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Linear Algebra and its Applications 413 (2006) 285–296

LINEAR ALGEBRA AND ITS APPLICATIONS

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On higher order linear systems: Reachability and feedback invariants

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Received 8 October 2004; accepted 19 October 2005
Available online 2 December 2005
Submitted by B.L. Shader

Abstract

We consider l-order linear control systems Σ with coefficients in a commutative ring R. The notion of reachability is studied for such systems and it is related to the reachability of the associated linearized system $\text{lin}(\Sigma)$.

We prove that reachability is a pointwise property, just as in the case of first order systems.

The feedback equivalence of l-order linear control systems over a commutative ring is also studied. We introduce some feedback invariants that generalize the Hermida–Pérez–SánchezGiralda invariant modules M_i. To conclude we apply results in order to give classification results in low dimension.

AMS classification: 93B10; 93B25; 93B55

Keywords: Linear system; Feedback; Pointwise study

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[1] Partially Supported by DGI (Spain) and Univ. León (Grupo de Álgebra Lineal y Computación Cuántica).

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doi:10.1016/j.laa.2005.10.016

1. Introduction

Consider the linear difference equation

$$x(t+l) = A_{l-1}x(t+l-1) + \cdots + A_1 x(t+1) + A_0 x(t) + Bu(t),$$

where $x(t) \in R^n$, $u(t) \in R^m$ for each t; $A_i \in R^{n\times n}$, $B \in R^{n\times m}$ are matrices with entries in R and R denotes a commutative ring with identity. For fixed initial values $(x_0, \ldots, x_{l-1}) = x_{\text{init}} \in (R^n)^l$ and a sequence of controls $u : \{0, 1, 2, \ldots\} \to R^m$ one has the evolution of system obtained from the difference equation

$$\begin{aligned}
\Phi_\Sigma(0, x_{\text{init}}, u) &= x_0 \\
\Phi_\Sigma(1, x_{\text{init}}, u) &= x_1 \\
&\ \vdots \\
\Phi_\Sigma(l-1, x_{\text{init}}, u) &= x_{l-1} \\
\Phi_\Sigma(l, x_{\text{init}}, u) &= \sum_{i=0}^{l-1} A_i \Phi_\Sigma(i, x_{\text{init}}, u) + Bu(0) \\
\Phi_\Sigma(l+1, x_{\text{init}}, u) &= \sum_{i=0}^{l-1} A_i \Phi_\Sigma(i+1, x_{\text{init}}, u) + Bu(1) \\
&\ \vdots \\
\Phi_\Sigma(l+t, x_{\text{init}}, u) &= \sum_{i=0}^{l-1} A_i \Phi_\Sigma(i+t, x_{\text{init}}, u) + Bu(t) \\
\Phi_\Sigma(l+t+1, x_{\text{init}}, u) &= \sum_{i=0}^{l-1} A_i \Phi_\Sigma(i+t+1, x_{\text{init}}, u) + Bu(t+1) \\
&\ \vdots
\end{aligned} \tag{1}$$

The difference equation gives rise to a higher order linear (dynamical) system. We denote that system by $\Sigma = (A_0, \ldots, A_{l-1}; B)$. In the case $l = 1$ we obtain the classical definition of linear system over a ring (see [2] or [7]).

Linearization is one of the usual ways to study an l-order linear system (see, for example [6]): One consider chains of l internal states $(x(t), x(t+1), \ldots, x(t+l-1))$ as new states and consequently the l-order linear system $\Sigma = (A_0, \ldots, A_{l-1}; B)$ is transformed in a first order linear system $\text{lin}(\Sigma) = (\widehat{A}, \widehat{B})$ (see Section 2 below).

In Section 2 we study the reachability property of l-order systems. This notion is characterized and we prove that the reachability of a system Σ is not equivalent to the reachability of the first order system $\text{lin}(\Sigma)$. Note that this notion is closely related to the notion of controllability given in [8,6] for higher order linear systems over $\mathbb{C}$.

Section 3 is devoted to study feedback actions on l-order linear systems. After a review of the feedback actions we introduce some feedback invariants associated to an l-order system following the Hermida–Pérez–SánchezGiralda invariants for the case of first order linear systems.

In Section 4 we discuss the notion of reachability of higher order linear systems from the pointwise point of view.

Finally, in Section 5 we give some classification results by using the invariants obtained in Section 3.

2. Reachability

Let Σ be an l-order, m-input linear system over R^n. Let $x_0, x_1, \dots, x_{l-1}, \omega$ be elements of R^n. Following Sontag [7] we say that the state ω is reachable from the initial condition $x_{\text{init}} = (x_0, \dots, x_{l-1})$ if there exists a control function $u : \{0, 1, \dots\} \to R^m$ such that $\Phi_\Sigma(t, x_{\text{init}}, u) = \omega$ for some $t \geqslant 0$. In this case we denote the fact by $(x_0, x_1, \dots, x_{l-1}) \underset{u}{\rightsquigarrow} \omega$, or simply by $(x_0, x_1, \dots, x_{l-1}) \rightsquigarrow \omega$, when the expression of the control function is not needed.

Definition 1. An l-order, m-input linear system Σ over R^n is reachable if for each $x_{\text{init}} \in (R^n)^l$ and each $\omega \in R^n$ there exists a control function u such that $(x_0, x_1, \dots, x_{l-1}) \underset{u}{\rightsquigarrow} \omega$.

Now we give a characterization of the property of reachability using the linearized system $\widehat{\Sigma}$ of a given system Σ.

Definition 2. Let Σ be an l-order, m-input linear system over R^n. Then we define the linearization of Σ as the system: $\text{lin}(\Sigma) = \widehat{\Sigma} = (\widehat{A}, \widehat{B})$ where

$$\widehat{A} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{1} \\ A_0 & A_1 & \dots & A_{l-2} & A_{l-1} \end{pmatrix} \qquad \widehat{B} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B \end{pmatrix}.$$

Note that the linearization $(\widehat{A}, \widehat{B})$ of the system $\Sigma = (A_0, \dots, A_{l-1}; B)$ is just the first order linear system

$$\begin{pmatrix} x(t+1) \\ x(t+2) \\ \vdots \\ x(t+l) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{1} \\ A_0 & A_1 & \dots & A_{l-2} & A_{l-1} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+1) \\ \vdots \\ x(t+l-1) \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B \end{pmatrix} u(t)$$

associated to the l-order system Σ. Denote also $(\widehat{A} * \widehat{B}) = (\widehat{B}, \widehat{A}\widehat{B}, \widehat{A}^2\widehat{B}, \dots)$.

Theorem 3. *Let R be a commutative ring with identity element. Let $\Sigma = (A_0, \dots, A_{l-1}; B)$ be an l-order, m-input linear system over R^n. The following statements are equivalent:*

(i) *System Σ is reachable.*
(ii) *For each $\omega \in R^n$ one has that $(0, 0, \dots, 0) \rightsquigarrow \omega$.*
(iii) *The linear map given by the block matrix $(\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})$ is onto.*

Proof. (i) $\Rightarrow$ (ii) is straightforward. To prove (ii) $\Rightarrow$ (iii) it is sufficient to show that the evolution of system Σ, with initial condition, $\vec{0} = (0, \ldots, 0)$ is given by the image of $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})$. In fact we prove that, for each t

$$\Phi_\Sigma(l + t, \vec{0}, u) = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})U_t,$$

where U_t will be the matrix $\begin{pmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ for the rest of the proof.

(Note that it is clear that $\Phi_\Sigma(i, \vec{0}, u) = 0$ for each $0 \leqslant i \leqslant l - 1$; i.e., for the initial conditions).

The first case $t = 0$ is clear because $\Phi_\Sigma(l, \vec{0}, u) = Bu(0) = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})U_0$.

Note that the block matrix $(\widehat{A} * \widehat{B})$ is the reachability matrix of the linearized system $\mathrm{lin}(\Sigma) = \widehat{\Sigma}$. For the induction step $((0, 1, \ldots, l + t) \Rightarrow l + t + 1)$ we need to show the structure of the matrix $(\widehat{A} * \widehat{B}) = (\widehat{B}, \widehat{A}\widehat{B}, \widehat{A}^2\widehat{B}, \ldots)$. Denote by $\vec{A}$ the last block-row of matrix $\widehat{A}$, that is, $\vec{A} = (A_0\ A_1\ \cdots\ A_{l-2}\ A_{l-1})$. Denote by C_i the ith block-column of matrix $(\widehat{A} * \widehat{B})$; that is, $C_i = \widehat{A}^{i-1}\widehat{B}$. Then it follows that the block-structure of the matrix $(\widehat{A} * \widehat{B})$ is given by the equality

$$(\widehat{A} * \widehat{B}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B & \vec{A}C_1 & \vec{A}C_2 & \cdots \\ \vdots & & & \iddots & B & \vec{A}C_1 & \vec{A}C_2 & \vdots & \cdots \\ \vdots & & \mathbf{0} & \iddots & \vec{A}C_1 & \vec{A}C_2 & \vdots & & \cdots \\ \vdots & \mathbf{0} & B & \iddots & \vec{A}C_2 & \vdots & & & \cdots \\ \mathbf{0} & B & \vec{A}C_1 & \iddots & \vdots & & & & \cdots \\ B & \vec{A}C_1 & \vec{A}C_2 & \cdots & & & & & \cdots \end{pmatrix}.$$

Hence, by the induction hypothesis, we have the equality

$$(\widehat{A} * \widehat{B})U_t = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B & \cdots \\ \vdots & & \iddots & B & \vec{A}C_1 & \cdots \\ \vdots & \mathbf{0} & \iddots & \vec{A}C_1 & \vec{A}C_2 & \cdots \\ \mathbf{0} & B & \iddots & \vdots & \vdots & \cdots \\ B & \vec{A}C_1 & \cdots & & & \cdots \end{pmatrix} \quad U_t = \begin{pmatrix} \Phi_\Sigma(t+1, \vec{0}, u) \\ \Phi_\Sigma(t+2, \vec{0}, u) \\ \vdots \\ \Phi_\Sigma(l+t-1, \vec{0}, u) \\ \Phi_\Sigma(l+t, \vec{0}, u) \end{pmatrix}. \tag{2}$$

By the discussion in Section 1 we have that

$$\Phi_\Sigma(l + t + 1, \vec{0}, u) = \sum_{i=0}^{l-1} A_i \Phi_\Sigma(i + t + 1, \vec{0}, u) + Bu(t + 1)$$

$$= (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}) \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \\ A_0 & \cdots & \cdots & A_{l-2} & A_{l-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \Phi_\Sigma(t+1, \vec{0}, u) \\ \Phi_\Sigma(t+2, \vec{0}, u) \\ \vdots \\ \Phi_\Sigma(l+t-1, \vec{0}, u) \\ \Phi_\Sigma(l+t, \vec{0}, u) \end{pmatrix} + Bu(t+1)$$

(by the induction hypothesis (equality 2))

$$\begin{aligned} &= (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})\widehat{A}(\widehat{A} * \widehat{B})U_t + Bu(t+1) \\ &= (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{B}, \widehat{A}\widehat{B}, \widehat{A}^2\widehat{B}, \widehat{A}^3\widehat{B}, \ldots)U_{t+1} \\ &= (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})U_{t+1} \end{aligned}$$

as desired.

Finally, (iii) $\Rightarrow$ (i) follows from the fact that for each t and each initial condition x_{init} one has that $\Phi_\Sigma(l+t, x_{\text{init}}, u) = \xi + \Phi_\Sigma(l+t, \vec{0}, u)$, for some $\xi \in R^n$, that depends only on initial conditions x_{init} and not on the controls u. If the linear map

$$(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B}) = \Phi(-, \vec{0}, -) : (R^m)^{\oplus\infty} \to R^n$$

is surjective (where $(R^m)^{\oplus\infty}$ denotes the direct sum of countably many copies of R^m), then for some $t \geqslant 0$ the linear map

$$(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B}) = \Phi(l+t, \vec{0}, -) : \quad \begin{array}{ccc} (R^m)^{\oplus(t+1)} & \to & R^n \\ u = (u(0), \ldots, u(t)) & \to & \Phi(l+t, \vec{0}, u) \end{array}$$

is surjective (where $(R^m)^{\oplus(t+1)}$ denotes the direct sum of $t+1$ copies of R^m) and therefore any state $\omega - \xi$ can be reached from $\vec{0}$ (i.e., $\vec{0} \underset{u}{\rightsquigarrow} (\omega - \xi)$). Consequently any state ω can be reached from x_{init} (i.e., $x_{\text{init}} \underset{u}{\rightsquigarrow} \omega$). □

As a first consequence of the above result we point out that if the linearized system $\widehat{\Sigma}$ is reachable (i.e., the linear map $(\widehat{A} * \widehat{B})$ is onto) then the l-order linear system Σ is reachable, but the converse is not true in general. This is because of the reachability character of $\widehat{\Sigma}$ requires that every sequence of l states can be reached, but for the reachability of Σ it is sufficient to reach every single state. Note that in [6] the controllability is defined in this terms, that is, by the ability of reaching, in the future, every sequence of l states from every sequence of l initial conditions. This is the main difference between controllability and reachability for higher order linear systems. We may see this in the following example:

Example 4. Let R be any commutative ring with $1 \neq 0$. Consider the second order single input linear system over R^2 given by

$$\Sigma = \left(A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

We have that system Σ is reachable because

$$(\mathbf{0},\mathbf{1})\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ A_0 & A_1\end{pmatrix}*\begin{pmatrix}\mathbf{0}\\ B\end{pmatrix}=(B, A_1B, A_0B+A_1^2B,\ldots)=\left(\begin{array}{c|c|c|c}1&0&0&\multirow{2}{*}{$\cdots$}\\0&1&0&\end{array}\right).$$

But, on the other hand, its linearized system

$$\operatorname{lin}(\Sigma)=\left(\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ A_0 & A_1\end{pmatrix},\begin{pmatrix}\mathbf{0}\\ B\end{pmatrix}\right)$$

is not reachable because the linear map

$$\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ A_0 & A_1\end{pmatrix}*\begin{pmatrix}\mathbf{0}\\ B\end{pmatrix}=\left(\begin{array}{c|c|c|c|c}\mathbf{0} & B & A_1B & A_0B+A_1^2B & \cdots\\ B & A_1B & A_0B+A_1^2B & A_0A_1B+A_1A_0B+A_1^3B & \end{array}\right)$$

$$=\left(\begin{array}{c|c|c|c|c}0&1&0&0&\\0&0&1&0&\cdots\\1&0&0&0&\\0&1&0&0&\end{array}\right)$$

is not onto.

3. Feedback equivalence of higher order systems

Let $\Sigma=(A_0,\ldots,A_{l-1};B)$ be an l-order, m-input linear system over R^n. The feedback group acting on such systems is the group generated by the following elementary actions:

A1 Change of basis $P^{-1}\in \mathrm{GL}_n(R)$ in the state-space R^n, which transforms:

$$A_i\to A_i'=PA_iP^{-1},$$
$$B\to B'=PB.$$

A2 Change of basis $Q\in \mathrm{GL}_m(R)$ in the input-space R^m, which transforms:

$$A_i\to A_i'=A_i,$$
$$B\to B'=BQ.$$

A3 Generalized feedback actions $(F_0,\ldots,F_{l-1})\in(R^{m\times n})^l$, which transforms:

$$A_i\to A_i'=A_i+BF_i,$$
$$B\to B'=B.$$

Note that in the case $l=1$ we have the standard feedback group.

Definition 5. We say that the systems Σ and Σ' are feedback equivalent if Σ can be transformed to Σ' by one element of the feedback group acting on Σ.

Next we will introduce the invariants N_i^{Σ} and M_i^{Σ}. Associated to the standard linearization $\widehat{\Sigma}$ of Σ we consider the block matrices

$$(\widehat{A}*\widehat{B})_{(i)}=(\widehat{B},\widehat{A}\widehat{B},\widehat{A}^2\widehat{B},\ldots,\widehat{A}^{i-1}\widehat{B}).$$

Let N_i^Σ be the R-module (submodule of R^n) generated by the columns of the block matrix $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{1})(\widehat{A} * \widehat{B})_{(i)}$. Let M_i^Σ be the quotient R-module $M_i^\Sigma = R^n/N_i^\Sigma$.

As in the case of first order linear systems (see [3,5]) we have the following invariance theorem:

Theorem 6. *Let R be a commutative ring and let $\Sigma = (A_0, \ldots, A_{l-1}; B)$ be an l-order, m-input linear system over R^n. With the above notations, the R-modules N_i^Σ and the quotient R-modules M_i^Σ are feedback invariants, up to isomorphism, associated to Σ. That is, if Σ and Σ' are feedback equivalent then*:

(a) *N_i^Σ is isomorphic to $N_i^{\Sigma'}$ for each $i = 1, 2, \ldots$*
(b) *M_i^Σ is isomorphic to $M_i^{\Sigma'}$ for each $i = 1, 2, \ldots$*

Proof. It is sufficient to prove that the modules N_i^Σ and M_i^Σ are invariants, up to isomorphism, when we consider actions of type A1, A2, and A3.

First suppose that Σ and $\Sigma' = (A_0', \ldots, A_{l-1}'; B')$ are equivalent via a change of basis $P^{-1} \in \mathrm{GL}_n(R)$; that is $A_i' = PA_iP^{-1}$ and $B' = PB$. Then it follows that

$$\widehat{A'} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \\ A_0' & \cdots & \cdots & A_{l-2}' & A_{l-1}' \end{pmatrix}$$

$$= \begin{pmatrix} P & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & P \end{pmatrix} \widehat{A} \begin{pmatrix} P^{-1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & P^{-1} \end{pmatrix}$$

and

$$\widehat{B'} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B' \end{pmatrix} = \begin{pmatrix} P & & & \\ & \ddots & & \\ & & \ddots & \\ & & & P \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B \end{pmatrix}.$$

Therefore

$$\widehat{A'}\widehat{B'} = \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix} \widehat{A}\widehat{B}$$

and it is now clear that

$$(\widehat{A'} * \widehat{B'})_{(i)} = (\widehat{B'}, \widehat{A'}\widehat{B'}, \ldots, \widehat{A'}^{i-1}\widehat{B'}) = \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix} (\widehat{A} * \widehat{B})_{(i)}$$

and

$$(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{A'}*\widehat{B'})_{(i)}=(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{P})(\widehat{A}*\widehat{B})_{(i)}=P(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{A}*\widehat{B})_{(i)}.$$

Since P is invertible it follows that $N_i^{\Sigma}\cong N_i^{\Sigma'}$ and $M_i^{\Sigma}\cong M_i^{\Sigma'}$.

Now suppose that Σ and Σ' are equivalent via a change of basis $Q\in\mathrm{GL}_m(R)$; that is $A_i'=A_i$ and $B'=BQ$. It follows that $\widehat{A'}\widehat{B'}=\widehat{A}\widehat{B}Q$ and consequently

$$\begin{aligned}(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{A'}*\widehat{B'})_{(i)}&=(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{B'},\widehat{A'}\widehat{B'},\ldots,\widehat{A'}^{i-1}\widehat{B'})\\&=(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{B},\widehat{A}\widehat{B},\ldots,\widehat{A}^{i-1}\widehat{B})\begin{pmatrix}Q&&\\&\ddots&\\&&Q\end{pmatrix}.\end{aligned}$$

Since $\begin{pmatrix}Q&&\\&\ddots&\\&&Q\end{pmatrix}$ is invertible, it follows that $N_i^{\Sigma}\cong N_i^{\Sigma'}$ and $M_i^{\Sigma}\cong M_i^{\Sigma'}$.

To conclude suppose that Σ and Σ' are equivalent via an action of type A3; that is $A_i'=A_i+BF_i$ and $B'=B$. We claim that $N_i^{\Sigma}=N_i^{\Sigma'}$ (not only isomorphic but equal). From this point it will be straightforward that $M_i^{\Sigma}=M_i^{\Sigma'}$.

Note that the above equalities hold when $\Sigma=\Gamma=(F,G)$ and $\Sigma'=\Gamma'=(F',G')$ are first order linear systems (see [2] or [5, Lemma 2.1]); that is, if Γ and Γ' are feedback equivalent via a feedback action K (i.e., $F'=F+GK$ and $G'=G$) then $\mathrm{Im}((F*G)_{(i)})=\mathrm{Im}((F'*G')_{(i)})$.

Since linearizations $\mathrm{lin}(\Sigma)$ and $\mathrm{lin}(\Sigma')$ are first order linear systems it follows that $\mathrm{Im}((\widehat{A'}*\widehat{B'})_{(i)})=\mathrm{Im}((\widehat{A}*\widehat{B})_{(i)})$, when a type A3 action is considered. Thus the equalities

$$\begin{aligned}N_i^{\Sigma'}&=\mathrm{Im}((\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})(\widehat{A'}*\widehat{B'})_{(i)})=(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})\mathrm{Im}((\widehat{A'}*\widehat{B'})_{(i)})\\&=(\mathbf{0},\mathbf{0},\ldots,\mathbf{0},\mathbf{1})\mathrm{Im}((\widehat{A}*\widehat{B})_{(i)})=N_i^{\Sigma}.\end{aligned}$$

are now clear. This proves the claim and the result. □

Note that in Theorem 3 it is shown that system Σ is reachable if and only if the quotient R-module M_i^{Σ} is zero for some i. Thus the following result is now straightforward.

Corollary 7. *Reachability is a feedback invariant also for l-order linear control systems.*

Now we give a reduction result which will be useful in Section 5. Let R be a commutative ring and let $\Sigma=(A_0,\ldots,A_{l-1};B)$ be an l-order, m-input linear system over R^n. Suppose that both $N_1^{\Sigma}=\mathrm{Im}(B)$ and $M_1^{\Sigma}=R^n/\mathrm{Im}(B)$ are free (if $R=k$ is a field, this is always the case); first of all we put B in Hermite form, using actions A1 and A2, thus Σ is transformed into the system

$$\left(\left(\begin{array}{c|c}X_0&Y_0\\\hline G_0&H_0\end{array}\right),\ldots,\left(\begin{array}{c|c}X_{l-1}&Y_{l-1}\\\hline G_{l-1}&H_{l-1}\end{array}\right);\left(\begin{array}{c|c}\mathbf{1}_r&\mathbf{0}\\\hline\mathbf{0}&\mathbf{0}\end{array}\right)\right)$$

and then, we can use generalized feedback actions A3 with

$$F_i=\left(\begin{array}{c|c}-X_i&-Y_i\\\hline\mathbf{0}&\mathbf{0}\end{array}\right)$$

so we have that Σ is equivalent to the system

$$\left(\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_0 & H_0\end{array}\right),\ldots,\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_{l-1} & H_{l-1}\end{array}\right);\left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\right), \tag{3}$$

where $r = \operatorname{rank}(B) = \dim(N_1^\Sigma)$. Now we look at the r-input first order systems $(H_0, G_0)\ldots(H_{l-1}, G_{l-1})$ over R^{n-r} to give our result:

Theorem 8. *Let R be a commutative ring with identity element. Let Σ and Σ' be two m-input l-order linear systems over R^n in the above reduced form. Then Σ and Σ' are feedback equivalent if and only if there exist a feedback action (P, Q, F) such that (H_i, G_i) and (H_i', G_i') are equivalent via (U, V, K). That is to say*

$$G_i' = UG_iV,$$
$$H_i' = UH_iU^{-1} + UG_iK$$

for $i = 0, 1, \ldots, l-1$.

Proof. Assume that system $\Sigma = (A_0, \ldots, A_{l-1}; B)$ is on its reduced form (3).

First suppose that (H_i, G_i) is equivalent to (H_i', G_i') via (U, V, K) for all i. Then consider the generalized feedback action (on Σ) given by

$$\left(P = \left(\begin{array}{c|c}V^{-1} & -V^{-1}KU\\ \hline \mathbf{0} & U\end{array}\right), Q = \left(\begin{array}{c|c}V & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{1}\end{array}\right), F = \left(\begin{array}{c|c}F_{11} & F_{12}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\right).$$

Since $UF_iU^{-1} + UG_iK = F_i'$, and $UG_iV = G_i'$, it follows that

$$\begin{aligned}PA_iP^{-1} + PBF &= \left(\begin{array}{c|c}V^{-1} & -V^{-1}KU\\ \hline \mathbf{0} & U\end{array}\right)\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_i & H_i\end{array}\right)\left(\begin{array}{c|c}V & K\\ \hline \mathbf{0} & U^{-1}\end{array}\right)\\ &\quad + \left(\begin{array}{c|c}V^{-1} & -V^{-1}KU\\ \hline \mathbf{0} & U\end{array}\right)\left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\left(\begin{array}{c|c}F_{11} & F_{12}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\\ &= \left(\begin{array}{c|c}* & *\\ \hline UG_iV & UH_iU^{-1} + UG_iK\end{array}\right) + \left(\begin{array}{c|c}V^{-1}F_{11} & V^{-1}F_{12}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\\ &= \left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline UG_iV & UH_iU^{-1} + UG_iK\end{array}\right) = \left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_i' & H_i'\end{array}\right) = A_i'\end{aligned}$$

for a suitable F. In the same way we can prove that $UBV = B'$. Thus l-order systems Σ and Σ' are equivalent via (P, Q, F).

To prove the converse suppose that the l-order systems

$$\left(\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_0 & F_0\end{array}\right),\ldots,\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_{l-1} & F_{l-1}\end{array}\right);\left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\right) \quad \text{and}$$

$$\left(\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_0' & F_0'\end{array}\right),\ldots,\left(\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline G_{l-1}' & F_{l-1}'\end{array}\right);\left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\right)$$

are feedback equivalent via (P, Q, K). Since $P\left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right) = \left(\begin{array}{c|c}\mathbf{1}_r & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)Q^{-1}$, it follows that P has the block structure $\left(\begin{array}{c|c}P_{11} & P_{12}\\ \hline \mathbf{0} & P_{22}\end{array}\right)$. Therefore (H_i, G_i) and (H_i', G_i') are equivalent via the action $(P_{22}, P_{11}^{-1}, -P_{11}^{-1}P_{12}P_{22}^{-1})$. $\square$

4. Reachability is a local and pointwise property

Let R be a conmutative ring with identity, let $\mathfrak{p}$ be any prime ideal of R and let Σ be an l-order, m-input system over R^n. We denote by $(A_i)_\mathfrak{p} : (R^n)_\mathfrak{p} \to (R^n)_\mathfrak{p}$ and $B_\mathfrak{p} : (R^m)_\mathfrak{p} \to (R^n)_\mathfrak{p}$ the linear maps obtained by natural extension of scalars from R to the local ring $R_\mathfrak{p}$. The system $\Sigma_\mathfrak{p} = ((A_i)_\mathfrak{p}, B_\mathfrak{p})$ is the localization of Σ at $\mathfrak{p}$. On the other hand, the linear maps $A_i(\mathfrak{p})$ and $B(\mathfrak{p})$, obtained by natural extension of scalars from R to the residual field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, are considered in order to obtain $\Sigma(\mathfrak{p}) = ((A_i)(\mathfrak{p}), B(\mathfrak{p}))$, which is the residual system of Σ at $\mathfrak{p}$ (see [3] for details).

Pointwise study can be applied in some interesting cases (see, for example [4]). In fact if we consider a compact topological space X and systems with coefficients in $\mathscr{C}(X; \mathbb{R})$ (the ring of continuous real functions defined on X), then residual systems are the evaluations $\Sigma(x)$ of Σ at points $x \in X$. Hence the word pointwise, in this case, has its "natural" sense.

A property of a system Σ is local if it is verified by all localizations $\Sigma_\mathfrak{p}$ of Σ. A property of a system Σ is pointwise if it is verified by all residual systems. In the particular case of $\mathscr{C}(X; \mathbb{R})$ a property of Σ is pointwise if and only if is verified by the evaluation $\Sigma(x)$ for all $x \in X$.

Because of the reachability is characterized in terms of the surjectivity of a linear map and the surjectivity of a linear map is a local and pointwise property (see [1]), it follows that the reachability is a local and pointwise property. Thus the following results are straightforward:

Theorem 9. *Σ is reachable (over R) if and only if $\Sigma(\mathfrak{p})$ is reachable (over the residual field $k(\mathfrak{p})$ of R at the prime ideal $\mathfrak{p}$) for each prime ideal $\mathfrak{p}$ of R.*

Corollary 10. *Let X be a compact topological space. Then an l-order system Σ over the ring $R = \mathscr{C}(X; \mathbb{R})$ is reachable if and only if all the evaluations $\Sigma(x)$ at points $x \in X$ are reachable over $\mathbb{R}$.*

5. Reachability and equivalence for low dimension systems over fields

In this section we will show, for low dimension systems over fields, how to verify if two systems are feedback equivalent and if a system is reachable. As every system is feedback equivalent to a system in the reduced form (3) and by means of Corollary 7 it is sufficient to study systems in reduced form.

5.1. The case $l = m = n = 2$

Next, we study the case $l = m = n = 2$ for $R = k$ a field. All the systems with these parameters belong to one of the following three types (distinguished by the invariant $\sigma_1 = \dim(M_1) = 2 - \operatorname{rank}(B)$).

$\sigma_1 = 2$. Systems with no controls. The classification problem here is equivalent to the simultaneous similarity of pairs of 2×2 matrices problem.

$\sigma_1 = 0$. There is only one system of this type (up to feedback actions):

$$\Sigma = \left(A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Thus all these systems are equivalent to each other.

$\sigma_1 = 1$. All this systems have the following reduced structure.

$$\Sigma = \left(A_0 = \begin{pmatrix} 0 & 0 \\ b_0 & a_0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ b_1 & a_1 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right). \tag{4}$$

According to Theorem 3, the reachability of this system is given by the matrix

$$(\mathbf{0}, \mathbf{1}) \left(\widehat{A} * \widehat{B} \right) = \left(\begin{array}{cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & b_1 & 0 & b_0 + a_1 b_1 & 0 & \end{array} \right).$$

This matrix has full rank (that is, Σ is reachable) if and only if $b_0 \neq 0$ or $b_1 \neq 0$.

The equivalence of two systems of the form (4) can be studied by means of Theorem 8 by checking the k-equivalence of the systems $\Gamma = (a_0 + a_1 x\ ,\ b_0 + b_1 x)$ and $\Gamma' = (a'_0 + a'_1 x\ ,\ b'_0 + b'_1 x)$. Systems Γ and Γ' are feedback equivalent if we can find $p \neq 0$, $q \neq 0$ and f in the field k satisfying:

$$p(a_0 + a_1 x) p^{-1} + p(b_0 + b_1 x) q f = a'_0 + a'_1 x,$$

$$p(b_0 + b_1 x) q = (b'_0 + b'_1 x)$$

and we obtain the equivalence condition $\frac{b_0}{b_1} = \frac{b'_0}{b'_1} = \frac{a_0 - a'_0}{a_1 - a'_1}$. We can use these conditions for reachable system because we have shown that for these systems $b_0 \neq 0$ or $b_1 \neq 0$. Note that the above results for systems with $l = m = n = 2$ can be easily extended for $m = n = 2$ and arbitrary l.

5.2. Systems with $l = m = 2$ and $n = 3$

As before we divide the systems depending on the invariant $\sigma_1 = \dim(M_1) = 3 - \operatorname{rank}(B)$. We will consider the case $\sigma_1 = 1$. These systems have the structure

$$\Sigma = \left(A_0 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline b_{01} & b_{02} & a_0 \end{array} \right), A_1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline b_{11} & b_{12} & a_1 \end{array} \right); B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

To study the reachability we study the rank of the matrix

$$(\mathbf{0}, \mathbf{1}) \left(\widehat{A} * \widehat{B} \right) = \left(\begin{array}{cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & b_{11} & b_{12} & b_{01} + a_1 b_{11} & b_{02} + a_1 b_{12} & \end{array} \right)$$

and we can see that Σ is reachable if and only if at least one of the b_{ij} is not equal to 0.

Again we can use the reduction result so we can study the reachability of the systems studying the k-equivalence of systems of the type

$$\Gamma = \left(a_0 + a_1 x; \begin{pmatrix} b_{01} & b_{02} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \end{pmatrix} x \right). \tag{5}$$

And from here we obtain that Σ is k-equivalent to Σ' if and only if one can find (f, g) in k solving the following linear equation:

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a'_0 \\ a'_1 \end{pmatrix} + \begin{pmatrix} b'_{01} & b'_{02} \\ b'_{11} & b'_{12} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

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