



A Note on the Controllability of Higher-Order Linear Systems

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Abstract—In this paper, a new condition for the controllability of higher-order linear dynamical systems is obtained. The suggested test contains rank conditions of suitably defined matrices and is based on the notion of compound matrices and the Binet-Cauchy formula. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

Consider the higher-order linear system

$$A_l q^{(l)}(t) + A_{l-1} q^{(l-1)}(t) + \cdots + A_1 q^{(1)}(t) + A_0 q(t) = Bu(t), \quad (1)$$

where $A_j \in \mathbb{C}^{n \times n}$ ($j = 0, 1, \dots, l$), t is the independent time variable, $q(t) \in \mathbb{C}^n$ is the unknown vector function, $u(t) \in \mathbb{C}^m$ is the piecewise continuous *input (control) vector* and $B \in \mathbb{C}^{n \times m}$ is the *input matrix*. (The indices on $q(t)$ denote derivatives with respect to t .) Applying the Laplace transformation to (1) yields the *matrix polynomial*

$$L(\lambda) = A_l \lambda^l + A_{l-1} \lambda^{l-1} + \cdots + A_1 \lambda + A_0, \quad (2)$$

where λ is a complex variable. As a consequence, the spectral analysis of $L(\lambda)$ leads to solutions of (1). The suggested references on matrix polynomials and their applications to differential equations are [1,2].

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A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $L(\lambda)$ in (2) if the system $L(\lambda_0)y = 0$ has a nonzero solution $y_0 \in \mathbb{C}^n$. This solution y_0 is known as an *eigenvector* of $L(\lambda)$ corresponding to λ_0 , and the set of all eigenvalues of $L(\lambda)$ is the *spectrum* of $L(\lambda)$, namely, $\sigma(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$. At this point and for the remainder of this paper, we shall assume that the matrix polynomial $L(\lambda)$ in (2) has a nonsingular leading coefficient A_l , and thus, $L(\lambda)$ has exactly nl eigenvalues, counting multiplicities.

The dynamical system (1) is equivalent to the first-order system

$$x^{(1)}(t) = C_L x(t) + \tilde{B}u(t), \tag{3}$$

where

$$C_L = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -A_l^{-1}A_0 & -A_l^{-1}A_1 & -A_l^{-1}A_2 & \dots & -A_l^{-1}A_{l-1} \end{bmatrix} \in \mathbb{C}^{nl \times nl}, \tag{4}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_l^{-1}B \end{bmatrix} \in \mathbb{C}^{nl \times m}, \quad \text{and} \quad x(t) = \begin{bmatrix} q(t) \\ q^{(1)}(t) \\ \vdots \\ q^{(l-1)}(t) \end{bmatrix} \in \mathbb{C}^{nl}. \tag{5}$$

The $nl \times nl$ matrix C_L in (4) is known as the (*block*) *companion matrix* of $L(\lambda)$ and its spectrum coincides with $\sigma(L)$, and the vector $x(t)$ in (5) is called the *state vector* of system (3) [1-3]. As a consequence, for a given initial condition $x_0 = x(0)$, the general solution of (1) is given by [1, Theorem 1.5]

$$q(t) = Z_r e^{tC_L} x_0 + Z_r \int_0^t e^{(t-s)C_L} Z_c B u(s) ds,$$

where $Z_r = [I \ 0 \ \dots \ 0] \in \mathbb{C}^{n \times nl}$ and $Z_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_l^{-1} \end{bmatrix} \in \mathbb{C}^{nl \times n}$.

The notion of the *controllability* of dynamical systems has attracted attention for some years. It refers to the ability of a system to transfer the state vector from one specified vector value to another in finite time. In particular, systems (1) and (3) are called *controllable* if for every $x_0, \omega \in \mathbb{C}^{nl}$, there exist an input vector $u(t)$ and a real $t_0 > 0$, such that $x(0) = x_0$ and $x(t_0) = \omega$.

In this article, we obtain an alternative test for the controllability of higher-order linear dynamical systems. The important feature of the new rank condition (Theorem 2) is that it is independent of λ and requires no computation of the eigenvalues of $L(\lambda)$ (see Statements (ii) and (iii) in Theorem 1) or the inverse of the leading coefficient A_l (see Statement (v) in Theorem 1). It is assumed that the reader is familiar with the notion of compound matrices and the Binet-Cauchy formula. The suggested reference is [4].

2. THE NEW TEST

One of the major concerns for a control engineer is to maintain the stability of certain systems. For this reason, in many cases, the behavior of the dynamical system (1) is modified by applying state feedback (i.e., input vector) of the form

$$u(t) = v(t) - F_{l-1}q^{(l-1)}(t) - \dots - F_1q^{(1)}(t) - F_0q(t),$$

where $F_j \in \mathbb{C}^{m \times n}$ ($j = 1, 2, \dots, l-1$) [3,5-7]. The new *closed loop* system

$$A_l q^{(l)}(t) + (A_{l-1} + BF_{l-1})q^{(l-1)}(t) + \dots + (A_0 + BF_0)q(t) = Bv(t) \tag{6}$$

is associated with the matrix polynomial

$$L_F(\lambda) = L(\lambda) + BF(\lambda), \quad (7)$$

where $F(\lambda) = F_{l-1}\lambda^{l-1} + \dots + F_1\lambda + F_0$.

Classical results on the controllability of first-order dynamical systems (see for example [2,8,9] and the references therein) have been generalized to the higher-order systems (1) and (6) in a natural way.

THEOREM 1. (See [3, Theorem 2.5 and Proposition 2.1], [7, Theorem V.2].) *The following statements are equivalent.*

- (i) System (1) is controllable.
- (ii) $\text{Ker } B^* \cap \text{Ker } L(\lambda)^* = \{0\}$, for all $\lambda \in \sigma(L)$.
- (iii) $\text{rank } [L(\lambda) \ B] = n$, for all $\lambda \in \sigma(L)$.
- (iv) System (6) is controllable.
- (v) $\text{rank } [\tilde{B} \ C_L \tilde{B} \ \dots \ C_L^{n_l-1} \tilde{B}] = nl$.

Observe now that the matrix polynomial $L_F(\lambda)$ in (7) is written

$$L_F(\lambda) = [L(\lambda) \ B\lambda^{l-1} \ \dots \ B\lambda \ B] \begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix}.$$

By taking the n^{th} compound matrix $C_n(\cdot)$ of both sides in the above equation, and using the Binet-Cauchy formula for compound matrices [4], it follows

$$\begin{aligned} \det L_F(\lambda) &= C_n(L_F(\lambda)) \\ &= C_n([L(\lambda) \ B\lambda^{l-1} \ \dots \ B\lambda \ B]) C_n \left(\begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix} \right). \end{aligned} \quad (8)$$

Moreover, it is clear that

$$C_n([L(\lambda) \ B\lambda^{l-1} \ \dots \ B\lambda \ B]) = [p_1(\lambda) \ p_2(\lambda) \ \dots \ p_\xi(\lambda)],$$

where $\xi = \binom{n+lm}{n}$, and $p_1(\lambda), p_2(\lambda), \dots, p_\xi(\lambda)$ are scalar polynomials of degree no more than nl . For every $j = 1, 2, \dots, \xi$,

$$p_j(\lambda) = [1 \ \lambda \ \lambda^2 \ \dots \ \lambda^{nl}] r_j,$$

where $r_j \in \mathbb{C}^{nl+1}$ is the vector of the (corresponding) coefficients of $p_j(\lambda)$. Hence, for the $(nl+1) \times \xi$ complex matrix

$$P(L(\lambda), B) = [r_1 \ r_2 \ \dots \ r_\xi], \quad (9)$$

which is known as the *Plücker matrix* of system (1) [10,11], it follows

$$C_n([L(\lambda) \ B\lambda^{l-1} \ \dots \ B\lambda \ B]) = [1 \ \lambda \ \lambda^2 \ \dots \ \lambda^{nl}] P(L(\lambda), B). \quad (10)$$

THEOREM 2. *The higher-order dynamical system (1) is controllable if and only if the Plücker matrix $P(L(\lambda), B)$ in (9) has full (row) rank, i.e.,*

$$\text{rank } P(L(\lambda), B) = nl + 1.$$

PROOF. Suppose that system (1) is controllable. Then by applying [9, Theorem 2.1] to the first-order system (3), we have that system (1) is controllable if and only if the spectrum of the closed loop system (6) can be assigned arbitrarily by suitable choice of F_0, F_1, \dots, F_{l-1} . Hence, for every monic scalar polynomial $d(\lambda)$ of degree nl , there exist $m \times n$ matrices F_0, F_1, \dots, F_{l-1} such that the matrix polynomial $L_F(\lambda)$ in (7) satisfies (recall that $\det A_l \neq 0$)

$$\det L_F(\lambda) = \det A_l d(\lambda).$$

Hence, equation (8) yields

$$C_n ([L(\lambda) \quad B\lambda^{l-1} \quad \dots \quad B\lambda \quad B]) C_n \begin{pmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{pmatrix} = \det A_l [1 \quad \lambda \quad \dots \quad \lambda^{nl}] z_d,$$

where z_d is the vector of the (corresponding) coefficients of $d(\lambda)$. Denoting

$$g_F = C_n \begin{pmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{pmatrix},$$

by the above discussion, it follows that

$$[1 \quad \lambda \quad \lambda^2 \quad \dots \quad \lambda^{nl}] P(L(\lambda), B) g_F = \det A_l [1 \quad \lambda \quad \lambda^2 \quad \dots \quad \lambda^{nl}] z_d,$$

for every $\lambda \in \mathbb{C}$, which implies that

$$P(L(\lambda), B) g_F = \det A_l z_d. \tag{11}$$

This system has $nl + 1$ equations and $\xi = \binom{n+lm}{n}$ unknowns, and since $n, m, l \geq 1$, one can see that $\xi \geq nl + 1$. As a consequence, (11) has solutions for every vector $z_d \in \mathbb{C}^{nl+1}$ (with its first coordinate equal to 1) if and only if the Plücker matrix $P(L(\lambda), B)$ has full (row) rank.

Conversely, assume that $\text{rank } P(L(\lambda), B) = nl + 1$ and that the dynamical system (1) is not controllable. Then by Theorem 1, there is a $\lambda_0 \in \mathbb{C}$, such that $\text{rank } [L(\lambda_0) \quad B] < n$. Moreover,

$$\text{rank } [L(\lambda_0) \quad B\lambda_0^{l-1} \quad \dots \quad B\lambda_0 \quad B] < n,$$

which means that all the $n \times n$ minors of the matrix $[L(\lambda_0) \quad B\lambda_0^{l-1} \quad \dots \quad B\lambda_0 \quad B]$ are zero. Hence,

$$C_n ([L(\lambda_0) \quad B\lambda_0^{l-1} \quad \dots \quad B\lambda_0 \quad B]) = 0,$$

and by (10),

$$[1 \quad \lambda_0 \quad \lambda_0^2 \quad \dots \quad \lambda_0^{nl}] P(L(\lambda), B) = 0.$$

Since $\xi \geq nl + 1$ and $[1 \quad \lambda_0 \quad \lambda_0^2 \quad \dots \quad \lambda_0^{nl}] \neq 0$, it is clear that

$$\text{rank } P(L(\lambda), B) < nl + 1,$$

that is a contradiction. Thus, system (1) is controllable. ■

Notice that the above method involves no computation of the spectrum $\sigma(L)$ or the matrix A_l^{-1} . Our result is illustrated in the following example.

EXAMPLE. Let $L(\lambda)$ be the 2×2 matrix polynomial

$$L(\lambda) = I\lambda^2 + A_1\lambda + A_0 = \begin{bmatrix} \lambda^2 + 1 & \lambda - 1 \\ \lambda - 1 & \lambda^2 - 1 \end{bmatrix},$$

and let $B_1 = [0 \ 1]^\top$ and $B_2 = [1 \ 0]^\top$. Consider the second-order linear systems

$$q^{(2)}(t) + A_1q^{(1)}(t) + A_0q(t) = B_1u(t) \quad (12)$$

and

$$q^{(2)}(t) + A_1q^{(1)}(t) + A_0q(t) = B_2u(t), \quad (13)$$

where $u(t)$ is the 1×1 input vector. For every $\lambda \in \mathbb{C}$,

$$\text{rank} [L(\lambda) \ B_1] = \text{rank} \begin{bmatrix} \lambda^2 + 1 & \lambda - 1 & 0 \\ \lambda - 1 & \lambda^2 - 1 & 1 \end{bmatrix} = 2.$$

On the other hand, for $\lambda_0 = 1$,

$$\text{rank} [L(\lambda_0) \ B_2] = \text{rank} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1 < 2.$$

Thus, by Theorem 1, system (12) is controllable but system (13) is not. Furthermore, we can see that

$$\begin{aligned} \mathcal{C}_2([L(\lambda) \ B_1 \lambda \ B_1]) &= \mathcal{C}_2 \left(\begin{bmatrix} \lambda^2 + 1 & \lambda - 1 & 0 & 0 \\ \lambda - 1 & \lambda^2 - 1 & \lambda & 1 \end{bmatrix} \right) \\ &= [\lambda^4 - \lambda^2 + 2\lambda - 2, \ \lambda^3 + \lambda, \ \lambda^2 + 1, \ \lambda^2 - \lambda, \ \lambda - 1, \ 0] \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_2([L(\lambda) \ B_2 \lambda \ B_2]) &= \mathcal{C}_2 \left(\begin{bmatrix} \lambda^2 + 1 & \lambda - 1 & \lambda & 1 \\ \lambda - 1 & \lambda^2 - 1 & 0 & 0 \end{bmatrix} \right) \\ &= [\lambda^4 - \lambda^2 + 2\lambda - 2, \ -\lambda^2 + \lambda, \ -\lambda + 1, \ -\lambda^3 + \lambda, \ -\lambda^2 + 1, \ 0]. \end{aligned}$$

As a consequence,

$$\text{rank } P(L(\lambda), B_1) = \text{rank} \begin{bmatrix} -2 & 0 & 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 5$$

and

$$\text{rank } P(L(\lambda), B_2) = \text{rank} \begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 4 < 5,$$

confirming Theorem 2. ■

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