big|_1^3 \quad = \quad -176 + 8i$$

Then adding, the required value $= (20+30i) + (-176+8i) = -156 + 38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz \quad = \quad (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} \quad = \quad -156 + 38i$$

It is clear that Method 2 is easier.

26. Show that $\displaystyle\int \frac{dz}{z^2+a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2.$

Let $z = a\tan u$. Then

$$\int \frac{dz}{z^2+a^2} \quad = \quad \int \frac{a\sec^2 u\,du}{a^2(\tan^2 u + 1)} \quad = \quad \frac{1}{a}\int du \quad = \quad \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1$$

Also,

$$\frac{1}{z^2+a^2} \quad = \quad \frac{1}{(z-ai)(z+ai)} \quad = \quad \frac{1}{2ai}\left(\frac{1}{z-ai} - \frac{1}{z+ai}\right)$$

and so

$$\int \frac{dz}{z^2+a^2} \quad = \quad \frac{1}{2ai}\int \frac{dz}{z-ai} - \frac{1}{2ai}\int \frac{dz}{z+ai}$$

$$= \quad \frac{1}{2ai}\ln(z-ai) - \frac{1}{2ai}\ln(z+ai) + c_2 \quad = \quad \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2$$

36. Evaluate $\int_C (z^2+3z)\,dz$ along (a) the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction, (b) the straight line from $(2,0)$ to $(0,2)$, (c) the straight lines from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$. *Ans.* $-\frac{44}{3}-\frac{8}{3}i$ for all cases

37. If $f(z)$ and $g(z)$ are integrable, prove that

$$\text{(a)} \quad \int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

$$\text{(b)} \quad \int_C \{2f(z) - 3i\,g(z)\}\,dz = 2\int_C f(z)\,dz - 3i\int_C g(z)\,dz.$$

38. Evaluate $\int_i^{2-i} (3xy+iy^2)\,dz$ (a) along the straight line joining $z=i$ and $z=2-i$, (b) along the curve $x=2t-2,\ y=1+t-t^2$. *Ans.* (a) $-\frac{4}{3}+\frac{8}{3}i$, (b) $-\frac{1}{3}+\frac{79}{30}i$

39. Evaluate $\oint_C \bar{z}^2\,dz$ around the circles (a) $|z|=1$, (b) $|z-1|=1$. *Ans.* (a) 0, (b) $4\pi i$

40. Evaluate $\oint_C (5z^4-z^3+2)\,dz$ around (a) the circle $|z|=1$, (b) the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, (c) the curve consisting of the parabolas $y=x^2$ from $(0,0)$ to $(1,1)$ and $y^2=x$ from $(1,1)$ to $(0,0)$. *Ans.* 0 in all cases

41. Evaluate $\int_C (z^2+1)^2\,dz$ along the arc of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ from the point where $\theta=0$ to the point where $\theta=2\pi$. *Ans.* $(96\pi^5a^5+80\pi^3a^3+30\pi a)/15$

42. Evaluate $\int_C \bar{z}^2\,dz + z^2\,d\bar{z}$ along the curve C defined by $z^2+2z\bar{z}+\bar{z}^2 = (2-2i)z+(2+2i)\bar{z}$ from the point $z=1$ to $z=2+2i$. *Ans.* 248/15

43. Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$, (b) the circle $|z-1|=5$, (c) the square with vertices at $3\pm 3i$, $-3\pm 3i$. *Ans.* $2\pi i$ in all cases

44. Evaluate $\oint_C (x^2+iy^2)\,ds$ around the circle $|z|=2$ where s is the arc length. *Ans.* $8\pi(1+i)$

29. Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$ **where C is the circle $|z|=4$.**

The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ are at $z=\pm\pi i$ inside C and are both of order two.

Residue at $z=\pi i$ is $\displaystyle\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}.$

Residue at $z=-\pi i$ is $\displaystyle\lim_{z\to-\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}.$

Then $\displaystyle\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i(\text{sum of residues}) = 2\pi i\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}.$

30. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ if C is (*a*) the circle $|z|=3$, (*b*) the circle $|z|=1$. *Ans.* (*a*) e^2, (*b*) 0

31. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$. *Ans.* $2\pi i$

32. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (*a*) the circle $|z-1|=4$, (*b*) the ellipse $|z-2|+|z+2|=6$.
Ans. (*a*) $-2\pi i$, (*b*) 0

33. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices at: (*a*) $2\pm i$, $-2\pm i$; (*b*) $-i$, $2-i$, $2+i$, i.
Ans. (*a*) 0, (*b*) $-\frac{1}{2}$

34. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t>0$ and C is the circle $|z|=3$.

35. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$. *Ans.* $-\pi i$

36. Prove that $f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^4}$ if C is a simple closed curve enclosing $z=a$ and $f(z)$ is analytic inside and on C.

37. Prove Cauchy's integral formulas for all positive integral values of n. [*Hint*: Use mathematical induction.]

38. Find the value of (*a*) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$, (*b*) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ if C is the circle $|z|=1$.
Ans. (*a*) $\pi i/32$, (*b*) $21\pi i/16$

39. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=3$. *Ans.* $\frac{1}{2}(\sin t - t\cos t)$

40. Prove Cauchy's integral formulas for the multiply-connected region of Fig. 4-26, Page 115.

79. Evaluate $\frac{1}{2\pi i}\oint_C \frac{z^2\,dz}{z^2+4}$ where C is the square with vertices at ± 2, $\pm 2 + 4i$. *Ans.* i

80. Evaluate $\oint_C \frac{\cos^2 tz}{z^3}\,dz$ where C is the circle $|z| = 1$ and $t > 0$. *Ans.* $-2\pi i t^2$

81. (*a*) Show that $\oint_C \frac{dz}{z+1} = 2\pi i$ if C is the circle $|z| = 2$.

(*b*) Use (*a*) to show that

$$\oint_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} = 0, \qquad \oint_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} = 2\pi$$

and verify these results directly.

82. Find all functions $f(z)$ which are analytic everywhere in the entire complex plane and which satisfy the conditions (*a*) $f(2-i) = 4i$ and (*b*) $|f(z)| < e^2$ for all z.

83. If $f(z)$ is analytic inside and on a simple closed curve C, prove that

$$(a)\quad f'(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} f(a + e^{i\theta})\,d\theta$$

$$(b)\quad \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta} f(a + e^{i\theta})\,d\theta$$

84. Prove that $8z^4 - 6z + 5 = 0$ has one root in each quadrant.

85. Show that (*a*) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 0$, (*b*) $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = 2\pi$.

86. Extend the result of Problem 23 so as to obtain formulas for the derivatives of $f(z)$ at any point in $\mathcal{R}$.

87. Prove that $z^3 e^{1-z} = 1$ has exactly two roots inside the circle $|z| = 1$.

88. If $t > 0$ and C is any simple closed curve enclosing $z = -1$, prove that

$$\frac{1}{2\pi i}\oint_C \frac{ze^{zt}}{(z+1)^3}\,dz = \left(t - \frac{t^2}{2}\right)e^{-t}$$

27. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for (*a*) $1 < |z| < 3$, (*b*) $|z| > 3$, (*c*) $0 < |z+1| < 2$, (*d*) $|z| < 1$.

(*a*) Resolving into partial fractions, $\dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left(\dfrac{1}{z+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{z+3}\right)$.

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(*b*) If $|z| > 1$, we have as in part (*a*),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(*c*) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2,\ u \neq 0$ or $0 < |z+1| < 2$.

(*d*) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \tfrac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \tfrac{1}{2} - \tfrac{1}{2}z + \tfrac{1}{2}z^2 - \tfrac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (*a*),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e. $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series.*

27. Let $u(x, y) = \alpha$ and $v(x, y) = \beta$, where u and v are the real and imaginary parts of an analytic function $f(z)$ and α and β are any constants, represent two families of curves. Prove that the families are orthogonal (i.e. each member of one family is perpendicular to each member of the other family at their point of intersection).

Consider any two members of the respective families, say $u(x, y) = \alpha_1$ and $v(x, y) = \beta_1$ where α_1 and β_1 are particular constants [Fig. 3-10].

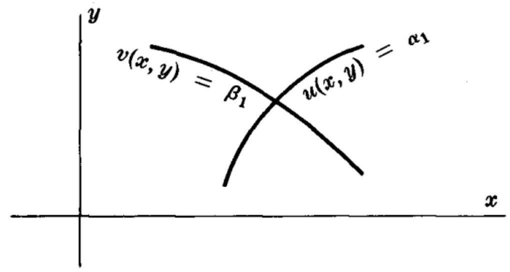

Fig. 3-10

Differentiating $u(x, y) = \alpha_1$ with respect to x yields

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

Then the slope of $u(x, y) = \alpha_1$ is

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}$$

Similarly the slope of $v(x, y) = \beta_1$ is

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}$$

The product of the slopes is, using the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\Big/\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -1$$

Thus the curves are orthogonal.

LAURENT'S THEOREM

91. Expand $f(z) = 1/(z-3)$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

Ans. (a) $-\frac{1}{3} - \frac{1}{9}z - \frac{1}{27}z^2 - \frac{1}{81}z^3 - \cdots$ (b) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots$

92. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$, (e) $0 < |z-2| < 1$.

Ans. (a) $-\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \cdots$ (b) $\cdots + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \cdots$

(c) $-\frac{1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \cdots$ (d) $-(z-1)^{-1} - 2(z-1)^{-2} - 2(z-1)^{-3} - \cdots$

(e) $1 - 2(z-2)^{-1} - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4 - \cdots$

93. Expand $f(z) = 1/z(z-2)$ in a Laurent series valid for (a) $0 < |z| < 2$, (b) $|z| > 2$.

94. Find an expansion of $f(z) = z/(z^2+1)$ valid for $|z-3| > 2$.

95. Expand $f(z) = 1/(z-2)^2$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$

96. Expand each of the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case.

(a) $(1-\cos z)/z$, (b) e^{z^2}/z^3, (c) $z^{-1}\cosh z^{-1}$, (d) $z^2 e^{-z^4}$, (e) $z \sinh \sqrt{z}$.

Ans. (a) $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$; removable singularity

(b) $\frac{1}{z^3} + \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \frac{z^5}{4!} + \frac{z^7}{5!} + \cdots$; pole of order 3

(c) $\frac{1}{z} - \frac{1}{2!\,z^3} + \frac{1}{4!\,z^5} - \cdots$; essential singularity

(d) $z^2 - z^6 + \frac{z^{10}}{2!} - \frac{z^{14}}{3!} + \cdots$; ordinary point

(e) $z^{3/2} + \frac{z^{5/2}}{3!} + \frac{z^{7/2}}{5!} + \frac{z^{9/2}}{7!} + \cdots$; branch point

97. Show that if $\tan z$ is expanded into a Laurent series about $z = \pi/2$, (a) the principal part is $-1/(z-\pi/2)$, (b) the series converges for $0 < |z-\pi/2| < \pi/2$, (c) $z = \pi/2$ is a simple pole.

99. (a) Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and (b) determine the region of convergence of this series. (c) Classify the singularities of $f(z)$.

Ans. (a) $e\left\{1 + 2(z-2)^{-1} + \frac{2^2(z-2)^{-2}}{2!} + \frac{2^3(z-2)^{-3}}{3!} + \cdots\right\}$ (b) $|z-2| > 0$ (c) $z = 2$; essential singularity, $z = \infty$; removable singularity

10. Show that $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

The poles of $\frac{z^2}{(z^2+1)^2(z^2+2z+2)}$ enclosed by the contour C of Fig. 7-5 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is $\lim_{z \to i} \frac{d}{dz}\left\{(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$.

Residue at $z = -1+i$ is $\lim_{z \to -1+i} (z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$.

Then
$$\oint_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or
$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_{\Gamma} \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 7, we obtain the required result.

14. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

Letting $z = e^{i\theta}$, we have $\sin\theta = (z - z^{-1})/2i$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \oint_C \frac{dz/iz}{\{5 - 3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2 - 10iz - 3)^2}$$

where C is the contour of Fig. 7-6.

The integrand has poles of order 2 at $z = \dfrac{10i \pm \sqrt{-100+36}}{6} = \dfrac{10i \pm 8i}{6} = 3i,\ i/3$. Only the pole $i/3$ lies inside C.

$$\text{Residue at } z = i/3 = \lim_{z\to i/3} \frac{d}{dz}\left\{(z-i/3)^2 \cdot \frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$= \lim_{z\to i/3} \frac{d}{dz}\left\{(z-i/3)^2 \cdot \frac{z}{(3z-i)^2\,(z-3i)^2}\right\} = -\frac{5}{256}.$$

Then

$$-\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$

Another method.

From Problem 12, we have for $a > |b|$,

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Then by differentiating both sides with respect to a (considering b as constant) using Leibnitz's rule, we have

$$\frac{d}{da}\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{\partial}{\partial a}\left(\frac{1}{a+b\sin\theta}\right)d\theta = -\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2}$$

$$= \frac{d}{da}\left(\frac{2\pi}{\sqrt{a^2-b^2}}\right) = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$$

i.e.,

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Letting $a = 5$ and $b = -3$, we have

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{2\pi(5)}{(5^2-3^2)^{3/2}} = \frac{5\pi}{32}$$

DEFINITE INTEGRALS

49. Prove that $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

50. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$. *Ans.* $5\pi/288$

51. Evaluate $\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta}\, d\theta$. *Ans.* 0

52. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\, d\theta$.

53. Prove that $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta = \frac{3\pi}{8}$.

54. Prove that if $m > 0$, $\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{4}$.

55. (*a*) Find the residue of $\frac{e^{iz}}{(z^2+1)^5}$ at $z = i$. (*b*) Evaluate $\int_0^\infty \frac{\cos x}{(x^2+1)^5}\, dx$.

56. If $a^2 > b^2 + c^2$, prove that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$.

57. Prove that $\int_0^{2\pi} \frac{\cos 3\theta}{(5 - 3\cos\theta)^4}\, d\theta = \frac{135\pi}{16{,}384}$.

58. Evaluate $\int_0^\infty \frac{dx}{x^4+x^2+1}$. *Ans.* $\pi\sqrt{3}/6$

59. Evaluate $\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2}$. *Ans.* $\pi/2$

60. Prove that $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.

61. Discuss the validity of the following solution to Problem 19. Let $u = (1+i)x/\sqrt{2}$ in the result $\int_0^\infty e^{-u^2}\, du = \frac{1}{2}\sqrt{\pi}$ to obtain $\int_0^\infty e^{-ix^2}\, dx = \frac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

62. Show that $\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\, dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi/\sqrt{3}}$.

24. (*a*) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ and (*b*) determine the region of convergence of this series.

(*a*) $f(z) = \sin z,\ f'(z) = \cos z,\ f''(z) = -\sin z,\ f'''(z) = -\cos z,\ f^{IV}(z) = \sin z,\ \ldots$

$f(\pi/4) = \sqrt{2}/2,\ f'(\pi/4) = \sqrt{2}/2,\ f''(\pi/4) = -\sqrt{2}/2,\ f'''(\pi/4) = -\sqrt{2}/2,\ f^{IV}(\pi/4) = \sqrt{2}/2,\ \ldots$

Then, since $a = \pi/4$,

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \cdots \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

Another method.

Let $u = z - \pi/4$ or $z = u + \pi/4$. Then we have,

$$\begin{aligned} \sin z &= \sin(u+\pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4) \\ &= \frac{\sqrt{2}}{2}(\sin u + \cos u) \\ &= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

(*b*) Since the singularity of $\sin z$ nearest to $\pi/4$ is at infinity, the series converges for all finite values of z, i.e. $|z| < \infty$. This can also be established by the ratio test.

توجه: شماره مسایل به ترتیب نیست.