24 BESSEL FUNCTIONS

BESSEL'S DIFFERENTIAL EQUATION

24.1 $$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *Bessel functions of order n*.

BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.2 $$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

24.3 $$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} - \cdots\right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{2k-n}}{k!\,\Gamma(k+1-n)}$$

24.4 $$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ and $J_{-n}(x)$ are linearly independent.

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ is bounded at $x = 0$ while $J_{-n}(x)$ is unbounded.

For $n = 0, 1$ we have

24.5 $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

24.6 $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

24.7 $$J_0'(x) = -J_1(x)$$

BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.8 $$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, \ldots \\[2ex] \lim\limits_{p\to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, \ldots \end{cases}$$

This is also called *Weber's function* or *Neumann's function* [also denoted by $N_n(x)$].

For $n = 0, 1, 2, \ldots$, L'Hospital's rule yields

24.9 $$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\}J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(x/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\Phi(k) + \Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!}$$

where $\gamma = .5772156\ldots$ is Euler's constant [page 1] and

24.10 $$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0$$

For $n = 0$,

24.11 $$Y_0(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\}J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \cdots\right\}$$

24.12 $$Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0, 1, 2, \ldots$$

For any value $n \geqq 0$, $J_n(x)$ is bounded at $x = 0$ while $Y_n(x)$ is unbounded.

GENERAL SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

24.13 $$y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

24.14 $$y = AJ_n(x) + BY_n(x) \qquad \text{all } n$$

24.15 $$y = AJ_n(x) + BJ_n(x)\int\frac{dx}{xJ_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $J_n(x)$

24.16 $$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

RECURRENCE FORMULAS FOR BESSEL FUNCTIONS

24.17 $$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

24.18 $$J_n'(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

24.19 $$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

24.20 $$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

24.21 $$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

24.22 $$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

The functions $Y_n(x)$ satisfy identical relations.

BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of sines and cosines.

24.23 $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

24.24 $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

24.25 $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

24.26 $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right)$

24.27 $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x\right\}$

24.28 $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1\right) \cos x\right\}$

For further results use the recurrence formula. Results for $Y_{1/2}(x), Y_{3/2}(x), \ldots$ are obtained from 24.8.

HANKEL FUNCTIONS OF FIRST AND SECOND KINDS OF ORDER n

24.29 $H_n^{(1)}(x) = J_n(x) + iY_n(x)$

24.30 $H_n^{(2)}(x) = J_n(x) - iY_n(x)$

BESSEL'S MODIFIED DIFFERENTIAL EQUATION

24.31 $$x^2y'' + xy' - (x^2 + n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *modified Bessel functions of order n*.

MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.32 $$I_n(x) = i^{-n} J_n(ix) = e^{-n\pi i/2} J_n(ix)$$
$$= \frac{x^n}{2^n \Gamma(n+1)} \left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} + \cdots\right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

24.33
$$I_{-n}(x) = i^n J_{-n}(ix) = e^{n\pi i/2} J_{-n}(ix)$$
$$= \frac{x^{-n}}{2^{-n} \Gamma(1-n)} \left\{1 + \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} + \cdots\right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-n}}{k!\,\Gamma(k+1-n)}$$

24.34 $$I_{-n}(x) = I_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots,$ then $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

For $n = 0, 1,$ we have

24.35 $I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

24.36 $I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$

24.37 $I_0'(x) = I_1(x)$

MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.38 $$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p\to n} \dfrac{\pi}{2\sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

For $n = 0, 1, 2, \ldots$, L'Hospital's rule yields

24.39 $$K_n(x) = (-1)^{n+1}\{\ln(x/2) + \gamma\}I_n(x) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\,(x/2)^{2k-n}$$
$$+ \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k) + \Phi(n+k)\}$$

where $\Phi(p)$ is given by 24.10.

For $n = 0$,

24.40 $$K_0(x) = -\{\ln(x/2) + \gamma\}I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2\cdot 4^2\cdot 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

24.41 $$K_{-n}(x) = K_n(x) \qquad n = 0, 1, 2, \ldots$$

GENERAL SOLUTION OF BESSEL'S MODIFIED EQUATION

24.42 $$y = AI_n(x) + BI_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

24.43 $$y = AI_n(x) + BK_n(x) \qquad \text{all } n$$

24.44 $$y = AI_n(x) + BI_n(x)\int\frac{dx}{xI_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $I_n(x)$

24.45 $$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x)t^n$$

RECURRENCE FORMULAS FOR MODIFIED BESSEL FUNCTIONS

24.46 $$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n(x)$$

24.47 $$I_n'(x) = \tfrac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$$

24.48 $$xI_n'(x) = xI_{n-1}(x) - nI_n(x)$$

24.49 $$xI_n'(x) = xI_{n+1}(x) + nI_n(x)$$

24.50 $$\frac{d}{dx}\{x^nI_n(x)\} = x^nI_{n-1}(x)$$

24.51 $$\frac{d}{dx}\{x^{-n}I_n(x)\} = x^{-n}I_{n+1}(x)$$

24.52 $$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x}K_n(x)$$

24.53 $$K_n'(x) = \tfrac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}$$

24.54 $$xK_n'(x) = -xK_{n-1}(x) - nK_n(x)$$

24.55 $$xK_n'(x) = nK_n(x) - xK_{n+1}(x)$$

24.56 $$\frac{d}{dx}\{x^nK_n(x)\} = -x^nK_{n-1}(x)$$

24.57 $$\frac{d}{dx}\{x^{-n}K_n(x)\} = -x^{-n}K_{n+1}(x)$$

MODIFIED BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of hyperbolic sines and cosines.

24.58 $I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$

24.59 $I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$

24.60 $I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{\sinh x}{x}\right)$

24.61 $I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{\cosh x}{x}\right)$

24.62 $I_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\left(\frac{3}{x^2} + 1\right) \sinh x - \frac{3}{x} \cosh x\right\}$

24.63 $I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\left(\frac{3}{x^2} + 1\right) \cosh x - \frac{3}{x} \sinh x\right\}$

For further results use the recurrence formula 24.46. Results for $K_{1/2}(x), K_{3/2}(x), \ldots$ are obtained from 24.38.

Ber AND Bei FUNCTIONS

The real and imaginary parts of $J_n(xe^{3\pi i/4})$ are denoted by $\text{Ber}_n(x)$ and $\text{Bei}_n(x)$ where

24.64 $$\text{Ber}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \cos \frac{(3n+2k)\pi}{4}$$

24.65 $$\text{Bei}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \sin \frac{(3n+2k)\pi}{4}$$

If $n = 0$,

24.66 $$\text{Ber}(x) = 1 - \frac{(x/2)^4}{2!^2} + \frac{(x/2)^8}{4!^2} - \cdots$$

24.67 $$\text{Bei}(x) = (x/2)^2 - \frac{(x/2)^6}{3!^2} + \frac{(x/2)^{10}}{5!^2} - \cdots$$

Ker AND Kei FUNCTIONS

The real and imaginary parts of $e^{-n\pi i/2} K_n(xe^{\pi i/4})$ are denoted by $\text{Ker}_n(x)$ and $\text{Kei}_n(x)$ where

24.68 $$\begin{aligned} \text{Ker}_n(x) = {} & -\{\ln(x/2) + \gamma\} \text{Ber}_n(x) + \tfrac{1}{4}\pi \, \text{Bei}_n(x) \\ & + \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \cos \frac{(3n+2k)\pi}{4} \\ & + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!} \{\Phi(k) + \Phi(n+k)\} \cos \frac{(3n+2k)\pi}{4} \end{aligned}$$

24.69 $$\begin{aligned} \text{Kei}_n(x) = {} & -\{\ln(x/2) + \gamma\} \text{Bei}_n(x) - \tfrac{1}{4}\pi \, \text{Ber}_n(x) \\ & - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \sin \frac{(3n+2k)\pi}{4} \\ & + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!} \{\Phi(k) + \Phi(n+k)\} \sin \frac{(3n+2k)\pi}{4} \end{aligned}$$

and Φ is given by 24.10, page 137.

If $n = 0$,

24.70 $$\text{Ker}(x) = -\{\ln(x/2) + \gamma\} \text{Ber}(x) + \frac{\pi}{4} \text{Bei}(x) + 1 - \frac{(x/2)^4}{2!^2}(1 + \tfrac{1}{2}) + \frac{(x/2)^8}{4!^2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) - \cdots$$

24.71 $$\text{Kei}(x) = -\{\ln(x/2) + \gamma\} \text{Bei}(x) - \frac{\pi}{4} \text{Ber}(x) + (x/2)^2 - \frac{(x/2)^6}{3!^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

DIFFERENTIAL EQUATION FOR Ber, Bei, Ker, Kei FUNCTIONS

24.72 $$x^2y'' + xy' - (ix^2 + n^2)y = 0$$

The general solution of this equation is

24.73 $$y = A\{\mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x)\} + B\{\mathrm{Ker}_n(x) + i\,\mathrm{Kei}_n(x)\}$$

GRAPHS OF BESSEL FUNCTIONS

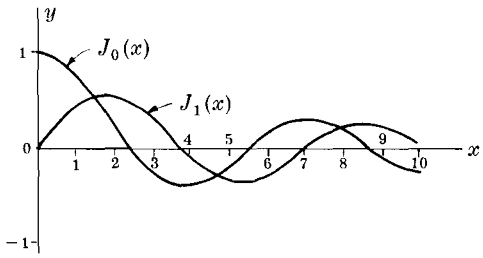

Fig. 24-1

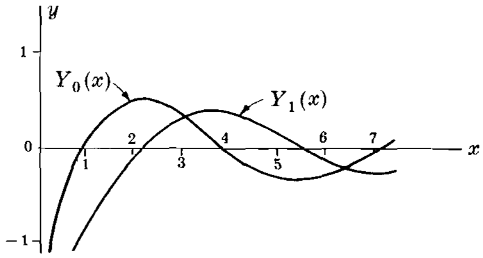

Fig. 24-2

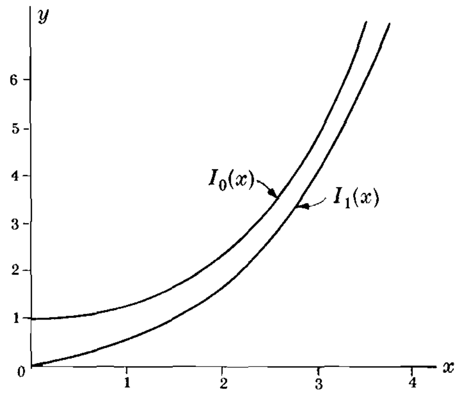

Fig. 24-3

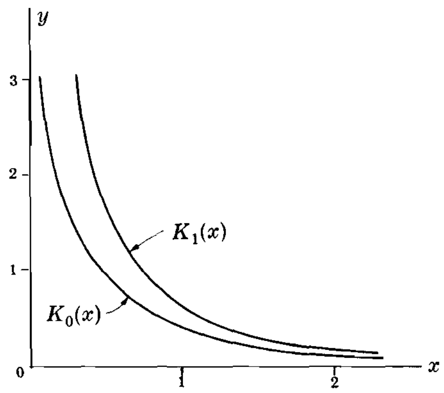

Fig. 24-4

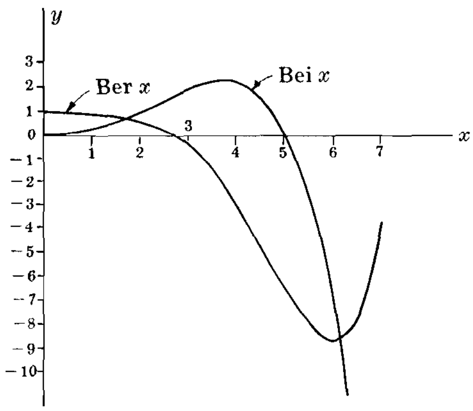

Fig. 24-5

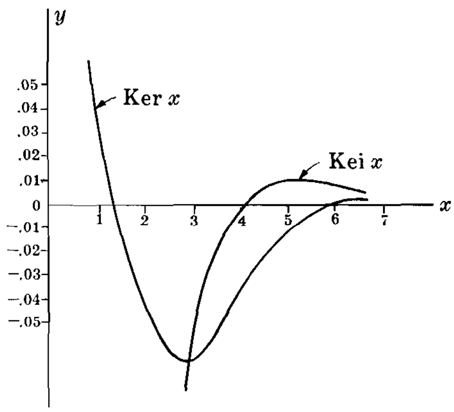

Fig. 24-6

INDEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.74 $\int x J_0(x)\,dx = x J_1(x)$

24.75 $\int x^2 J_0(x)\,dx = x^2 J_1(x) + x J_0(x) - \int J_0(x)\,dx$

24.76 $\int x^m J_0(x)\,dx = x^m J_1(x) + (m-1)x^{m-1} J_0(x) - (m-1)^2 \int x^{m-2} J_0(x)\,dx$

24.77 $\int \frac{J_0(x)}{x^2}\,dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\,dx$

24.78 $\int \frac{J_0(x)}{x^m}\,dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2} \int \frac{J_0(x)}{x^{m-2}}\,dx$

24.79 $\int J_1(x)\,dx = -J_0(x)$

24.80 $\int x J_1(x)\,dx = -x J_0(x) + \int J_0(x)\,dx$

24.81 $\int x^m J_1(x)\,dx = -x^m J_0(x) + m \int x^{m-1} J_0(x)\,dx$

24.82 $\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$

24.83 $\int \frac{J_1(x)}{x^m}\,dx = -\frac{J_1(x)}{m x^{m-1}} + \frac{1}{m} \int \frac{J_0(x)}{x^{m-1}}\,dx$

24.84 $\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$

24.85 $\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$

24.86 $\int x^m J_n(x)\,dx = -x^m J_{n-1}(x) + (m+n-1) \int x^{m-1} J_{n-1}(x)\,dx$

24.87 $\int x J_n(\alpha x) J_n(\beta x)\,dx = \frac{x\{\alpha J_n(\beta x) J_n'(\alpha x) - \beta J_n(\alpha x) J_n'(\beta x)\}}{\beta^2 - \alpha^2}$

24.88 $\int x J_n^2(\alpha x)\,dx = \frac{x^2}{2}\{J_n'(\alpha x)\}^2 + \frac{x^2}{2}\left(1 - \frac{n^2}{\alpha^2 x^2}\right)\{J_n(\alpha x)\}^2$

The above results also hold if we replace $J_n(x)$ by $Y_n(x)$ or, more generally, $A\,J_n(x) + B\,Y_n(x)$ where A and B are constants.

DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.89 $\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$

24.90 $\int_0^\infty e^{-ax} J_n(bx)\,dx = \frac{(\sqrt{a^2+b^2}-a)^n}{b^n\sqrt{a^2+b^2}} \qquad n > -1$

24.91 $\int_0^\infty \cos ax\, J_0(bx)\,dx = \begin{cases} \frac{1}{\sqrt{a^2-b^2}} & a > b \\ 0 & a < b \end{cases}$

24.92 $$\int_0^\infty J_n(bx)\,dx = \frac{1}{b} \qquad n > -1$$

24.93 $$\int_0^\infty \frac{J_n(bx)}{x}\,dx = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$$

24.94 $$\int_0^\infty e^{-ax} J_0(b\sqrt{x})\,dx = \frac{e^{-b^2/4a}}{a}$$

24.95 $$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{\alpha\,J_n(\beta)\,J_n'(\alpha) - \beta\,J_n(\alpha)\,J_n'(\beta)}{\beta^2 - \alpha^2}$$

24.96 $$\int_0^1 x\,J_n^2(\alpha x)\,dx = \tfrac{1}{2}\{J_n'(\alpha)\}^2 + \tfrac{1}{2}(1 - n^2/\alpha^2)\{J_n(\alpha)\}^2$$

24.97 $$\int_0^1 x\,J_0(\alpha x)\,I_0(\beta x)\,dx = \frac{\beta\,J_0(\alpha)\,I_0'(\beta) - \alpha\,J_0'(\alpha)\,I_0(\beta)}{\alpha^2 + \beta^2}$$

INTEGRAL REPRESENTATIONS FOR BESSEL FUNCTIONS

24.98 $$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta$$

24.99 $$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = \text{integer}$$

24.100 $$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^\pi \cos(x\sin\theta)\cos^{2n}\theta\,d\theta, \qquad n > -\tfrac{1}{2}$$

24.101 $$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh u)\,du$$

24.102 $$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{x\sin\theta}\,d\theta$$

ASYMPTOTIC EXPANSIONS

24.103 $$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad \text{where } x \text{ is large}$$

24.104 $$Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad \text{where } x \text{ is large}$$

24.105 $$J_n(x) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{2n}\right)^n \qquad \text{where } n \text{ is large}$$

24.106 $$Y_n(x) \sim -\sqrt{\frac{2}{\pi n}}\left(\frac{ex}{2n}\right)^{-n} \qquad \text{where } n \text{ is large}$$

24.107 $$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \qquad \text{where } x \text{ is large}$$

24.108 $$K_n(x) \sim \frac{e^{-x}}{\sqrt{2\pi x}} \qquad \text{where } x \text{ is large}$$

ORTHOGONAL SERIES OF BESSEL FUNCTIONS

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the positive roots of $R\,J_n(x) + Sx\,J_n'(x) = 0,\ n > -1$. Then the following series expansions hold under the conditions indicated.

$S = 0,\ R \neq 0$, i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are positive roots of $J_n(x) = 0$

24.109
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

24.110
$$A_k = \frac{2}{J_{n+1}^2(\lambda_k)} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

24.111
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

24.112
$$A_k = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx$$

$R/S > -n$

24.113
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

24.114
$$A_k = \frac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k)\, J_{n+1}(\lambda_k)} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

24.115
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

24.116
$$A_k = \frac{2}{J_0^2(\lambda_k) + J_1^2(\lambda_k)} \int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx$$

$R/S = -n$

24.117
$$f(x) = A_0 x^n + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

24.118
$$\begin{cases} A_0 = 2(n+1) \displaystyle\int_0^1 x^{n+1} f(x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k)\, J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx \end{cases}$$

In particular if $n = 0$ so that $R = 0$ [i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$],

24.119
$$f(x) = A_0 + A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + \cdots$$

where

24.120
$$\begin{cases} A_0 = 2 \displaystyle\int_0^1 x\, f(x)\, dx \\ A_k = \dfrac{2}{J_0^2(\lambda_k)} \displaystyle\int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx \end{cases}$$

$R/S < -n$

In this case there are two pure imaginary roots $\pm i\lambda_0$ as well as the positive roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ and we have

24.121 $$f(x) = A_0 I_n(\lambda_0 x) + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

24.122 $$\begin{cases} A_0 = \dfrac{2}{I_n^2(\lambda_0) + I_{n-1}(\lambda_0) I_{n+1}(\lambda_0)} \displaystyle\int_0^1 x f(x) I_n(\lambda_0 x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k) J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x f(x) J_n(\lambda_k x)\, dx \end{cases}$$

MISCELLANEOUS RESULTS

24.123 $$\cos(x \sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$$

24.124 $$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots$$

24.125 $$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y) \qquad n = 0, \pm 1, \pm 2, \ldots$$

This is called the *addition formula* for Bessel functions.

24.126 $$1 = J_0(x) + 2J_2(x) + \cdots + 2J_{2n}(x) + \cdots$$

24.127 $$x = 2\{J_1(x) + 3J_3(x) + 5J_5(x) + \cdots + (2n+1)J_{2n+1}(x) + \cdots\}$$

24.128 $$x^2 = 2\{4J_2(x) + 16J_4(x) + 36J_6(x) + \cdots + (2n)^2 J_{2n}(x) + \cdots\}$$

24.129 $$\frac{x J_1(x)}{4} = J_2(x) - 2J_4(x) + 3J_6(x) - \cdots$$

24.130 $$1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + 2J_3^2(x) + \cdots$$

24.131 $$J_n''(x) = \tfrac{1}{4}\{J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\}$$

24.132 $$J_n'''(x) = \tfrac{1}{8}\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\}$$

Formulas 24.131 and 24.132 can be generalized.

24.133 $$J_n'(x)J_{-n}(x) - J_{-n}' J_n(x) = \frac{2\sin n\pi}{\pi x}$$

24.134 $$J_n(x)J_{-n+1}(x) + J_{-n}(x)J_{n-1}(x) = \frac{2\sin n\pi}{\pi x}$$

24.135 $$J_{n+1}(x)Y_n(x) - J_n(x)Y_{n+1}(x) = J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{2}{\pi x}$$

24.136 $$\sin x = 2\{J_1(x) - J_3(x) + J_5(x) - \cdots\}$$

24.137 $$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots$$

24.138 $$\sinh x = 2\{I_1(x) + I_3(x) + I_5(x) + \cdots\}$$

24.139 $$\cosh x = I_0(x) + 2\{I_2(x) + I_4(x) + I_6(x) + \cdots\}$$

25 LEGENDRE FUNCTIONS

LEGENDRE'S DIFFERENTIAL EQUATION

25.1 $$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Solutions of this equation are called *Legendre functions of order n.*

LEGENDRE POLYNOMIALS

If $n = 0, 1, 2, \ldots,$ solutions of 25.1 are Legendre polynomials $P_n(x)$ given by *Rodrigue's formula*

25.2 $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

SPECIAL LEGENDRE POLYNOMIALS

25.3 $P_0(x) = 1$

25.4 $P_1(x) = x$

25.5 $P_2(x) = \frac{1}{2}(3x^2 - 1)$

25.6 $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

25.7 $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

25.8 $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

25.9 $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

25.10 $P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

LEGENDRE POLYNOMIALS IN TERMS OF θ WHERE $x = \cos\theta$

25.11 $P_0(\cos\theta) = 1$

25.12 $P_1(\cos\theta) = \cos\theta$

25.13 $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

25.14 $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

25.15 $P_4(\cos\theta) = \frac{1}{64}(9 + 20\cos 2\theta + 35\cos 4\theta)$

25.16 $P_5(\cos\theta) = \frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$

25.17 $P_6(\cos\theta) = \frac{1}{512}(50 + 105\cos 2\theta + 126\cos 4\theta + 231\cos 6\theta)$

25.18 $P_7(\cos\theta) = \frac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$

GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

25.19 $$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

RECURRENCE FORMULAS FOR LEGENDRE POLYNOMIALS

25.20 $$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

25.21 $$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$$

25.22 $$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

25.23 $$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

25.24 $$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

25.25 $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad m \neq n$$

25.26 $$\int_{-1}^{1} \{P_n(x)\}^2\,dx = \frac{2}{2n+1}$$

Because of 25.25, $P_m(x)$ and $P_n(x)$ are called *orthogonal* in $-1 \leqq x \leqq 1$.

ORTHOGONAL SERIES OF LEGENDRE POLYNOMIALS

25.27 $$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots$$

where

25.28 $$A_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx$$

SPECIAL RESULTS INVOLVING LEGENDRE POLYNOMIALS

25.29 $P_n(1) = 1$ **25.30** $P_n(-1) = (-1)^n$ **25.31** $P_n(-x) = (-1)^nP_n(x)$

25.32 $$P_n(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2}\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & n \text{ even} \end{cases}$$

25.33 $$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x+\sqrt{x^2-1}\cos\phi)^n\,d\phi$$

25.34 $$\int P_n(x)\,dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

25.35 $$|P_n(x)| \leqq 1$$

25.36 $$P_n(x) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz$$

where C is a simple closed curve having x as interior point.

GENERAL SOLUTION OF LEGENDRE'S EQUATION

The general solution of Legendre's equation is

25.37 $$y = A\,U_n(x) + B\,V_n(x)$$

where

25.38 $$U_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots$$

25.39 $$V_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots$$

These series converge for $-1 < x < 1$.

LEGENDRE FUNCTIONS OF THE SECOND KIND

If $n = 0, 1, 2, \ldots$ one of the series 25.38, 25.39 terminates. In such cases,

25.40 $$P_n(x) = \begin{cases} U_n(x)/U_n(1) & n = 0, 2, 4, \ldots \\ V_n(x)/V_n(1) & n = 1, 3, 5, \ldots \end{cases}$$

where

25.41 $$U_n(1) = (-1)^{n/2}\,2^n\left[\left(\frac{n}{2}\right)!\right]^2 \Big/ n! \qquad n = 0, 2, 4, \ldots$$

25.42 $$V_n(1) = (-1)^{(n-1)/2}\,2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2 \Big/ n! \qquad n = 1, 3, 5, \ldots$$

The nonterminating series in such case with a suitable multiplicative constant is denoted by $Q_n(x)$ and is called *Legendre's function of the second kind of order n*. We define

25.43 $$Q_n(x) = \begin{cases} U_n(1)\,V_n(x) & n = 0, 2, 4, \ldots \\ -V_n(1)\,U_n(x) & n = 1, 3, 5, \ldots \end{cases}$$

SPECIAL LEGENDRE FUNCTIONS OF THE SECOND KIND

25.44 $$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

25.45 $$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

25.46 $$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

25.47 $$Q_3(x) = \frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}$$

The functions $Q_n(x)$ satisfy recurrence formulas exactly analogous to 25.20 through 25.24.

Using these, the general solution of Legendre's equation can also be written

25.48 $$y = A\,P_n(x) + B\,Q_n(x)$$

26 ASSOCIATED LEGENDRE FUNCTIONS

LEGENDRE'S ASSOCIATED DIFFERENTIAL EQUATION

26.1 $$(1-x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

Solutions of this equation are called *associated Legendre functions.* We restrict ourselves to the important case where m, n are nonnegative integers.

ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

26.2 $$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) = \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

where $P_n(x)$ are Legendre polynomials [page 146]. We have

26.3 $$P_n^0(x) = P_n(x)$$

26.4 $$P_n^m(x) = 0 \qquad \text{if } m > n$$

SPECIAL ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

26.5 $P_1^1(x) = (1-x^2)^{1/2}$

26.6 $P_2^1(x) = 3x(1-x^2)^{1/2}$

26.7 $P_2^2(x) = 3(1-x^2)$

26.8 $P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$

26.9 $P_3^2(x) = 15x(1-x^2)$

26.10 $P_3^3(x) = 15(1-x^2)^{3/2}$

GENERATING FUNCTION FOR $P_n^m(x)$

26.11 $$\frac{(2m)!\,(1-x^2)^{m/2}t^m}{2^m m!\,(1-2tx+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x)t^n$$

RECURRENCE FORMULAS

26.12 $$(n+1-m)\,P_{n+1}^m(x) - (2n+1)x\,P_n^m(x) + (n+m)\,P_{n-1}^m(x) = 0$$

26.13 $$P_n^{m+2}(x) - \frac{2(m+1)x}{(1-x^2)^{1/2}}\,P_n^{m+1}(x) + (n-m)(n+m+1)\,P_n^m(x) = 0$$

ORTHOGONALITY OF $P_n^m(x)$

26.14 $$\int_{-1}^{1} P_n^m(x)\, P_l^m(x)\, dx = 0 \qquad \text{if } n \neq l$$

26.15 $$\int_{-1}^{1} \{P_n^m(x)\}^2\, dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

ORTHOGONAL SERIES

26.16 $$f(x) = A_m P_m^m(x) + A_{m+1} P_{m+1}^m(x) + A_{m+2} P_{m+2}^m(x) + \cdots$$

where

26.17 $$A_k = \frac{2k+1}{2}\frac{(k-m)!}{(k+m)!}\int_{-1}^{1} f(x)\, P_k^m(x)\, dx$$

ASSOCIATED LEGENDRE FUNCTIONS OF THE SECOND KIND

26.18 $$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

where $Q_n(x)$ are Legendre functions of the second kind [page 148].

These functions are unbounded at $x = \pm 1$, whereas $P_n^m(x)$ are bounded at $x = \pm 1$.

The functions $Q_n^m(x)$ satisfy the same recurrence relations as $P_n^m(x)$ [see 26.12 and 26.13].

GENERAL SOLUTION OF LEGENDRE'S ASSOCIATED EQUATION

26.19 $$y = A\, P_n^m(x) + B\, Q_n^m(x)$$